510.5
CUL-H06345-1-P/17547/003/5
P17547

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY.

CONTINUATION OF THE BULLETIN OF THE NEW YORK
MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

F. N. COLE, A. ZIWET, F. MORLEY, E. O. LOVETT.

VOL. VI.

OCTOBER 1899 TO JULY 1900.

PUBLISHED FOR THE SOCIETY
BY
THE MACMILLAN COMPANY,
LANCASTER, PA., AND NEW YORK,
1900.

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

1

BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY

THE SIXTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The Sixth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY was held at the Ohio State University, Columbus, Ohio, on Friday and Saturday, August 25th and 26th, 1899. As has been the custom for several years, the meeting was held in conjunction with the American Association for the Advancement of Science. Aside from the material and incidental advantages, it has been found that the freedom of intercourse between the members of the Society and of Sections A and B of the Association, afforded by simultaneous meetings, serves to increase the interest in both gatherings, while the two-fold attachment of many members tends to enlarge the attendance. During the recent meeting of the American Association the initial steps were taken looking toward the still closer affiliation of the two societies for the purposes of the summer meeting, and the Committee of Section A was authorized to arrange for a joint meeting hereafter, if this should be deemed advisable.

The number of persons present during the meeting was over fifty, and included the following thirty-two members of the Society:

Professor W. W. Beman, Professor Henry Benner, Professor R. D. Bohannan, Professor C. H. Chandler, Professor L. E. Dickson, Professor C. L. Doolittle, Professor L. W. Dowling, Professor H. T. Eddy, Mr. H. B. Evans, Professor B. F. Finkel, Professor G. B. Halsted, Professor Thomas F. Holgate, Professor E. W. Hyde, Mr. H. G. Keppel, Professor T. E. McKinney, Professor A. MacFarlane, Professor W. H. Maltbie, Professor J. L. Markley, Professor Mansfield Merriman, Dr. G. A. Miller, Professor E. H. Moore, Professor A. B. Nelson, Professor H. B. Newson, Professor Alexander Pell, Professor G. T. Sellew, Professor J. B. Shaw, Dr. H. F. Stecker, Professor Ormond Stone, Dr. W. M. Strong, Professor A. L. P. Wernicke, Professor H. S. White, Professor R. S. Woodward.

V4
1899

The first session opened at ten o'clock on Friday morning, and was presided over by Professor R. S. Woodward, President of the Society. On taking the chair the presiding officer congratulated the Society on the manifest interest in mathematical study and investigation in this country as indicated by the long list of papers to be presented, and contrasted the present rapid growth of mathematical knowledge with the conditions twenty or twenty-five years ago.

By holding extended sessions the full programme was completed by Saturday noon. On Friday evening nearly all the members in attendance dined together at the Chittenden Hotel. This informal dinner has come to be a very interesting feature of the summer meeting and affords an opportunity for friendly interchange of thought upon mathematical topics that do not come up in the discussions of the regular programme.

At the Saturday morning session the Council announced the election of the following ten persons to membership in the Society: Professor Edgar Buckingham, Bryn Mawr College; Dr. Warren G. Bullard, University of Vermont; Mr. Walter B. Ford, Albany, N. Y.; Miss Fanny C. Gates, Woman's College, Baltimore; Miss Carrie Hammerslough, Columbia University; Professor Thomas F. Nichols, Hamilton College; Mr. Herbert W. Richmond, King's College, Cambridge, England; Professor George T. Sellew, Knox College, Galesburg, Illinois; Professor E. B. Skinner, University of Wisconsin; Dr. George P. Starkweather, Yale University. Nine applications for membership were received.

The following papers were read:

(1) Dr. A. S. CHESSIN: "Note on relative motion."

(2) Mr. F. R. MOULTON: "On surfaces of zero relative velocity and a certain class of special solutions in the problem of four bodies."

(3) Professor J. E. DAVIES: "On the use of generalized differentiation in the solution of physical problems."

(4) Professor ARNOLD EMCH: "A new class of link works."

(5) Mr. J. V. COLLINS: "A relation between point and vector analysis."

(6) Professor FLORIAN CAJORI: "John Speidell's 'New Logarithmes.'"

(7) Mr. H. W. RICHMOND: "On analogues of the property of the orthocenter."

(8) Professor H. S. WHITE: "Irrational covariant conics of a plane cubic."

(9) Professor Frank Morley: "On the generalization of Desargues's theorem."

(10) Professor E. H. Moore: "On certain crinkly curves."

(11) Dr. W. M. Strong: "Note on non-quaternion number systems."

(12) Professor H. B. Newson: "On mixed groups."

(13) Dr. Edward Kasner: "The invariant theory of the inversion group."

(14) Miss E. N. Martin: "Note on the imprimitive substitution groups of degree fifteen, and on the primitive substitution groups of degree eighteen."

(15) Professor L. E. Dickson: "A new definition of the general abelian linear group."

(16) Professor L. E. Dickson: "Definition of various linear groups as groups of isomorphisms."

(17) Mr. W. B. Fite: "On the groups of cogredient isomorphisms that are abelian."

(18) Dr. G. A. Miller: "On the groups that are direct products of two subgroups."

(19) Dr. G. H. Ling: "A proof that there are no simple groups of order 1440, 1512, 1680, or 1800."

(20) Dr. H. F. Stecker: "On a relation between the totality of the elliptic functions and a line complex."

(21) Dr. Virgil Snyder: "Geometric construction of the elliptic integral of the second kind, and of the Weierstrass sigma-function."

(22) Professor Alexander Pell: "Some applications of elliptic functions."

(23) Dr. L. T. More: "On Fresnel's wave surface."

Mr. Moulton's paper was offered to the Society through Professor Moore, Mr. Collins's through Professor Benner, Dr. Kasner's through Professor Fiske, Miss Martin's through Professor Harkness, and Mr. Fite's through Dr. Miller.

On account of the absence of the authors, Dr. Chessin's paper was read by Professor Woodward, Mr. Moulton's and Professor Morley's by Professor Moore, Professor Davies's by Professor Hyde, Professor Emch's by Professor Newson, Professor Cajori's by the Secretary, Miss Martin's, Dr. Ling's and Dr. Snyder's by Dr. Miller, and Dr. More's by Professor Brace of the University of Nebraska. The papers by Mr. Richmond, Dr. Kasner, and Professor Dickson (No. 15) were read by title. Abstracts of the papers are given below.

Considering space motion in the theory of relative motion as a perturbation of the motion which the given system would have if space motion did not exist, we arrive at the notion of a "perturbative function of relative motion." This function is the sum of three homogenous functions of p, q, r (components of the angular velocity ω of space rotation) of the respective degrees 0, 1, and 2. The purpose of Dr. Chessin's paper is to give the mechanical significance of the third of these three functions. The author shows that it is always possible to consider ω as the diagonal of a parallelogram constructed on two angular velocities ω_1 and ω_2, the direction of ω_1 being one about which relative rotation is *possible*, whereas such a rotation is *impossible* about the direction of ω_2. In this connection it was remarked that it is sometimes useful to express the kinetic energy of rotation of a body referred not to its *principal* axes of inertia, but to a system of *conjugate* axes. If then G denote the kinetic energy of space rotation,

$$2G = A'\omega_x^2 + B'\omega_y^2 + C'\omega_z^2,$$

where ω_x, ω_y, ω_z are the components of ω along and A', B', C' the moments of inertia about the conjugate (generally oblique) axes XYZ of the ellipsoid of inertia. Now, if we take for X the direction of ω_1, then

$$G = G_1 + G_2,$$
$$G_1 = \tfrac{1}{2} A'\omega_1^2, \qquad G_2 = \tfrac{1}{2} I\omega_2^2,$$

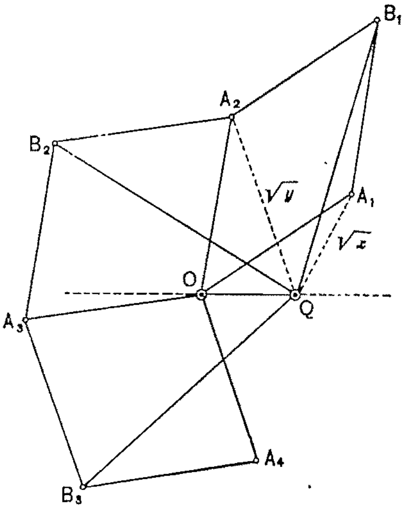
I denoting the moment of inertia about an axis in the plane conjugate with the direction of ω_1. Likewise, if we take for X the direction of ω_2, then

$$G_2 = \tfrac{1}{2} A'\omega_2^2, \qquad G_1 = \tfrac{1}{2} I\omega_1^2.$$

G_2 is exactly the third of the three terms of which ω is composed. Hence its mechanical significance.

Mr. Moulton's paper discussed the motions of four bodies, three of finite masses and one infinitesimal. The finite bodies were supposed to move according to one or other of the Lagrangian solutions, in circles around their common center of gravity. It was shown that the surfaces of zero relative motion separate those portions of space in which the infinitesimal body may move from those in which it cannot move. There are just twenty-eight solutions for which the system remains of constant form and size, and revolves in an arbitrary period.

The purpose of Professor Davies's paper was to call attention to some of the earlier investigations on the subject of generalized differentiation, and in particular to an apparently little known paper by Buchwaldt, a captain in the Danish army, who critically reviewed the work of Liouville, previously published in the *Journal de l'École Polytechnique*, for 1832, *et seq.*. In view of some of the recent papers of Oliver Heaviside, wherein the usefulness of generalized differentiation in problems in electricity is pointed out, it was shown that attention had been called, long ago, by Liouville to the great value of this process in the solution of physical problems generally and that it is likewise valuable in analysis for certain inverse problems.



Professor Emch's paper, which is intended for publication in the *Annals of Mathematics*, is in abstract as follows:— The element of the linkwork is the *cell* $OA_1B_1A_2$-$O \cdot QB_1$ in which O, Q are fixed points, and $OA_1 = A_1B_1 = B_1A_2 = A_2O$. Putting $QA_1 = \sqrt{x}$, $QA_2 = \sqrt{y}$, $QB_1 = R$, $OA_1 = r$, $OQ = c$, and $(R + r)^2 = \alpha$, $(r + c)^2 = \beta$, $(r - c)^2 = \gamma$, $(R - r)^2 = \delta$, the relation exists

$$\int^{y} \frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} - \int^{x} \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}} = a$$

where a is a constant. Let v and u designate the values of these integrals, so that $v - u = a$. The variables x and y may be expressed by elliptic functions

$$x = \lambda(u), \quad y = \lambda(v).$$

A second cell may be added having the same fixed points O and Q and the link OA_2 in common with the first cell. In the same manner any number of cells may be added, thus forming a link-work of one degree of freedom. If the link-work consists of n cells, $\overline{OA_{n+1}}$ will be the last link and the argument belonging to QA_{n+1} will be $u + n\cdot a$. Supposing w_1 and w_2 as the periods of $\lambda(u)$, the quantities R, r, and c may be chosen in such a manner that

$$a = \frac{m_1 w_1 + m_2 w_2}{n}$$

where m_1 and m_2 designate integers. *In this case the link-work closes, i. e., the point A_{n+1} will coincide with A_1, and the condition of closing is independent of the value of u, or of the original position of the link-work.* In the special cases of closed link-works of three and four cells the condition reduces to $R = r$ and $R^2 - r^2 = r^2 - c^2$, respectively. A certain transformation of the link-work leads to Jacobi's construction of elliptic functions and Poncelet's poristic polygons.

In his Quaternions (second edition, page 22) Tait asserts that "The theory of 'geometrical nets' is very fully developed in Hamilton's Elements, and the method is shown to include as particular cases the processes of Grassmann's Ausdehnungslehre and Möbius's Barycentrische Calcul." This passage is criticized by Mr. Collins on two grounds: first, that it is too sweeping; and secondly, that if point and vector analysis are so closely connected, this relationship calls for a careful examination. The question arises, then, how can the two be brought together in a harmonious system. In his Elements, instead of studying point-vector analysis in its simplest form, Hamilton makes out of the simple expressions of point analysis very complex ones, anharmonic ratios in fact. In this way he limits the application of this analysis to the so-called modern geometry.

By Hamilton's ρ and Grassmann's p is or should be meant a complex concept consisting of a point p held in position by a vector from an origin ρ. Here p we will say denotes an infinitesimal portion of a line and $p\rho$ denotes p at the end of ρ whose initial point is at the origin. Thus $p\rho$ standing now for both Hamilton's ρ and Grassmann's p is really complex, just as ϵ $(= T\epsilon U\epsilon$ or $S\epsilon + V\epsilon)$ is complex in quaternions, or as $p_1\epsilon_1\epsilon_2$ is complex in the Ausdehnungslehre. Using this notation the fundamental equations of point analysis become

$$(1) \qquad m_1 p\rho_1 + m_2 p\rho_2 = (m_1 + m_2)\, p\left(\frac{m_1\rho_1 + m_2\rho_2}{m_1 + m_2}\right),$$

$$\Sigma mp\rho = \Sigma m \cdot p\left(\frac{\Sigma m\rho}{\Sigma m}\right), \tag{2}$$

$$p\rho_2 - p\rho_1 = 0\,p\left(\frac{\rho_2 - \rho_1}{0}\right). \tag{3}$$

Hyde in his Directional Calculus writes these equations in duplicate, *i. e.*, first in the simple point analysis form, and then in the vector analysis form, the vectors being indicated as the difference of units points. If the above notation and equations be used in the solution of problems, the point equations and vector equations correspond and agree at each step of the solution. If p be divided out of any equation, a vector equation is left, while if the p's be given the subscripts of the ρ's, and the ρ's be removed, we have the regular point analysis equations. Thus this notation throws light on any solution or equation from two directions. The vector equations have the advantage that the origin can be taken anywhere; the point equations that they are always symmetrical. Because the fundamental concept dealt with in point analysis is complex, the notation used to present it should be complex until the subject is well grasped.

Professor Cajori remarked in his paper that no general history of mathematics published in England or in Europe makes mention of John Speidell, though it is quite generally known to English readers through certain special articles, for example, J. W. L. Glaisher's report on "Tables" in the British Association Report for 1873, that Speidell is the author of the earliest table of natural logarithms. The book appeared in 1619, only five years after Napier's famous work. A fac-simile of the title-page of the edition of 1622 was exhibited to the Society.

Mr. Richmond's memoir deals first with the systems of $2n$ points in a space of n dimensions S_n, such that the S_{n-1} determined by any n points is perpendicular to that determined by the remaining n points. Here for 'perpendicular' we may substitute 'conjugate as to a fixed quadric,' the absolute. More generally in S_n let there be p points $u_1 = 0, \cdots, u_p = 0$ and let the absolute be

$$\sum_1^p k_r u_r^2 = 0\,;$$

then any two spaces S_{n-1} which contain all the p points are necessarily perpendicular. For a proper quadric the values of p range from $n+1$ to $2n$; for the euclidean absolute (passing over the case of points all at infinity) p ranges from $n+2$ to $2n$. Such a system of p points is called an orthogenic system. The system of $n+1$ orthogenic points is the self-conjugate pyramid; accordingly the system of $n+2$ points is selected for special consideration. This is the 'polviereek' in the case of the conic ($n=2$), and the 'polfünfeck' in that of the quadric in 3 dimensions. In the euclidean case, it is pointed out, *inter alia*, that the centroids of all pairs formed from the $n+2$ points lie on a sphere; and if further n be even, it is remarked that $n+1$ points determine uniquely $\frac{1}{2}n$ others, the whole forming an orthogenic system.

Professor White's paper embodied the results of some recent investigations upon a plane cubic curve and the systems of conics connected with it, and will be published in the *Transactions of the American Mathematical Society*. Two doubly infinite systems of conics are known (Hilbert in *Liouville*, 1888) which are related to the general plane cubic by an invariant property, namely, their equations are reproduced (to a factor) by an operation which transforms the cubic into its Hessian. Professor White finds each of these systems of conics to be the system of conic polars to a new cubic curve, and (1) these two cubics are coördinate with the fundamental cubic by virtue of a *cyclic* relation—each gives rise to the other two in the same way as the first did to the second and third; (2) all three lie in the syzygetic sheaf containing the first and its Hessian; (3) finally, the set of three are characterized by having their Cayleyan (curve of third class) in common. Two class-cubics defined by the self-reproductive property with respect to the transverse of this same operation prove to be coördinate to the conjugate class-cubic of the fundamental, and to be themselves conjugate to the two above described in the sense of Aronhold (*Crelle*, 55). They have the Cayleyan for their common Hessian. From these results, using the equation of the conjugate class-cubic, the covariant equations are written for conic polars of four cubics related to the Hessian just as the first four are to the Cayleyan.

Professor Morley communicated a proof of the theorem that all full spreads of order n in a linear space S_m which pass through

$$\frac{(m+n-1)!}{m!\,(n-1)!}+1

given points are apolar with a determinate full spread of class n in an intersecting space S_{n-1}. This paper is intended for publication in the *Transactions.*

Professor Moore's paper treats (1) of the area filling curves $x = \varphi(t)$, $y = \psi(t)$ of Peano and Hilbert, and in particular (2) of the curve $y = \psi(t)$, connected with the Peano curve, which is continuous and has at no point a progressive or a regressive non-vertical tangent, and at no point a progressive and a regressive vertical tangent. This curve from the simplicity of its geometric definition and from the intuitive clearness of its properties appears to be fit to replace the classical Weierstrass curve as the standard example of continuous curves having no tangents. The paper will be published in the *Transactions.*

Dr. Strong announced the following theorem: Any degenerate non-quaternion number system whose multiplication table contains m independent products can be reduced to a form such that the independent products will be $l_1, l_2, l_3, \cdots, l_m$, and will occur in the order here given. No independent product will be the product of units which themselves enter products. By "degenerate system" is meant a system from which the modulus has been deleted; the term 'independent product' denotes a product of two units which is linearly independent of the products preceding it; and the 'order' of products in the multiplication table of non-quaternion number systems is defined thus: $l_i \cdot l_{i+a}$ and $l_{i+a} \cdot l_i$ precede $l_{i+b}\, l_{i+c}$ where $b, c \neq 0$; $l_i \cdot l_{i+a}$ and $l_{i+a} \cdot l_i$ precede $l_i \cdot l_{i+b}$ and $l_{i+b} \cdot l_i$ if $a < b$; $l_{i+a}\, l_i$ precedes $l_i\, l_{i+a}$ where $a \neq 0$.

The object of Professor Newson's paper was to exhibit a method for treating mixed groups of projective transformations (or collineations). The method was applied to one mixed group as an example. Mixed groups contain two distinct sets of transformations, one set forms a continuous group and the other set forms a multitude of discontinuous groups. All projective transformations of the plane which leave a pair of points A and B separately invariant form a continuous group $G\,(AB)$; add to these all transformations which interchange A and B, and we have the mixed group $mG\,(AB)$. All real projective transformations which interchange A and B were determined synthetically; they fall into four distinct classes. Those of the first class are involutoric, *i. e.*, of period two; those of the other three classes are non-periodic and are designated as elliptic, parabolic, and hyperbolic, respectively. The paper will be published in the *Kansas University Quarterly.*

To every group of transformations corresponds a geometry, or theory of invariants (Klein, Lie). The group considered by Dr. Kasner is the six-parameter group generated by the inversions of the plane; the corresponding inversion geometry is formally identical with projective geometry on a quadric surface. Two principal methods are developed for studying the invariant properties of geometric configuration. 1° Quaternary.—A curve whose equation in tetracyclic coördinates x_i connected by the quadratic identity $Q = 0$ is $f_n = 0$ is also represented by $f_n + MQ = 0$, where M is any form of order $n - 2$; but of the entire system of forms $f_n + MQ$, there is one and only one (F) which is apolar to Q. The final result may be stated: the (quaternary) inversion concomitants of the curve are the projective concomitants of F, Q. 2° Double binary.—A point being defined by the parameters $\lambda_1 : \lambda_2$, $\mu_1 : \mu_2$ of the pair of minimal lines passing through it, the equation of the curve to be studied takes the form $\varphi\ (\lambda_1{}^n\lambda_2,\ \mu_1{}^n\mu_2) = 0$; the (binary) inversion concomitants of the curve are the concomitants of the double binary form φ, with respect to independent linear transformations of λ and μ. The relations between 1° and 2° were considered in detail. It was shown for example that the binary and quarternary polar theories coincide; and a series of methods was given of which 2° is the first member. Application was made to circles and cyclic curves.

In Miss Martin's paper it was shown that there are 68 imprimitive groups of degree fifteen. Of these 28 have their elements divisible into three systems of imprimitivity, 27 into five systems, while the remaining 13 have their elements divisible in both ways. In addition to the symmetric and to the alternating groups of degree eighteen only two other primitive groups have been found. One is a triply transitive group of order 18. 17. 16; the other is a doubly transitive simple group of order 16. 17. 8 contained self-conjugately in the former.

Professor Dickson's papers are intended for publication in the *Transactions*. An abstract giving some of the results of both papers was published in the BULLETIN, vol. 5, p. 331. Mr. Dickson also presented a preliminary communication containing an elementary proof of a theorem by Professor Moore on the abstract form of the symmetric group on k letters.

Mr. Fite's paper presented the following considerations: The group of cogredient isomorphisms of any group G cannot be the direct product of cyclical groups whose orders are such that one of them is not a divisor of the least common multiple of the others. If the group of cogredient isomorphisms is abelian, the order of every one of its operators is the order of a commutator of G. If G is of order p^m and contains an abelian subgroup of order p^{m-a}, its group of cogredient isomorphisms contains no operator of order greater than p^a, if it is abelian; if the largest abelian subgroup of G is of order p^{m-a}, where $a \neq \frac{1}{2}(m-1)$, G cannot have an abelian group of cogredient isomorphisms of order p^{m-1}. If G is of order p^m and contains an abelian subgroup of order p^{m-1}, it cannot have an abelian group of cogredient isomorphisms of order p^{m-a}, where $a < \frac{1}{2}(m-1)$.

Dr. Miller's paper, which will be offered for publication to the *Transactions*, is in abstract as follows:

If a group G contains two self-conjugate subgroups G_1, G_2 which have just K common operators, and if we form all the possible commutators $s^{-1}t^{-1}st$ by taking s from G_1 and t from G_2 (or vice versa) then these commutators generate a self-conjugate subgroup of G which is contained in K, and each operator of one of the two subgroups G_1, G_2 transforms every operator of the other into itself multiplied by some one of the given commutators. If G is generated by G_1 and G_2, and if $K = 1$, then G is said to be the direct product of G_1, G_2. Two theorems are proposed:

THEOREM I. If the order g of a group G is divisible by p^a but not by p^{a+1}, p being any prime number, and if all the conjugates of any operator (s) contained in a subgroup P of order p^a may be obtained by transforming s by means of operators of P, then G is the direct product of P and some other subgroup of G, and it contains a self-conjugate subgroup of each of the orders $g \div p^\beta$ ($\beta = 0, 1, 2, \ldots, a-1$).

THEOREM II. If the order of a group is mn (m being prime to n) and if it contains a subgroup of order m which has the property that each of its operators is transformed by some one of the operators of this subgroup in the same way as it is transformed by any given operator of the entire group, then the group is the direct product of two subgroups whose orders are m and n respectively.

Hölder, Cole, and Burnside have examined all the orders up to 1093 in regard to possible simple groups and proved that there are no simple groups of any of these orders with exception of those already known whose orders are 60, 168, 360, 504, 660 and 1092. At the February meeting of this Society Dr. Ling presented a paper in which all the other orders that do not exceed 2000, except four, were examined. The orders 1440, 1512, 1680 and 1800 presented difficulties which could not then be overcome. With the assistance of Dr. Miller, Dr. Ling has now examined these orders and the result reached is that *there is no simple group whose order lies between the numbers* 1092 *and* 2001. These results will be incorporated in the paper previously presented to the Society and will be offered to the *American Journal of Mathematics* for publication.

The fundamental considerations in Dr. Stecker's paper were these: (1) the six line coördinates of a complex satisfy the identical relation $p_1p_2 + p_3p_4 + p_5p_6 = 0$, and conversely, any six quantities satisfying such an identical relation may be treated as line coördinates of a complex; (2) the biquadratic which enters into an elliptic integral has three irrational invariants each of which is the product of two invariant determinants. These three irrational invariants, A, B, C, satisfy the relation $A + B + C = 0$. Hence the above-mentioned six determinants may be treated as line coördinates of a complex. Several interesting results were pointed out.

Dr. Snyder's paper describes the models of the functions $\sigma(u)$ and $\zeta(u)$, recently prepared by the author under the direction of Professor Klein. These models, which are to be reproduced in plaster, are constructed with the same constants ($g_2 = 0$, $g_3 = 4$) employed by Professor Burkhardt in the case of $\wp(u)$ and $\wp'(u)$, thus completing the set to a certain extent.

Professor Pell's paper dealt with the curves on an ellipsoid, called by E. Cosserat the 'D' lines. The author expressed the coördinates of these lines for the ellipsoid as uniform functions of one parameter; by means of these expressions he traced the path of the curves, and deduced some interesting properties of these lines. The paper will be offered for publication to the *Transactions*.

Dr. More pointed out in his paper that it is generally assumed that unless a ray of light in a biaxial crystal is refracted along an optic axis, it will break up into two rays—the ordinary and the extraordinary—but the fact is that there are an infinite number of directions along which the ray is not divided. Suppose the section of Fresnel's wave surface by a coördinate plane to consist of a circle and an ellipse and let O be the center of a wave disturbance in the surface of a crystal. Draw any radius vector OBC, cutting the circle at B and the ellipse at C; the intersection of tangents at B and C will be S, a point on the tangent plane at the point of incidence. Then SB and SC will be the wave fronts of an ordinary and an extraordinary ray. Since these two lines are tangents at the points of intersection of a single radius vector OBC, the latter is the path of both an ordinary and an extraordinary ray. Therefore from the indices of refraction we may compute an incident ray which will not be divided but will move along OBC, and since there are an infinite number of points S there are an infinite number of directions for which the ray is not divided. The locus of the point S is a curve of the eighth degree and may be expressed in a simple and symmetrical form. By a simple transformation of the equation uniaxial crystals are included in this treatment.

THOMAS F. HOLGATE,
Acting Secretary.

EVANSTON, ILLINOIS.

REPORT ON THE RECENT PROGRESS IN THE THEORY OF LINEAR GROUPS.

BY PROFESSOR L. E. DICKSON.

(Read before Section A of the American Association for the Advancement of Science, Columbus, Ohio, August 22, 1899.)

THIS report is intended to supplement in certain directions the report* of Dr. Miller on group theory read at the last summer meeting of the American Association. In order to allow space for important details, I will consider neither continuous nor discontinuous linear groups of infinite order, but confine my report to finite linear groups. Of these I shall consider first the finite collineation groups and afterwards the linear congruence groups and the more general linear groups in Galois fields.

PART I.—FINITE GROUPS OF LINEAR SUBSTITUTIONS.

§1. *General Theorems on Finite Linear Groups.*

The following theorem was stated by Jordan in 1878 and proven in various ways by Lipschitz, Kronecker, Rost,

---

* BULLETIN, Feb., 1899, p. 227.

Maschke, Moore,* and recently by Burnside† in a very elementary manner :

A linear homogeneous substitution of finite period p,

$$x_i' = \sum_{i=1}^{n} a_{ij} x_j, \qquad (j = 1, 2, \cdots, n)$$

may be linearly transformed to the canonical form

$$y_i' = \varepsilon_i y_i, \qquad (i = 1, 2, \cdots, n)$$

where each ε_i *is a* p^{th} *root of unity.*

Concerning the nature of the irrationalities which may enter into the coefficients of the substitutions of a finite linear group, we have the theorem due to Professor Maschke :—‡

If a linear substitution group of finite order contain at least one substitution, the roots of whose characteristic equation are all distinct, the group can be so transformed that all the coefficients of the substitutions of the transformed group are cyclotomic, i. e., are rationally expressible by roots of unity.

The existence of a universal invariant for finite linear groups is expressed by the theorem :—

A finite group of n*-ary linear homogeneous substitutions leaves absolutely invariant a positive Hermitian form. By suitable linear transformation of the group this invariant becomes*

$$\sum_{i=1}^{n} x_i \bar{x}_i .$$

This theorem was stated by Loewy § and proven by Moore.‖ The first part of the theorem was established earlier by Picard ¶ for the cases $n = 2, 3$. The complete theorem was proven by Valentiner ** for $n = 2$ and 3.

H. Minkowski †† has shown that every finite group G of n-ary linear homogeneous substitutions with integral coefficients is isomorphic with the group of substitutions resulting upon reducing its coefficients modulo 2, and derives the corollary that the order of G is a divisor of 2^nN, where $N \equiv (2-1)(2^n-2)\cdots(2^n-2^{n-1})$ is the order of the group of all n-ary substitutions taken modulo 2.

---

* *Math. Annalen*, vol. 50, pp. 213–219. Full references are given in the paper by Professor Moore.

† *Messenger of Math*, no. 331 (November, 1898).

‡ *Math. Annalen*, vol. 50, p. 492.

§ *Comptes rendus*, vol. 123 (1896), pp. 168–171.

‖ *Math. Annalen*, vol. 50, pp. 213–219.

¶ *Bull. Soc. Math de France*, vol. 15 (1887), pp. 152–156.

** "De endelige Transformations-Gruppers Theori," avec un resumé en français, *Danish Academy*, 1889.

†† *Crelle*, vol. 100 (1887), pp. 449–458.

The generators of the group of n-ary linear substitutions of determinant unity and with integral coefficients were given by Kronecker * as follows:

$$(1) \qquad x_1 = -x_k', \quad x_k = x_1', \quad x_i = x_i' \quad (i \gtrless k,\ i > 1),$$

$$(2) \qquad x_1 = x_1' + x_2', \qquad x_i = x_i' \qquad (i = 2, 3, \cdots, n).$$

In place of these n generators, Krazer † uses three generators. But Kronecker had already shown that two generators would suffice, while the simple form of the generators would thereby be lost. For the case $n = 3$, W. Burnside ‡ proves that the generator (2) together with the substitution permuting x_1, x_2, x_3 in a cycle suffices to generate the group. In Kronecker's paper the generators of the abelian linear groups are also given.

§ 2. *Special Groups of Collineations.*

The problem of the determination of all finite groups of binary linear substitutions has been solved geometrically by Klein § and algebraically by Gordan.‖ Essentially the groups thus found are those connected with the regular solids. The finite ternary groups were considered at great length by Jordan,¶ who, however, missed the group of 168 collineations in the plane, which was derived by Klein ** by the consideration of the transformation of the seventh order of elliptic functions. He also missed the simple group of 360 collineations, set up by Valentiner †† and proven simply isomorphic to the alternating group on six letters by Wiman. ‡‡ Certain configurations connected with this G_{360} had been given earlier by Gerbaldi. §§

---

* *Monatsber. der Berl. Akad.* (1866), p. 597.

† *Annali di Matematica*, 2d series, vol. 12, pp. 283–300.

‡ *Messenger of Math.*, 2d series, vol. 24 (1894), pp. 109–112.

§ *Math. Annalen*, vol. 9 (1876), pp. 183–188.

‖ *Math. Annalen*, vol. 12 (1877), pp. 23–46.

¶ *Crelle*, vol. 84 (1878), pp. 89–215; *Atti della Reale Accademia di Napoli*, vol. 8 (1879).

** *Math. Annalen*, vol. 14, pp. 428–471 (Jordan's omission is indicated in the foot-note to p. 438); *Math. Annalen*, vol. 15, pp. 251–282.

†† "De endelige Transformations-Gruppers Theori," avec un résumé en français; 1889, *Danish Academy*.

‡‡ *Math. Annalen*, vol. 47.

§§ "Sul gruppo semplice di 360 collineazioni piane," *Math. Annalen*, vol. 50, p. 472.

In view of the evident independence of the work of Jordan and Valentiner and their difference of method, we may feel certain of the completeness of the determination of the finite ternary linear groups. Of these groups the ones which leave a straight line invariant as well as those permuting three points cyclically are regarded as trivial. Besides these there exist only the groups of orders 216, 168, 360, together with their subgroups of orders 72, 36, and 60. We will briefly characterize the former set of three groups.

The Hesse group of 216 collineations is defined by a pencil $\lambda F + \mu H$, where F is a ternary cubic and H its Hessian covariant. The G_{216} has subgroups of orders 36 and 72. It has been studied at length by Maschke.*

The simple group of 168 collineations leaves invariant

$$x^3y + y^3z + z^3x = 0.$$

As mentioned above, it was first met by Klein. It is studied in detail in Klein-Fricke's Modulfunctionen. The types of substitutions and of subgroups of the G_{168} are given by Gordan † (*Mathematische Annalen*, vol. 25, p. 462).

The simple group of 360 collineations has as its simplest invariant form the function

$$10\,x^3y^3 + 9\,z\,(x^5 + y^5) - 45\,x^2y^2z^2 - 135\,xyz^4 + 27z^6.$$

It has two systems, each of 15 conjugate octahedral subgroups, and two systems each of six conjugate icosahedral subgroups. The even permutations of the icosahedral groups of one such system form the alternating group on six letters (Wiman). The G_{360} has ten conjugate subgroups of order 36. It is not possible to set up a group of 360 ternary homogenous substitutions holoedrically isomorphic with the group of 360 collineations.‡ This result follows from the corresponding impossibility for the subgroup G_{36}, as shown by Jordan.

Quaternary linear homogenous groups of orders 120 and 720 have been given by Klein,§ one of order 2·16·6! was studied by Reichardt.¶ All ternary and quaternary linear groups simply isomorphic to the symmetric or alternating group of substitutions have been determined by Maschke.**

---

* *Math. Annalen*, vol. 33.

† "Ueber Gleichungen siebenten Grades mit einer Gruppe von 168 Substitutionen:" I. *Math. Annalen*, vol. 20, pp. 516–530 ; II. *Math. Annalen*, vol. 25, pp. 459–521.

‡ Wiman, *Math. Annalen*, vol. 47, p. 538.

§ Ikosaeder, p. 198 ; *Math. Annalen*, vol. 4.

¶ *Leipzig Ber.* (1885), pp. 419–426.

** *Math. Annalen*, vol. 51.

The problem emphasized by Klein of representing every group as a collineation group in the fewest possible number of variables has been solved for the symmetric and alternating groups by A. Wiman.[*] He shows that, for $n>7$, no collineation group exists in space of $<n-2$ dimensions, which is isomorphic with the symmetric or alternating group on n letters. Except for the case $n=9$, there is only one group in R_{n-2} isomorphic with the symmetric and one with the alternating group on n letters.

A quaternary linear homogeneous group of order 51840 arising in the theory of the transformation of hyperelliptic functions has been studied by Klein, Burkhardt, Maschke, Witting, and others. Jordan[†] had already determined the Galois group of the equation for the 27 lines on a general cubic surface and proved[‡] it isomorphic with the abelian group for the trisection of hyperelliptic functions of four periods. The possibility of the solution by hyperelliptic functions of the equation for the 27 lines on a cubic surface having been thus established, Klein[§] proceeded to sketch a method of reduction of the problem of the equation of the 27 straight lines to the problem of the trisection of the hyperelliptic function of 1st order. The actual calculations were effected by Burkhardt,[||] who gives full references to the work of Witting[¶] on a configuration in space connected with the G_{51840} and to that of Maschke[**] on the complete form system of the group.

From the obscurity of a passage on p. 214 of a note by G. G. Morrice,[††] some readers have inferred that the simplicity of the group of 51840 linear substitutions had been established. The group probably meant was the group of order 4 and the word "simplicity" is not used in its technical group sense.

Various linear congruence groups will be found in Part II which are isomorphic with the present G_{51840}. The latter group is a subgroup of index 28 under the group of order 1,451,520, the group of the equation of the 28th degree on which depends the theory of the double tangents to a curve of the fourth order.[*]

---

[*] *Math. Annalen*, vol. 52 (1899), pp. 243-270; *Gött. Nach.* 1897.

[†] Traité des substitutions, pp. 316-329; also Kühnen's Dissertation, Marburg, 1888.

[‡] *Ibid*, pp. 365-369; a shorter proof has been given by the writer in *Comptes rendus*, vol. 128 (1899), pp. 873-5.

[§] *Liouville's Journal* (4), vol. 4 (1888), pp. 169-176.

[||] *Math. Annalen*, vol. 38, and vol. 41, p. 312.

[¶] Gött. Diss., 1887; *Math. Annalen*, vol. 29.

[**] *Math. Annalen*, vol. 33, pp. 317-344.

[††] *Proc. Lond. Math. Soc.*, vol. 22 (1892), pp. 213-217.

The determination of all surfaces of the third order which admit of collineations has been made by Karl Bobek.[†] The only surfaces found were those studied by Eckhardt.[‡] Among these is Clebsch's diagonal surface[§] with the group G_{120}; also surfaces with a G_{648} and a G_{108}.

In the paper of Biermann,[||] "Ueber die regelmässigen Punktgruppen in Räumen höherer Dimension und die zugehörigen linearen Substitutionen mehrerer Variabeln," Klein's method for ordinary space is extended to n dimensions and worked out for the case $n = 5$. Schlegel had already determined the regular bodies in R_5. Biermann reached linear substitutions in two complex variables previously studied by Jordan and Klein.

Two special classes of finite linear groups of orders n^3 and $n^4(n^2 - 1)$ are given by Jordan.[¶]

PART II.—LINEAR GROUPS IN A GALOIS FIELD.

§3. *Theory of the Galois Field.*

In a great many investigations in group theory, including the classic ones of Galois, Betti, Mathieu, Jordan, as well as in numerous more recent articles, the theory of Galois's imaginaries has proved to be an instrument of great power. Galois first studied the properties of the polynomials built on a root of a congruence of degree n belonging to and irreducible modulo p. These p^n distinct polynomials combine according to the ordinary laws of algebra. Let for example j denote an (imaginary) root of the irreducible congruence

$$x^3 + x + 1 \equiv 0 \pmod{2}.$$

Then all rational functions of j can be reduced to one of the eight forms

$$0, \quad 1, \quad j, \quad j+1, \quad j^2, \quad j^2+1, \quad j^2+j, \quad j^2+j+1.$$

The sum, difference, product, or quotient (except by zero) of any two of these polynomials reduced to an unique polynomial belonging to the set.

---

[*] Geiser, *Math. Annalen*, vol. 1, pp. 129–138.
Jordan, Traité des substitutions, p. 330; *Comptes rendus*, vol. 67 (1868).
Weber, *Math. Annalen*, vol. 23, pp. 489–503.

[†] *Monatshefte für Math. und Physik.* (Wien), vol. 10 (1899), pp. 102–122.

[‡] *Math. Annalen*, vol. 10, p. 227.

[§] *Math. Annalen*, vol. 4, p. 336.

[||] *Wien Ber.*, vol. 95, pp. 523–548.

[¶] *Bull. S. M F.*, vol. 5, pp. 175–177.

Moore's abstract formulation of the theory is convenient for group investigations. In place of the above set of eight polynomials, Moore considers eight abstract *marks* constituting the Galois field of order 8, in notation the $GF[2^3]$.

Rather elaborate expositions of the theory have been given by Schönemann, Dedekind, Serret, and Moore. A very complete, but elementary, exposition of the abstract theory of Galois fields, with proof of their existence, will be published shortly by the writer. Owing to the lack of references in the papers on the subject, the following bibliography on Galois fields has been added:

Galois: "Sur la théorie des nombres," *Bulletin des sciences mathématiques*, de M. Férussac, 1830; reprinted in *Journal de mathématiques*, 1846, and in Oeuvres mathématiques d'Evariste Galois, Paris, 1897.

Schönemann: "Grundzüge einer allgemeinen Theorie der höheren Congruenzen deren Modul eine reelle Primzahl ist," *Crelle*, vol. 31 (1846), pp. 269–325.

Dedekind: "Abriss einer Theorie der höheren Congruenzen in Bezug auf einen reellen Primzahl-Modulus." *Crelle*, vol. 54 (1857), pp. 1–26.

Serret: "Sur les fonctions entières irréductibles suivant un module premier, dan le cas où le degré est une puissance du module," *Journal de Liouville*, 1873, p. 301 and p. 437; also in Cours d'algèbre supérieure, vol. 2, pp. 122–211.

Jordan: "Théorie de Galois," Traité des substitutions, 1870, pp. 14–18.

Pellet: "Sur les fonctions irréductibles suivant un module premier et une fonction modulaire," *Comptes rendus*, vol. 70 (1870), pp. 328–330, and vol. 93 (1881), p. 1065, also vol. 90 (1880), p. 1339.

Pellet: "Sur les fonctions réduites suivant un module premier," *Bulletin Soc. Math. de France*, vol. 17 (1889), pp. 156–167.

Pellet: "Sur la décomposition d'une fonction entière en facteurs irréductibles suivant un module premier p," *Comptes rendus*, vol. 86 (1878), p. 1071.

Moore: "A doubly-infinite system of simple groups," Congress Mathematical Papers, 1893.

Borel et Drach: Théorie des nombres et algèbre supérieure, 1895.

Dickson: "Higher irreducible congruences," Bulletin, July, 1897, pp. 381–389.

§4. *The General Linear Homogeneous Group.*

Consider the p^{nm} letters $l_{\xi_1, \xi_2, \cdots, \xi_m}$ characterized by m indices, each being an arbitrary one of the p^n marks of $GF[p^n]$. A *linear homogeneous substitution* (A) on these letters replaces $l_{\xi_1, \cdots, \xi_m}$ by $l_{\xi_1', \cdots, \xi_m'}$ where

$$\text{(A)} \qquad \xi_i' = \sum_{j=1}^{m} a_{ij}\xi_j \qquad (i = 1, \cdots, m),$$

the a_{ij} being certain marks of the $GF[p^n]$, such that the determinant $| a_{ij} | \neq 0$.

Since the compound of two linear homogeneous substitutions is again a linear homogeneous substitution, we say that the totality of substitutions (A) form a group, the *general linear homogeneous group of degree* p^{nm}. In the linear form, this group is due to Jordan (Traité des substitutions, § 119, § 169), special forms being due to Galois. In the writer's dissertation* it was proven to be identical with a group of substitutions in one variable investigated by E. Betti in 1852–1855 and in more detail by E. Mathieu in 1861. Its structure (factors of composition) was determined by Jordan for the case $n = 1$, and for general n by the writer in his dissertation and somewhat later by Burnside.†

§ 5. *The Linear Fractional Group.*

It is obtained analytically by taking the linear homogeneous group fractionally, so that its substitutions may be exhibited in the form

$$\xi_1' : \xi_2' : \cdots : \xi_m' = \sum_{j=1}^{m} a_{1j}\xi_j : \sum_{j=1}^{m} a_{2j}\xi_j : \cdots : \sum_{j=1}^{m} a_{mj}\xi_j$$

It is a concrete form of the quotient group of the general linear homogenous group by the invariant subgroup formed of the substitutions.

$$\xi_i' = \mu\xi_i, \qquad (i = 1, \cdots, m).$$

For the case $m = 2$, we have the ordinary linear fractional group in one variable. It was investigated by Galois, Mathieu,‡ etc.; its structure was determined for the case $n = 1$ by Gierster‖ and for general n by Professor Moore§

---

* *Annals of Math.*, 1897, pp. 65–120, pp. 161–183.

† The theory of groups, pp. 340–341.

‡ *Liouville's Journal* (2) vol. 5, pp. 9–42; vol. 6, pp. 241–323.

‖ *Math. Annalen*, vol. 18 (1881), pp. 319–365.

§ The Congress Math. Papers of 1893; abstract in the BULLETIN, Dec. 1893.

and later, but imperfectly, by Professor Burnside.* The subgroups of the linear fractional group in the $GF[p^n]$ were determined by Gierster for $n = 1$ (*Mathematische Annalen*, vol. 18 (1881), pp. 319–365) and for general n by Moore (Abstract in the BULLETIN, Oct., 1898). It was shown that the subgroups of the modular group are either solvable groups or certain modular groups or finally groups of double the order of certain modular groups. The decomposition of the subgroups of the modular groups thus leads only to cyclic groups and modular groups.

The linear homogeneous group was proven by Moore (BULLETIN, Nov., 1895) to be the group of holoedric transformation into itself of the regular abelian group of degree p^n. In the same paper were determined certain abstract defining invariants of the linear homogeneous and the linear fractional groups.

§ 6. *The Abelian Linear Group.*

Consider two sets of $2m$ indices

$$\xi_{i1}, \quad \eta_{i1}; \quad \xi_{i2}, \; \eta_{i2} \qquad (i = 1, 2, \cdots, m).$$

Of the substitutions of the general linear homogeneous group on $2m$ indices in the $GF[p^n]$, those which, if operating simultaneously on the two sets of indices, multiply the function

$$\zeta \equiv \sum_{i=1}^{m} \begin{vmatrix} \xi_{i1} & \xi_{i2} \\ \eta_{i1} & \eta_{i2} \end{vmatrix}$$

by a constant factor, form a group, the general abelian group. We exclude the case $m = 1$, when the group is merely the general binary linear homogeneous group treated in § 4.

Its structure was determined by Jordan † for the case $n = 1$ and by the writer ‡ for general n. Indeed, its factors of composition are the prime factors of $p^n - 1$ together with the factors of composition of the special abelian group of those substitutions leaving ζ absolutely invariant. For $p = 2$, the latter group is simple except when $m = 2$, $n = 1$, when it is isomorphic with the symmetric group on six letters. For $p > 2$, it has a maximal self-conjugate subgroup of order 2, so that the quotient group is simple.

§ 7. An immediate generalization of the abelian group consists in the group of substitutions on mq variables

---

* *Proc. Lond. Math. Soc.*, vol. 25 (Feb., 1894), pp. 113–139.

† Traité des substitutions, pp. 171–179.

‡ *Quar. Jour. of Math.*, July, 1897, pp. 169–178, and April, 1899, pp. 383–4.

p17547

$$x_{ij}' = \sum_{k=1}^{m} (a_{k1}^{ij} x_{k1} + \cdots + a_{kq}^{ij} x_{kq}) \qquad \binom{i=1,\cdots,m}{j=1,\cdots,q}$$

which, if operating cogrediently upon q independent sets of mq variables, the j th set of which is exhibited thus

$$x_{i1}^{(j)}, \quad x_{i2}^{(j)}, \quad \cdots, \quad x_{iq}^{(j)} \qquad (i = 1 \cdots m)$$

will leave invariant up to a constant factor the function

$$\varphi = \sum_{i=1}^{m} \begin{vmatrix} x_{i1}^{(1)} & x_{i2}^{(1)} \cdots & x_{iq}^{(1)} \\ x_{i1}^{(2)} & x_{i2}^{(2)} \cdots & x_{iq}^{(2)} \\ \cdots & \cdots & \cdots \\ x_{i1}^{(q)} & x_{i2}^{(q)} & x_{iq}^{(q)} \end{vmatrix}.$$

This group was indicated by Kronecker.† The continuous group of linear substitutions leaving φ invariant was investigated by the writer in the BULLETIN for May, 1897. The structure of the discontinuous group of substitutions in the $GF[p^n]$ which leave φ absolutely invariant has been determined by the writer.* For $q > 2$, a substitution leaving φ invariant must leave each determinant in φ separately invariant or at most permute these determinants.

§8. *The First and Second Hypoabelian Groups.*

These groups are certain subgroups of the abelian linear group in the $GF[2^n]$. For $n = 1$, they were defined and investigated at length by Jordan making use of his theory of "exposants d'échange."

By a suitable generalization of Jordan (Traité, p. 440, § 578), we can give the following elementary definition of the generalized hypoabelian groups. The first and second hypoabelian groups in the $GF[2^n]$ are the groups of linear homogeneous substitutions

$$\begin{aligned} \xi_i' &= \sum_{j=1}^{m} (\alpha_{ij}\xi_j + \gamma_{ij}\eta_j) \\ \eta_i' &= \sum_{j=1}^{m} (\beta_{ij}\xi_j + \delta_{ij}\eta_j) \end{aligned} \qquad (i = 1, \cdots, m)$$

which leave invariant the respective functions

$$\sum_{i=1}^{m} \xi_i\eta_i, \quad \sum_{i=1}^{m} \xi_i\eta_i + \lambda(\xi_1^2 + \eta_1^2),$$

where λ is a mark such that $\xi_1\eta_1 + \lambda(\xi_1^2 + \eta_1^2)$ is an irreducible quadratic form in the field. The orders of the groups are respectively

---

† Compare Jordan, Traité des substitutions, p. 219.

* "A class of linear groups including the abelian group," *Quar. Jour. of Math.*, July, 1899.

$$2(2^{nm} \mp 1)[(2^{2n(m-1)} - 1)2^{2n(m-1)}]$$
$$[(2^{2n(m-2)} - 1)2^{2n(m-2)}] \cdots [(2^{2n} - 1)2^{2n}].$$

Each group has a subgroup of index two, defined by the invariant equation

$$\sum_{i,j}^{1 \cdots m} a_{ij}\delta_{ij} + \lambda^2 (\alpha_{11}^2 + \beta_{11}^2 + \gamma_{11}^2 + \delta_{11}^2) = m,$$

the terms in λ^2 not occurring for the first hypoabelian group. The subgroup is simple in the case of the first hypoabelian group if $m > 2$, and for the second hypoabelian group if $m > 1$. For $m = 2$,* the former has two factors of composition each equal to $(2^{2n} - 1)2^n$.

A definition of the abelian and the two hypoabelian groups as groups of isomorphisms of certain elementary groups has been given by the writer.† Consider, for example, the abstract group F generated by the operators $\theta, A_1, \cdots, A_m, B_1, \cdots, B_m$, all of period p and mutually commutative except the pairs A_i, B_i for which

$$A_iB_i = \theta\, B_iA_i \qquad (i = 1, 2, \cdots, m).$$

The group of isomorphisms of F is isomorphic with the abelian linear group for $p > 2$ and with the first hypoabelian group for $p = 2$.

§9. *The Orthogonal Group.*

This group is formed of the linear homogeneous substitutions on m indices $\xi_1, \ldots, \xi_m$ in the $GF[p^n]$, $p > 2$, which leave invariant the function $\sum_{i=1}^{m} \xi_i^2$. Its order and generators were determined by Jordan‡ for the case $n = 1$, certain exceptional cases being overlooked, and by the writer§ for general n. It is generated by the following substitutions [only those indices which are altered are written]:

$$C_i: \quad \xi_i' = -\xi_i,$$

$$O_{ij}^{\alpha\beta}: \quad \begin{cases} \xi_i' = \alpha\xi_i + \beta\xi_j \\ \xi_j' = -\beta\xi_i + \alpha\xi_j \end{cases} \qquad (\alpha^2 + \beta^2 = 1),$$

with two exceptional cases, when additional generators are necessary, viz., for $p^n = 5$, $m \geqq 3$, $$R: \left\{ \begin{array}{l} \xi_1' = \xi_1 + \xi_2 + 2\xi_3 \\ \xi_2' = \xi_1 + 2\xi_2 + \xi_3 \\ \xi_3' = 2\xi_1 + \xi_2 + \xi_3 \end{array} \right\} \qquad (R^2 = 1),$$

---

* *Amer. Jour. of Math.*, July, 1899, p. 249.

† Abstract in the BULLETIN, April, 1899; complete in the *Transactions.*

‡ Traité des substitutions, pp. 155–177.

§ BULLETIN, Feb., 1898.

for $p^n = 3$, $m \geqq 4$, $$W: \left\{ \begin{array}{l} \xi_1' = \xi_1 - \xi_2 - \xi_3 - \xi_4 \\ \xi_2' = \xi_1 - \xi_2 + \xi_3 + \xi_4 \\ \xi_3' = \xi_1 + \xi_2 - \xi_3 + \xi_4 \\ \xi_4' = \xi_1 + \xi_2 + \xi_3 - \xi_4 \end{array} \right. \quad (W^3 = 1).$$

Its structure has been fully determined by the writer in several distinct manners.* For m odd and > 1, the orthogonal group has the factors of composition 2, 2, and

$$\tfrac{1}{2}(p^{n(m-1)} - 1)\, p^{n(m-2)}\, (p^{n(m-3)} - 1)\, p^{n(m-4)} \cdots (p^{2n} - 1)\, p^n,$$

the case $p^n = 3$, $m = 3$ being an exception, the factors being 2, 2, 2, 2, 3.

For m even and > 4, it has the factors of composition 2, 2, 2, and

$$\tfrac{1}{4}[p^{n(m-1)} - \epsilon^{\frac{m}{2}} p^{n\left(\frac{m}{2}-1\right)}]\; [(p^{n(m-2)} - 1)\, p^{n(m-3)})]$$
$$\cdots [(p^{2n} - 1)\, p^n],$$

ϵ being ± 1 according to the form $4l \pm 1$ of p^n.

For $m = 4$, it has the factors of composition 2, 2, $\tfrac{1}{2}(p^{2n} - 1)\, p^n$, $\tfrac{1}{2}(p^{2n} - 1)\, p^n$ and 2, except in the cases $p^n = 2$ and 3, when the composite numbers 6 and 12 respectively are to be replaced by their prime factors.

§ 10. *Other Linear Groups with a Quadratic Invariant.*†

Every m-ary linear group in the $GF[p^n]$, $p > 2$, defined by an invariant which is a quadratic form of determinant not zero in the field, can be transformed into the orthogonal group if m be odd; while for m even, it can be transformed either into the orthogonal group or else into the group leaving invariant the form

$$\sum_{i=1}^{m-1} \xi_i^2 + \nu \xi_m^2 \qquad (\nu = \text{not-square}).$$

The latter group has for $m > 2$, the factors of composition 2, 2, and

$$\tfrac{1}{2}[p^{n(m-1)} + \epsilon^{\frac{m}{2}} p^{n\left(\frac{m}{2}-1\right)}]\; [(p^{n(m-2)} - 1)\, p^{n(m-3)})]$$

---

* *Proc. Calif. Acad. Sci.* (3), vol. 1, nos. 4–5; BULLETIN, May, 1898; *Amer. Jour. of Math.*, July, 1899; *Proc. Lond. Math. Soc.*, vol. 30, pp. 70–98.

† *Amer. Jour. of Math.*, July, 1899.

$$\cdots [(p^{2n}-1)\, p^n].$$

For $m=2$, it is simply isomorphic with the linear fractional group in the $GF[p^{2n}]$.

Every m-ary linear homogeneous group in the $GF[2^n]$ defined by a quadratic invariant, not expressible as a quadratic form in fewer than m variables, can be transformed linearly into a group which, for m even, is either the first or else the second hypoabelian group; while for m odd, $m \equiv 2M+1$, it is the group leaving invariant

$$\xi_0^2 + \sum_{i=1}^{M} \xi_i \eta_i.$$

The latter group is simply isomorphic with the abelian group on $2M$ indices in the $GF[2^n]$.

§ 11. *Linear Group defined by the Invariant* $\sum_{i=1}^{m} \xi_{i1}\xi_{i2}\cdots\xi_{iq}$.

For $q=2$, this group was studied at length by the writer (*Proc. Lond. Math. Soc.*, vol. 30, pp. 70–98), and for $q>2$ (Ibid., pp. 200-208). For $q=2$, $p>2$, the group is isomorphic with the orthogonal group, except in the case m odd and -1 a not-square in the field, when it is isomorphic with the group of § 10. For $q>2$, the substitutions leave the products $\xi_{i1}\xi_{i2}\cdots\xi_{iq}$ $(i=1, \cdots, m)$ separately invariant, or else merely permute them. Its structure is therefore evident.

§ 12. *The Hyperorthogonal Group.*

In a paper contributed to the *Mathematische Annalen*, the writer has determined the order, generators, and structure of the linear homogeneous m-ary group defined by the invariant

$$\varphi_r \equiv \sum_{r=1}^{m} \lambda_i \xi_i^r \qquad (r>2).$$

There exists no continuous group leaving invariant φ_r $(r>2)$, while the only collineations of $\varphi_r=0$ into itself are those leaving the terms $\lambda_1\xi_1^r, \cdots, \lambda_m\xi_m^r$ separately fixed, or at most permuting them. The latter result holds for linear substitutions in the $GF[p^n]$, if r be neither a multiple of p nor of the form p^s+1. If r be of either of these two forms, the determination of the group reduces to that of the group in the $GF[p^{2s}]$ leaving invariant $\sum_{i=1}^{m} \xi_i^{p^s+1}$. The latter group may be called the *Hyperorthogonal Group*, since those of its substitutions which belong to the $GF[p^s]$ form the orthogonal group in the $GF[p^s]$.

The hyperorthogonal group is generated by the substitutions

$$T_{i,\tau}: \qquad \xi_i' = \tau\xi_i \qquad\qquad (\tau^{p^s+1} = 1),$$

$$O_{ij}^{\alpha\beta}: \begin{cases} \xi_i' = \alpha\xi_i + \beta\xi_j \\ \xi_j' = -\beta^{p^s}\xi_i + \alpha^{p^s}\xi_j \end{cases} \qquad (\alpha^{p^s+1} + \beta^{p^s+1} = 1),$$

except for the case $p^s = 2$, $m \geqq 3$, when a new generator is necessary, for example,

$$W: \begin{pmatrix} 1 & 1 & 1 \\ 1 & i & i^2 \\ 1 & i^2 & i \end{pmatrix} \qquad [i^2 + i + 1 \equiv 0 \pmod 2].$$

Those of its substitutions having determinant unity form a self-conjugate subgroup of index $p^s + 1$. The factors of composition of this subgroup are the prime factors of d and the composite number

$$\frac{1}{d}\left[p^{sm} - (-1)^m\right] p^{s(m-1)} \left[p^{s(m-1)} - (-1)^{m-1}\right]$$
$$p^{s(m-2)} \cdots \left[p^{2s} - 1\right] p^s,$$

where d is the greatest common divisor of m and $p^s + 1$. The cases $m = 3$, $p^s = 2$; $m = 2$, $p^s = 2$; $m = 2$, $p^s = 3$ are exceptional. For $m = 2$, the linear substitutions in the $GF[p^{2s}]$ which have determined unity and which leave invariant $\xi_1^{p^s+1} + \xi_2^{p^s+1}$ form a group isomorphic with the linear fractional group in the $GF[p^s]$.

§ 13. *The Hyperabelian group.*

This group consists of the totality of linear homogeneous substitutions on $2m$ indices in the $GF[p^{2n}]$ which leave invariant

$$\psi \equiv \sum_{i=1}^{m} (\xi_{2i-1}\xi_{2i}^{p^n} - \xi_{2i}\xi_{2i-1}^{p^n}).$$

The totality of substitutions with coefficients in the $GF[p^n]$ which leave ψ invariant forms a subgroup which is readily recognized as the abelian linear group on $2m$ indices in the $GF[p^n]$. Hence the name *hyperabelian.** It is easily proven to be simply isomorphic with the hyperorthogonal group on $2m$ indices in the $GF[p^{2n}]$.

---

* Dickson: "Concerning the four known simple groups of order 25920, with an introduction to the hyperabelian group," *Proc. Lond. Math. Soc.*, vol. 30 (1899).

§ 14. *Isomorphisms between Various Linear Groups.*

The conception of "the compounds of a given linear homogeneous group," introduced in recent papers by the writer,[*] has proven to be a powerful means of setting up isomorphisms. The group of quaternary linear homogeneous substitutions of determinant unity in the $GF[p^n]$ is isomorphic with a subgroup of the senary linear group leaving invariant $\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$, and hence, according as $p^n = 2^n$, $4l + 1$, or $4l - 1$, to the senary first hypoabelian group, the senary orthogonal group, or the group of § 10.

The simple group of order $\frac{1}{2}(p^{4n} - 1)(p^{2n} - 1)p^{4n}$ derived from the quaternary abelian group and that from the quinary orthogonal group, each in the $GF[p^n], p > 2$, are simply isomorphic.

The following four simple groups of order 25920 are simply isomorphic:[†]

1°. The abelian group on four indices modulo 3.
2°. The second hypoabelian group on six indices modulo 2.
3°. The orthogonal group on five indices modulo 3.
4°. The hyperabelian group on four indices in the $GF[2^2]$.

In the paper cited an abstract group, a substitution-group on 36 letters, and one on 27 letters are given, each isomorphic to the above groups of order 25920.

SHORTER NOTICES.

Leçons sur la détermination des orbites, professées à la Faculté des Sciences de Paris. Par F. TISSERAND; redigées et developpées pour les calculs numériques, par J. PERCHOT; avec une Préface de H. POINCARÉ. Paris, Gauthier-Villars et Fils. 1899. 4to, xiv + 124 pp.

TISSERAND is chiefly known outside of France for his Traité de mécanique céleste—a monument of labor undertaken and carried out amid the pressure of many duties. He had in fact an enormous capacity for work and in his numerous memoirs left few parts of theoretical astronomy untouched. The volume of 120 quarto pages before us, edited by M. Perchot, shows him working for the practical astronomer. The problem of the determination of an orbit from three observations is not exactly an enticing one. It is a tiresome piece of analysis which appears to offer little scope at the present time for the aspiring mathematician. In the case of a parabolic orbit the solution can even be made without successive approximations as M. Poincaré points out in the preface. In the case of an elliptic orbit, almost the only point of mathematical interest is the presence of a transcendental equation.

---

[*] "Concerning a linear homogeneous group in $C_{m,2}$ variables isomorphic to the general linear homogeneous group in m variables," BULLETIN, Dec., 1898. "The structure of certain linear groups with quadratic invariants," *Proc. Lond. Math. Soc.*, vol. 30, pp. 70–98. "A new definition of the general abelian linear group," *Transactions* (read Feb. 1899).

[†] *Proc. Lond. Math. Soc.*, vol. 31. Abstract in BULLETIN, May, 1899, p. 384.

The treatises on this subject are well known, if not numerous. Gauss, in his Theoria motus, laid the foundations of the method at present in practical use, while Laplace a short time before had given a different form of solution. In the preface to the 'Leçons,' M. Poincaré has devoted some space to showing that Gauss's method is really equivalent to that of Laplace in spite of their apparent dissimilarity. The former has, however, been fully worked up with tables and formulæ for practical use in the great treatises of Oppolzer (of which there is a French translation) and Watson. In spite of the fact that a computer who has at his finger ends the notation and formulæ given in either of the last named volumes, will probably not care to change to another; M. Tisserand's work may nevertheless be found of use. The equations are developed quickly and easily, and moreover are all put together in a form which admits of immediate application. The young astronomer, unless he has acquired a firm grasp of his mathematics, may perhaps find it somewhat difficult to see the bearing of all the formulæ owing to the brevity of the explanations.

There are two chapters. The first on the method of Olbers for the determination of the orbit of a comet from three observations; the second on Gauss's method for the similar determination of the orbit of a planet. A numerical example of the latter, fully worked out, is set forth, showing the form in which the calculations would be actually made. Tables VIII and IX of Oppolzer's treatise are reproduced.

ERNEST W. BROWN.

Vorlesungen über mathematische Physik, Band I, Mechanik. Von GUSTAV KIRCHHOFF. 4te Auflage von W. WIEN. Teubner, Leipzig.

KIRCHHOFF's four volumes on mathematical physics which have appeared at various times since 1876 are so well known that little need be said about them here. The first volume on general mechanics has already reached its fourth edition. The second and third editions were issued under the superintendence of the author. Dr. Wien has had the advantage of the author's manuscript notes, found after his death, in preparing the volume for the fourth edition. No changes have, however, been made beyond the correction of printer's errors and the removal of slight obscurities. One of the best testimonies that we can give to the care and ability which Kirchhoff devoted to the original volume is to say that those who possess any one of the first three issues will not find it necessary to buy this last edition.

ERNEST W. BROWN.

Leçons élémentaires sur la théorie des formes et ses applications géometriques, a l'usage des candidats a l'agregation des sciences mathematiques. Par H. ANDOYER. Paris, Gauthier-Villars et Fils, 1898. 4to, 184 pp. Lithographed.

THIS title represents accurately the contents of the book. A restricted list of topics is adequately treated. Geometrical applications are plentiful but do not encroach upon nor obscure the purely algebraic theory. The treatment is perhaps elementary, but in style simplicity is less noticeable than brevity. The work as a whole is more nearly a syllabus than a textbook for the beginner, and as a syllabus it cannot fail to become widely known and valued.

Binary and ternary forms are introduced, but of binary forms only linear, bilinear, quadric, cubic, and quartic forms are discussed; and of ternary, only linear, bilinear, and quadric. The bilinear ternary form in cogredient variables I do not remember having seen in any earlier text, although Salmon's Higher plane curves gives a brief geometrical discussion of skew reciprocity; its inclusion here together with the form bilinear in contragredient variables must be commended by geometricians. Duality as a method stands certainly on a par with projectivity.

Two points of excellence are worthy of special mention. The first is that from the outset stem forms are assumed to contain several sets of variables; and this convention is observed in the case of each particular form, an unlimited number of sets of cogredient variables being adjoined. Of course this amplifies the complete system of each stem form, but the extension is easy, since no new types occur. The second novel merit is that the term *polar* is so defined as to include such operations as Aronhold's. If so-called variables are replaced by cogredient variables, the process is ordinarily called a polar operation, and M. Andoyer sees no reason why a different term should be needed when coefficients are replaced by cogredient coefficients. The reason why the operation is invariant is the same in both cases. As illustrating the degree of elegance attained by M. Andoyer may be cited from § 3, Chapter I, the proof (susceptible of improvement, though not of simplification in form) that absolute invariants exist; and the entire § 2 of chapter III, which sets forth the relation of a homography to a binary bilinear form.

In a text prepared especially to emphasize geometrical applications, the purely algebraic side of any problem is naturally of only secondary importance. Here the student learns to think of invariance as based on reasons rather than on specific normal forms; it is not important for the beginner to reduce the canonizant, for example, to typical form, and it is important to see clearly that every eliminant of a set of equations must be an invariant, regardless of the notation used to express it. This principle leads to a preponderance of reasoning over reckoning, and the postponement of purely algebraic problems. While complete form systems are given for each special stem form, Gordan's theorem is simply stated without proof. Nor is there any mention of Hilbert's theorems (though on p. 101 it is stated that "Le théorem de Gordan subsiste dans le domaine ternaire,") nor of the enumeration problems solved by Sylvester and Deruyts. This marks the elementary character of the work, and leaves much to be looked for from the larger compend which, the preface tells us, the author has had in preparation already for some years.

HENRY S. WHITE.

Sur les lois de réciprocité. Par X. STOUFF, professeur à la Faculté des Sciences de Besançon. Paris, Hermann, 1898. 8vo, 31 pp.

THE laws of reciprocity have been the object of numerous mathematical researches, the chief of which are the memoirs of Jacobi, Eisenstein, and Kummer in *Crelle's Journal.* Stouff proposes to apply n-dimensional geometry to the subject following the examples of Minkowski's geometry of numbers.* At bottom the question of the laws of reciprocity appears to have a natural and close connection with the fuchsian polygons of Poincaré generalized for space of any number of dimensions. The intervention of these polygons appears implicitly in Gauss's memoir on biquadratic residues; the works of Stieltjes appertain here also. Certain differences between the various orders of residues explain themselves by difference in properties of spaces of different dimensions. The geometric reasoning of the author presupposes familiarity with transcendental geometry on the part of the reader; where the theorems admit of immediate extension to space of any number of dimensions the exposition is made for five dimensional spaces. It is beyond the scope of this short notice to recapitulate the results, which, in the nature of the subject, resist compression.

---

* Minkowski, Geometrie der Zahlen, Leipzig, Teubner, 1896.

E. O. LOVETT.

Leçons sur la théorie analytique des équations aux dérivées partielles du premier ordre. Par ÉTIENNE DELASSUS, chargé de conférences à l'Université de Lille. Paris, Hermann, 1897. 8vo, 88 pp.

THE theory of partial differential equations has reached a high degree of perfection, but the theory in its present form lacks unity, its various parts resulting from particular properties and remaining almost independent of one another.

Thus the integration of linear systems rests on a partial extension of the theorem of Cauchy; in non-linear systems it is necessary to distinguish two cases according as the unknown function does or does not enter, since the method of Jacobi and Mayer experiences modifications in passing from one to the other; further the method of Lie is demonstrated only for systems in which the unknown does not figure, and is the consequence of a theorem which results from the theory of characteristics or from a method of Jacobi.

The object of the memoir of Delassus is to unify this system of details by determining a canonical form altogether general and establishing a general existence theorem. In the case of systems in which the unknown does not enter, this canonical system coincides with the form in involution, but it is no longer the same when the unknown enters explicitly. Upon this general canonical form Delassus establishes a fundamental existence theorem which he calls for short the generalized theorem of Cauchy; the demonstration of it yields the following theorem which plays a capital rôle throughout the theory: The integration of a system of partial differential equations of the first order having but one unknown can always be referred to that of a single equation of the first order. This theorem was already known for the case where the unknown function is absent, having been demonstrated by Lie by means of the general theory of characteristics and then by Mayer who employed a method of integration of Jacobi.

The author employs the generalized theorem of Cauchy to formulate the generalized problem of Cauchy which, if solved, would effect the integration completely; to study singular solutions which he classifies according as they are simply, doubly, and so on to n-ply singular; and finally to investigate precisely the transformation which changes a system where the unknown is present into one where it is absent.

The theorem on the reduction to a single equation furnishes immediately a method of integration of linear systems, the method of Mayer, the simplest known. For the integration of non-linear systems the author presents the theory of the complete integral in the ordinary manner, but adds the solution of the problem of Cauchy by means of such an integral without the geometric considerations relative to characteristics.

The exposition of the method of Jacobi and Mayer is made without using any algebraic properties of the bracket expressions and without any reference to the presence or absence of the unknown; moreover Delassus shows that the method leads always to a complete integral. He finds finally the method of Lie as an immediate consequence of the above reduction theorem, and without raising any question relative to the unknown function.

A list of forty examples of simultaneous systems forms the second part of this most valuable contribution to the theory of the linear partial differential equation.

E. O. LOVETT.

NOTES.

THE Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will meet at the University of Chicago on Thursday and Friday, December 28 and 29 next. Titles and abstracts of papers to be read should be in the hands of the Secretary of the Section not later than December 12.

AMONG the officers of the American association for the advancement of science for the coming year are: Professor R. S. WOODWARD, Columbia University, president; Professor CHARLES BASKERVILLE, University of North Carolina, general secretary; Professor ASAPH HALL, JR., University of Michigan, vice-president of section A, mathematics and astronomy; Dr. W. M. STRONG, Yale University, secretary of section A.

THE meeting of the British association for the advancement of science was held at Dover, September 13–21. Professor J. H. POYNTING was president of the mathematical section, and Mr. E. T. WHITTAKER secretary. The following papers were presented to the section: Report of the committee appointed to calculate tables of certain integrals; Report of the committee appointed to calculate tables of certain mathematical functions; Mr. FRANCIS GALTON, F.R.S., "The median estimate;" Professor A. R. FORSYTH, F.R.S., "A system of invariants for parallel configurations in space;" Professor J. D. EVERETT, F.R.S., "On the notation of the calculus of differences;" Professor A. C. DIXON, "On the partial differential equation of the second order;" Professor A. R. FORSYTH, F.R.S., "On singular solutions of ordinary differential equations;" Professor IRVING STRINGHAM, "On the fundamental differential equations of geometry;" Mr. E. T. WHITTAKER, "Report on recent progress in the problem of three bodies;" Professor E. O. LOVETT, "An application and interpretation of infinitesimal transformations;" Lieut. Col. A. J. C. CUNNINGHAM, "On Fermat's numbers."

THE German mathematical association held its annual meeting at Munich from the 17th to the 23d of September in connection with the association of scientists and physicians. The preliminary programme includes the following papers in the order of announcement:—K. HENSEL, Berlin, "On the analytico-arithmetical theory of algebraic functions of two variables;" K. HEUN, Berlin, "The kinetical problems of scientific technology;" E. STUDY, Greifswald, "Geometry of dynames;" R. MEHMKE, Stuttgart, and J. BAUSCHINGER, Berlin, "Discussion of the decimal division of angles and time;" H. WEBER, Strassburg, and G. HAUCK, Berlin, "On the organization of university mathematical instruction to accord with the new Prussian examination requirements;" K. DÖHLEMANN, Munich, "On hyperboloidal straight lines;" F. ENGEL, Leipzig, "On two remarkable groups in space of five dimensions;" P. GORDAN, Erlangen, "On symmetric functions;" D. HILBERT, Göttingen, "On the number concept;" "On Dirichlet's principle;" J. HORN, Charlottenburg, "Divergent series in the theory of differential equations;" G. KOHN, Vienna, "On the geometry of Klein's group of 360 collineations and 360 correlations;" M. LERCH, Freiburg, Switzerland, "Arithmetical considerations on infinite series;" L. MAURER, Tübingen, "Invariance under special groups of linear substitutions;" F.

MEYER, Königsberg, "On the notion of equivalence of geometrical propositions;" M. NOETHER, Erlangen, "Riemann's lectures of 1861–62 on abelian functions;" E. SCHIMPF, Bochum, "Introduction of a measure of convergence and divergence in the theory of infinite processes (series, products, continued fractions, etc.);" L. SCHLESINGER, Klausenburg, "On Riemann's posthumous paper on the theory of linear differential equations;" J. SOMMER, Göttingen, "On quadratic manifoldnesses in five-dimensional space;" A. SOMMERFELD, Clausthal, "On Weierstrass's criteria in the calculus of variations;" E. v. WEBER, Munich, "A fundamental classification of differential problems." Pedagogical papers from RUDEL, of Nürnberg, SCHOTTEN, of Halle, and TREUTLEIN, of Karlsruhe, were also announced.

ON the seventeenth of last June the statue erected to the memory of CARL FRIEDRICH GAUSS and WILHELM WEBER was unveiled at Göttingen with appropriate ceremonies. The monument in bronze, on a granite base, represents Gauss seated in an arm-chair and Weber standing by his side and apparently discussing with him their joint invention, the electric telegraph. It is the work of the Berlin sculptor Hartzer. The principal address of the celebration was delivered by Professor Woldemar Voigt who sketched in brief and pregnant words the scientific career of the two men. A large number of delegates from various scientific institutions and other guests were present at the unveiling and the succeeding banquet. An exhibition of manuscripts by Gauss and Weber, of instruments used by them, and of other relics was arranged in the Aula of the University and attracted much attention. The Festschrift published in commemoration of the day contains two valuable memoirs, one by Professor Hilbert, "On the foundations of euclidean geometry," the other by Professor Wiechert, "On the theory of electrodynamics." The University of Göttingen took this opportunity to confer the honorary degree of doctor of philosophy on Professor G. H. Darwin, of Cambridge, Professor Geitel, of Wolfenbüttel, Professor J. Hadamard, of Paris, Professor Lorentz, of Leiden, Professor E. H. Moore, of Chicago, Professor Righi, of Bologna, and Professor von Sterneck, of Vienna.

THE international congress of philosophy in connection with the Paris exposition of 1900 will be held from the 2d to the 7th of August. The section devoted to the history of the sciences has made liberal provision in its programme for the discussion of the fundamental notions of mathematics.

THE Paris academy of sciences has been authorized to increase the number of its national and foreign correspondents from one hundred to one hundred and sixteen.

A MATHEMATICAL society has been organized at Liverpool, England, to be known as the Liverpool mathematical society. Mr. Walter Stott is honorary secretary.

THE London mathematical society has decided to issue its *Proceedings* in two volumes per annum.

M. GAUTHIER-VILLARS, of Paris, announces the early appearance of the following mathematical works:—A two volume treatise on the theory of forms and higher analytic geometry, by ANDOYER; an elementary treatise on the theory of numbers, by CAHEN; a treatise on geometrical optics, by WALLON; the scientific works of G. ROBIN, including the memoirs on the theory of functions founded exclusively on the idea of number, and those on mathematical physics and general thermodynamics, edited by L. RAFFY; an index to the *Comptes rendus* from 1881 to 1895, including volumes 92 to 121.

THE fall announcements of M. Hermann, of Paris, include Recherches sur les fonctions de variables réelles, by R. BAIRE; Sur les fondements de la théorie des ensembles transfinis, by G. CANTOR, translation by F. MAROTTE, of the two well known memoirs of the author published in the *Mathematische Annalen*; Leçons nouvelles sur les applications géométriques du calcul différentiel, by W. de TANNENBERG.

MESSRS. Carré and Naud, of Paris, have a history of mathematics by BOYER in the press, and an annuaire of mathematicians in preparation.

PROFESSOR R. S. BALL is engaged on a treatise on the theory of screws which is to be considerably larger than his former book on that subject. It will be published by the Cambridge university press.

THE second volume of Professor A. R. FORSYTH's theory of differential equations is in the press.

The following courses of lectures have been announced for the winter semester 1899–1900:—

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Analytical geometry, four hours; Maxima and minima, two hours; Elliptic functions, four hours; Colloquia, two hours; Mathematical seminar, two hours.—By Professor E. R. HOPPE: Analytical geometry, four hours; Differential calculus and theory of series, four hours.—By Professor J. KNOBLAUCH: Differential calculus, four hours; Applications of elliptic functions, four hours; Exercises in differential calculus, one hour.—By Professor LEHMANN-FILHÉS: Integral calculus, four hours; Method of least squares, one hour.—By Professor G. FROBENIUS: Theory of determinants, four hours; Theory of numbers, four hours; Mathematical seminar, two hours.—By Professor K. HENSEL: Synthetic geometry, four hours; Geometry of numbers, three hours; Exercises in the geometry of numbers, one hour.—By Professor G. HETTNER: Theory of probability and of errors, four hours.

UNIVERSITY OF BRESLAU.—By Professor J. ROSANES: Exercises in mathematical seminar, one hour; Differential calculus and elements of integral calculus, four hours; Elements of the theory of determinants, two hours.—By Professor R. STURM: History of mathematics, one hour; Mathematical seminar, two hours; Theory of geometrical correspondences second part, three hours; Non-euclidean geometry, two hours.—By Professor F. LONDON: Introduction to the theory of functions, three hours; The technology of insurance, two hours.

UNIVERSITY OF ERLANGEN.—By Professor P. GORDAN: Differential and integral calculus, four hours; Algebra, four hours; Seminar, three hours.—By Professor M. NOETHER: Analytical geometry, four hours; Differential geometry, four hours; Mathematical exercises.

UNIVERSITY OF GÖTTINGEN.—By Professor F. KLEIN: Mechanics of deformable bodies, four hours; Mathematical seminar, two hours.—By Professor D. HILBERT: Integral calculus, four hours; Curvature of surfaces, two hours; Number-concept and quadrature of circle, two hours; Exercises in mathematical seminar, one hour.—By Dr. F. SCHILLING: Descriptive and projective geometry, four hours; Exercises in descriptive geometry, four hours; Exercises in differential calculus, one hour.—By Dr. G. BOHLMANN: Chapters in the mathematics of insurance, two hours; The mathematical foundations of insurance, three hours; Mathematical exercises in seminar, two hours.—By Dr. ZERMELO: Theory of functions, four hours.—By Dr. SOMMER: Theory of numbers, four hours; Exercises in the theory of numbers, one hour.

UNIVERSITY OF GREIFSWALD.—By Professor W. THOMÉ: Analytical geometry, two hours; Theory and applications of determinants, two hours; Exercises in mathematical seminar, two hours.—By Professor E. STUDY: Mechanics, four hours; Applications of the infinitesimal calculus to geometry, four hours; Exercises in seminar, one hour.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER: Plane analytical geometry, three hours; Theory of definite integrals, three hours; Mathematical seminar, one hour.—By Professor P. STÄCKEL: Integral calculus, three hours; Higher geometry, three hours; Seminar, one hour.

UNIVERSITY OF KOENIGSBERG.—BY Professor F. MEYER: Integral calculus, three hours; Exercises to preceding, one hour; Elliptic functions, four hours; Seminar, one hour.—By Professor A. SCHOENFLIES: Determinants, one hour; Projective geometry, four hours; Exercises to preceding, one hour; General introduction to higher mathematics, two hours.—By Professor L. SAALSCHÜTZ: Differential equations, three hours; Exercises in differential equations, one hour.—By Dr. T. VAHLEN; Calculus of variations, three hours; Exercises, one hour.

UNIVERSITY OF HALLE-WITTENBERG.—By Professor G. CANTOR: Analytical mechanics, four hours; Chapters of the infinitesimal calculus, two hours; Exercises in seminar, two hours.—By Professor A. WANGERIN: The most important partial differential equations of mathematical physics, one hour; Theory of surfaces and space curves, four hours; Synthetic geometry, three hours; Mathematical seminar, two hours.—By Professor V. EBERHARD: Chapters of analytical geometry, one hour; Theory of numbers, four hours.

UNIVERSITY OF HEIDELBERG.—By Professor L. KOENIGSBERGER: Analytical mechanics, four hours; Calculus of variations, differential equations, two hours; Elliptic functions, two hours; Seminar, two hours.—By Professor M. CANTOR: Infinitesimal calculus, four hours; Exercises in the calculus, one hour; Political arithmetic, two hours.—By Professor F. EISENLOHR: Theoretical optics, four hours; Infinitesimal calculus, five hours; Potential, two hours.—By Professor K. KOEHLER: Analytical geometry of space, three hours.—By Professor G. LANDSBERG: Descriptive geometry, three hours; Introduction to the theory of algebraic equations, two hours.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Algebraic analysis, four hours; Mathematical geography, three hours; Seminary, two hours.—By Professor G. FREGE: Analytical mechanics, four hours; Symbolic algebra, two hours.

UNIVERSITY OF LEIPZIG.—By Professor A. MAYER: Higher analytical mechanics, four hours; Exercises to the preceding, one hour.—By Professor C. NEUMANN: Infinitesimal calculus, four hours; Seminar, one hour.—By Professor O. HÖLDER: Introduction to the theory of functions of a complex variable, four hours; Chapters in the theory of substitution groups and equations, two hours; Seminar, one hour.—By Professor F. ENGEL: Analytical mechanics, four hours; Theory of continuous transformation groups, two hours; Seminar, one hour.—By Dr. F. HAUSDORFF: Chapters of higher geometry, four hours.—By Dr. G. KOWALEWSKI: Algebraic invariant theory.

UNIVERSITY OF MUNICH.—By Professor G. BAUER: Plane analytical geometry; Conduction of heat in fixed bodies; Seminar.—By Professor F. LINDEMANN: Differential calculus; Abelian functions; Mathematical seminar on theory of automorphic functions and application of elliptic functions.—By Professor A. PRINGSHEIM: Introduction to the theory of analytic functions; Elementary theory of infinite algorithms (series, products, continued fractions).—By Dr. H. BRUNN: Introduction to the mathematical foundations of natural science.—By Dr. K. DÖHLEMANN: Descriptive geometry and exercises; Synthetic geometry and exercises.—By Dr. E. v. WEBER: Introduction to analysis; Introduction to the Cauchy-Riemann theory of functions.

THE London mathematical society has conferred its De Morgan medal on Professor W. BURNSIDE for his mathematical researches in the theory of groups of a finite order.

THE Adams prize offered by the University of Cambridge for an essay on the theory of the aberration of light has been divided between Mr. J. LARMOR, fellow of St. John's college, senior wrangler 1880, and Mr. G. T. WALKER, fellow of Trinity college, senior wrangler in 1889.

THE Madrid academy of sciences recently announced the results of the competition for its mathematical prize set in 1897. There were three competitors. Professors GINO LORIA, of the University of Genoa, and GOMES TEIXEIRA, of the Polytechnic of Oporto, each received a prize, and JOAQUIN DE VARGAS Y AGUIRRE, of Salamanca, an honorable mention.

THE honorary degree of doctor of laws has been recently conferred on Professor A. R. FORSYTH by Dublin University, and that of doctor of civil law on Professor SIMON NEWCOMB, by Oxford University.

PROFESSOR L. FUCHS has been made rector of the University of Berlin for the current year, and Professor E. NETTO has received the same appointment at the University of Giessen.

MR. H. M. MACDONALD, of Clare College, Cambridge, fourth wrangler in 1889, has been appointed to the University lectureship in mathematics made vacant by Professor A. E. H. LOVE's recent resignation to accept the Sedleian professorship in mathematics at Oxford University.

LORD KELVIN has resigned the chair of natural philosophy in Glasgow University after a tenure of fifty-three years.

DR. MILTON UPDEGRAFF, professor of astronomy in the University of Missouri, has been appointed to a professorship in mathematics in the United States naval observatory.

DR. L. E. DICKSON has resigned an assistant professorship of mathematics in the University of California to accept an associate professorship in the University of Texas.

MR. PETER FIELD, fellow in Cornell University, has been appointed professor of mathematics in Carthage College.

THE following have recently qualified as privatdocenten: Dr. GRASSMANN, at the University of Halle; Dr. KOWALEWSKI, at the University of Leipzig; Dr. J. GOETTLER, at the University of Munich.

MESSRS. T. BIRTWHISTLE, of Pembroke College, and R. P. PARANJPYE, of St. John's College, were bracketed for the senior wranglership of this year at Cambridge University. The Smith's prizes for 1899 were awarded to Messrs. W. H. AUSTIN and G. W. WALKER, of Trinity College, senior and fourth wranglers in 1897. Mr. PARANJPYE is the first native of India to attain the honor and Mr. BIRTWHISTLE is the first member of Pembroke College to achieve it since Sir GEORGE STOKES who was senior wrangler in 1841.

AT Emmanuel College, Cambridge, Mr. G. T. BENNETT, senior wrangler, 1890, has been elected to a senior fellowship, and Mr. H. S. CARSLAW, fourth wrangler 1894, to a junior fellowship.

MR. A. H. WILSON, instructor in mathematics at Princeton University, has been granted a year's leave of absence, and Dr. L. W. REID has been appointed to an instructorship in the same institution.

THE following mathematicians have been elected to fellowships in the Reale Accademia dei Lincei: Professor P. TARDY and G. VERONESE, as ordinary fellows; Professor G. RICCI, as corresponding fellow; Professor G. MITTAG-LEFFLER and J. WEINGARTEN, as foreign fellows.

PROFESSORS F. KLEIN and W. NERNST, of Göttingen, have been elected to foreign membership in the academy of sciences of Budapest.

AT its recent decennial celebration Clark University conferred the honorary degree of doctor of laws on five scientists among whom were Professors PICARD, of Paris, and BOLTZMANN, of Vienna.

MR. J. D. THOMPSON, of Trinity College, Cambridge, has been elected to an assistant professorship of mathematics in the University of West Virginia.

DR. F. H. SAFFORD, of Harvard University, has been made assistant professor of mathematics at the University of Cincinnati.

MR. HOWARD OPDIKE has been promoted to an assistant professorship of mathematics at Union College.

DR. H. Y. BENEDICT and MR. I. N. PUTNAM have been appointed instructors in mathematics at the University of Texas.

THE Right Rev. CHARLES GRAVES, Lord Bishop of Limerick, professor of mathematics at Trinity College since 1843, died recently at the age of eighty-seven years.

THE deaths are announced of Professor ROBERT WILHELM BUNSEN, August 16th, 1899, at the age of eighty-eight years; of M. GRIESS, one of the editors of *L'éducation mathématique*, at the age of forty-two years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

APPELL [P.]. Eléments d'analyse mathématique à l'usage des ingénieurs et des physiciens. Cours professé à l'Ecole centrale des arts et manufactures. Paris, Carré et Naud, 1898. 8vo. 4+720 pp. Cloth. Fr. 24.00

BARBARIN (P.). Notions complémentaires sur les courbes usuelles. Paris, Nony, 1899. 8vo. 47 pp. Fr. 3.00

BERNOULLI (JACOB). Wahrscheinlichkeitsrechnung (Ars conjectandi). (1713.) Teil 1 und 2. Uebersetzt und herausgegeben von R. Haussner. Leipzig, Engelmann, 1899 12mo. 162 pp. (Ostwald's Klassiker der exakten Wissenschaften, No. 107.) Cloth. M. 2.50

——. Teil 3 und 4, mit dem Anhang: Brief an einen Freund über das Ballspiel (jeu de paume). Uebersetzt und herausgegeben von R. Haussner. Leipzig, Engelmann, 1899. 12mo. 172 pp. (Ostwald's Klassiker der exakten Wissenschaften, No. 108.) Cloth. M. 2 50

BIBLIOGRAPHIA mathematica rossica. List of mathematical works and papers published in Russia, edited by D. Sintsof. No. 1: 1896. Kazan, 1898. 8vo. 24 pp. M. 1.20

——. No. 2: 1897. Kazan, 1899. 8vo. 22 pp. M. 1.20

BLAKE (E. M.). Upon the ruled surfaces generated by the plane movements whose centrodes are congruent conics tangent at homologous points. (*American Journal of Mathematics*, Vol. 21, pp. 257–269.) 1898. 4to.

BOHLMANN (G.). See GENOCCHI (A.) and JAHRESBERICHT.

BONNEL (J.). Les hypothèses dans la géométrie. L'atome et le tome (fin). Paris, Gauthier-Villars, 1898. 8vo. Fr. 1.50

BOUTON (C. L.). Invariants of the general linear differential equation and their relation to the theory of continuous groups. Leipzig, 1899. 8vo. 82 pp. M. 2.00

BRAHY (E.). Exercices méthodiques de calcul différentiel. 2e édition. Paris, Gauthier-Villars, 1899. Fr. 5.00

BROCARD (H.). Notes de bibliographie des courbes géométriques. Partie complémentaire. (Lith.) Bar-le-Duc, Comte-Jacquet, 1899. 8vo. 8+243 pp.

BRUNEL (G.). See ENCYKLOPÄDIE.

BUBNOW (N.). See GERBERT.

CANTOR (G.). Sur les fondements de la théorie des ensembles transfinis. Traduit par F. Mariotte. Paris, 1899. 8vo. 100 pp. Fr. 4.50

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. Vol. II, first half: 1200-1550. 2te Auflage. Leipzig, Teubner, 1899. 8vo. 480 pp. M. 14.00

CARNOY. Cours de géométrie analytique. (En 2 volumes.) Vol. II: Géométrie de l'espace. 5e édition. Paris, Gauthier-Villars, 1899. 8vo. Fr. 11.00

CAUCHY (A.). Oeuvres complètes. 2e série, Vol. IV: Résumé des leçons données à l'Ecole polytechnique sur le calcul infinitésimal. Leçons sur le calcul différentiel. Paris, Gauthier-Villars, 1899. 4to. Fr. 20.00

CHRISTEN (T.). Beiträge zur Verwendung des freien Integrationsweges. Basel, 1897. 8vo. 35 pp. M. 1.80

CURTZE (M.). Nicolaus Coppernicus. Eine biographische Skizze. Mit dem Bildnis des Coppernicus. Berlin, Paetel, 1899. 8vo. 84 pp. (Sammlung populärer Schriften, herausgegeben von der Gesellschaft Urania zu Berlin, No. 54.) M. 2.00

——. See EUCLIDES.

CZUBER (E.). See JAHRESBERICHT.

DE MORGAN (A.). On the study and difficulties of mathematics. New corrected and annotated edition, with references to date, of the work published in 1831, by the Society for the diffusion of useful knowledge. Chicago, The Open Court Publishing Co., 1899. 8vo. 8 + 288 pp., portrait. Cloth. $1.25

DUPORCQ (E.). Premiers principes de géométrie moderne, à l'usage des élèves de mathématiques spéciales et des candidats à la licence et à l'agrégation. Paris, Gauthier-Villars, 1899. 8vo. 7 + 160 pp. Fr. 3.00

EUCLIDES. Opera omnia. Ediderunt J. L. Heiberg et H. Menge. (In 12 voluminibus.) Supplementum: Anaritii in decem libros priores Elementorum Euclidis Commentarii. Ex interpretatione Gherardi Cremonensis in codice Cracoviensi 569 servata. Edidit M. Curtze. Lipsiae, Teubner, 1899. 12mo. 29 + 389 pp. M. 6.00

FEHR (H.). Application de la méthode vectorielle de Grassmann à la géométrie infinitésimale. (Thèse.) Paris, Carré et Naud, 1899. 8vo. 100 pp.

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Teil I, Band II, Heft 1: Grundlagen der allgemeinen Funktionenlehre von A. Pringsheim; Differential- und Integralrechnung von A. Voss; Bestimmte Integrale von G. Brunel. Redigiert von H. Burkhardt. Leipzig, Teubner, 1899. 8vo. 160 pp. M. 4.80

FINSTERWALDER (S.). See JAHRESBERICHT.

FORMULAIRE de mathématiques. Vol. II, §2: Arithmétique, par G. Peano. Turin, Bocca, 1898. 8vo. 58 pp.

FRANSÉN (A. E.). Om Dirichlets problem for eqvationen $\Delta u = 0$ och liknande upgifter for allmännare eqvationer. Upsala, 1899. 8vo. 48 pp. M. 2.00

GANDTNER (J. O.). Elemente der analytischen Geometrie. 10te Auflage, herausgegeben von E. Gruhl. Berlin, 1899. 8vo. Cloth. M. 1.50

GANTER (H.) und RUDIO (F.). Die Elemente der analytischen Geometrie. Mit zahlreichen Uebungsbeispielen. Teil II: Die analytische Geometrie des Raumes, von F. Rudio. 2te, verbesserte Auflage. Leipzig, Teubner, 1899. 8vo. 10 + 184 pp. M. 2.40

Genocchi (A.). Differentialrechnung und Grundzüge der Integralrechnung. Herausgegeben von G. Peano. Autorisierte deutsche Uebersetzung von G. Bohlmann und A. Schepp. Mit einem Vorworte von A. Mayer. (In 2 Lief.) 2te Lieferung. Leipzig, Teubner, 1899. 8vo. 7 pp. and pp. 225–399. M. 5.00

Gerberti, postea Silvestri II, papae, opera mathematica (972–1003). Accedunt aliorum opera ad Gerberti libellos æstimandos intelligendosque necessaria per septem appendices distributa. Collegit, ad fidem codicum manuscriptorum partim iterum, partim primum edidit, apparatu critico instruxit, commentario auxit, figuris illustravit N. Bubnow. Berlin, Friedländer, 1899. 8vo. 119 + 620 pp. M. 24.00

Goettler (J.). Untersuchungen über den allgemeinen Raumconnex. (Progr.) München, Kellerer, 1899. 8vo. 4 + 52 pp. M. 1.00

Gruhl (E.). See Gandtner (J. O.).

Gould (E. S.). A primer of the calculus. 2d revised edition, enlarged. New York, Van Nostrand Co., 1899. 16mo. 122 pp. (Van Nostrand's science series, No. 112.) Boards. $0.50

Gutzmer (A.). See Jahresbericht.

Hantzsch (V.). Sebastian Münster. Leben, Werk, wissenschaftliche Bedeutung. Leipzig, Teubner, 1898. 8vo. 2 + 187 pp. (Schriften der K. Sächsischen Gesellschaft der Wissenschaften, Abhandlungen der philologisch-historischen Classe, Vol. 18, No. 3.) M. 6.00

Hauck (G.). See Jahresbericht.

Haussner (R.). See Bernoulli (Jakob.).

Hensel (K.). See Kronecker (L.).

Heronis Alexandrini opera quæ supersunt omnia. Vol I: Herons von Alexandria Druckwerke und Automatentheater griechisch und deutsch herausgegeben von W. Schmidt. Im Anhang Herons Fragment über Wasseruhren, Philons Druckwerke, Vitruvs Kapitel zur Pneumatik. Mit einer Einleitung über die Heronische Frage und Anmerkungen. Supplementheft: Die Geschichte der Textüberlieferung. Griechisches Wortregister. Leipzig, Teubner, 1899. 12mo. 72 + 514 + 182 pp. M. 12.00

Jahrbuch über die Fortschritte der Mathematik, begründet von C. Ohrtmann. Unter besonderer Mitwirkung von F. Müller und A. Wangerin herausgegeben von E. Lampe. Vol. 28, Jahrgang 1897, in 3 Heften. 1tes Heft. Berlin, Reimer, 1899. 8vo. 416 pp. M. 13.00

Jahresbericht der Deutschen Mathematiker-Vereinigung. Vol. VI, Heft II: Die geometrischen Grundlagen der Photogrammetrie. Bericht von S. Finsterwalder. — Mechanische Beziehungen bei der Flächen-Deformation. Bericht von demselben. — Uebersicht über die wichtigsten Lehrbücher der Infinitesimal-Rechnung von Euler bis auf die heutige Zeit. Bericht von G. Bohlmann. —Herausgegeben im Auftrage des Vorstandes von G. Hauck und A. Gutzmer. Leipzig. Teubner, 1899. 8vo. 4 + 110 pp. M. 4.00

——. Vol. VII, Heft 1: Die Chronik der Vereinigung für das Jahr 1898, sowie die auf der Versammlung in Düsseldorf gehaltenen Vorträge.—Herausgegeben im Auftrage des Vorstandes von G. Hauck und A. Gutzmer. Leipzig, Teubner, 1899. 8vo. 159 pp. M. 4.80

—— Vol. VII, Heft 2: Die Entwickelung der Wahrscheinlichkeitstheorie und ihrer Anwendungen. Bericht von E. Czuber. Herausgegeben im Auftrage des Vorstandes von G. Hauck und A. Gutzmer. Leipzig, Teubner, 1899. 8vo. 8 + 279 pp. M. 8.00

JUEL (G.). Inledning i læren om de grafiske kurver. Avec un résumé en français. Kjöbenhavn, 1899. 4to. 90 pp (*Vid.-Selsk. Skrifter.*) M. 3.50

JUNKER (F.). Höhere Analysis. Teil 2: Integralrechnung. Leipzig, Göschen, 1899. 12mo. 205 pp. (Sammlung Göschen, No. 88.) M. 0.80

KLEIN (F.). Lectures on selected parts of elementary geometry, reported by F. Tägert. Russian translation by N. Parfentief and D. Sintsof. Kazen, 1898. 8vo. M. 2.50

KRAHE (A.). Apuntes de álgebra superior, adaptados à los programas de las escuelas especiales de ingenieros por J. Salmeron y Garcia. Madrid, 1899. 4to. 288 pp. Fr. 12.50

KRONECKER (L.). Werke, herausgegeben auf Veranlassung der Königl. preussischen Akademie der Wissenschaften von K. Hensel. (In 4 Bänden.) Vol. III, Halbband 1. Leipzig, Teubner, 1899. 4to. 8 + 474 pp. M. 36.00

LACOUR (P.). Historisk Matematik. 2te, gennemsete udgave. (In 10 hefter.) Kjöbenhavn, 1899. 8vo. Heft 6-8: pp. 161–256. M. 2.40

LALANDE (P. A.). Quid de mathematica vel rationali, vel naturali senserit Baconus Verulamius. Parisiis, 1899. 8vo. 111 pp. Fr. 7.50

LAMPE (E.). See JAHRBUCH.

LANGE (J.). Jacob Steiners Lebensjahre in Berlin 1821–63. Nach seinen Personalakten dargestellt. Berlin, 1899. 4to. 70 pp., portrait. M. 2.00

MANSION (P.). Mélanges mathématiques (1883–1898). Paris, Gauthier-Villars, 1898. 8vo. Fr. 5.00

MARIOTTE (F.). See CANTOR (G.).

MASCHKE (H.). Die Reduction linearer homogener Substitutionen von endlicher Periode auf ihre kanonische Form. 8vo. (*Mathematische Annalen*, Vol. 50, pp. 220–224.)

——. Ueber den arithmetischen Charakter der Coefficienten der Substitutionen endlicher linearer Substitutionsgruppen. 8vo. (*Mathematische Annalen*, Vol. 50, pp. 492–498.)

MAYER (A.). See GENOCCHI (A.).

MENGE (H.). See EUCLIDES.

MÜLLER (F.). See JAHRBUCH.

MUTH (P.). Theorie und Anwendung der Elementartheiler. Leipzig, Teubner, 1899. 8vo. 16 + 236 pp. M. 8.00

NEWSON (H. B.). The five types of projective transformations in the plane. 8vo. (*Kansas University Quarterly*, Vol. 8, 1899, pp. 43–66.)

OCAGNE (M. D'). Traité de nomographie. Théorie des abaques; applications pratiques. Paris, Gauthier-Villars, 1899. 8vo. 14 + 480 pp. Fr. 14.00.

PARFENTIEF (N.). See KLEIN (F.).

PASCAL (E.). Die Variationsrechnung. Autorisirte deutsche Ausgabe von A. Schepp. Leipzig, Teubner, 1899. 8vo. 6 + 146 pp. Cloth. M. 3.60.

PEANO (G.). Analisi della teoria dei vettori. Torino, Clausen, 1898. 8vo. 24 pp. (*Accademia reale delle scienze di Torino*, anno 1897–98.)

——. See FORMULAIRE and GENOCCHI (A.).

PIETZKER (F.). Beiträge zur Funktionen-Lehre. Leipzig, Teubner, 1899. 8vo. 5 + 64 pp. M. 2.80.

PORETZKY (P.). Sept lois fondamentales de la logique mathématique. Kazan, 1899. 8vo. 158 pp. M. 3.50.

PRINGSHEIM (A.). See ENCYKLOPÄDIE.

RUDIO (F.). See GANTER (H.).

SAINT-LOUP. Note sur les carrés magiques. Besançon, Jacquin [1899]. 8vo. 10 pp.

SALMERON Y GARCIA. See KRAHE (A.).

SCHEFFERS (G.). Ueber gewisse zweifach unendliche Curvenschaaren in der Ebene. (*Berichte der mathematisch-physischen Classe der K. Sächsischen Gesellschaft der Wissenschaften zu Leipzig*, 24 October, 1898.) 8vo. Pp. 261–294.

——. Synthetische Bestimmung aller Berührungstransformationen der Kreise in der Ebene. 8vo. (*Berichte der mathematisch-physischen Classe der K. Sächsischen Gesellschaft der Wissenschaften zu Leipzig*, 6 März, 1899). Pp. 145–160.

SCHEPP (A.). See GENOCCHI (A.) and PASCAL (E.).

SCHMIDT (W.). Heron von Alexandria; Uebersicht über seine Schriften und seine physikalisch-technischen Apparate. Leipzig, Teubner, 1899. 8vo. 15 pp., 3 plates. M. 0.80

——. See HERO.

SINTSOF (D.). See BIBLIOGRAPHIA and KLEIN (F.).

SOMMER (J.). Ueber die Bestimmung ausgezeichneter Punktgruppen auf Curven vom Geschlecht p. Tübingen, 1898. 8vo. 33 pp. M. 1.50

SONIN (N. I.). Integral calculus. Part I. (Russian.) St. Petersburg, 1898. 4to. 24+414 pp. (Lith.) M. 5.00

SPINDLER (K.). Ein Beitrag zur Einführung in das Gebiet der räumlichen Configurationen. Teil III. Diedenhofen, 1898. 8vo. 28 pp. M. 1.50

STOUFF (X.). Sur les lois de réciprocité. Paris, Hermann, 1898. 8vo. 31 pp.

THÖLDTE. Das logarithmische Potential von ν einander sich ausschliessenden Kreisflächen. Dessau, 1899. 4to. 20 pp. M. 1.50

VOLKOF (M.). Development of the ideas of number. (Russian.) St. Petersburg, 1899. 8vo. 127 pp. M. 2.80

VOSS (A.). See ENCYKLOPÄDIE.

WANGERIN (A.). See JAHRBUCH.

WETH (R.). Ueber eine Verallgemeinerung der Gauss'schen Differentialgleichung. (Progr.) Basel, Schwabe, 1899. 4to. 37 pp. M. 1.60

WILCZYNSKI (E. J.). On systems of multiform functions belonging to a group of linear substitutions with uniform coefficients. (*American Journal of Mathematics*, Vol. 21, pp. 85–106.) 1898. 4to.

WILLSON (F. N.). Some mathematical curves and their graphical construction. A brief treatise on the properties, methods of construction and practical applications of conic sections, trochoids, link-motion curves, centroids, spirals, the helix, and other important curves. For students in mathematical, engineering, or architectural courses, draughtsmen, etc. New York, The Macmillan Co., 1899. 4to. Cloth. (Part III of Theoretical and practical graphics.) $1.50

ZEUTHEN (H. G.). Forelæsninger over trekantskoordinater. Kjöbenhavn, 1898. 8vo. 64 pp. M. 2.50

ZIGNAGO (J.). Equazioni funzionali. Genova, 1899. 31 pp.

II. ELEMENTARY MATHEMATICS.

AUGUST (E. F.). Vollständige logarithmische und trigonometrische Tafeln. 23ste Auflage, besorgt von F. August. Leipzig, 1899. 12mo. 8 + 204 pp. Cloth. M. 1.60

BACKHAUS (K.). See WIESE (B.).

BARDEY (E.). Methodisch geordnete Aufgabensammlung, mehr als 8000 Aufgaben enthaltend über alle Teile der Elementar-Arithmetik, vorzugsweise für Gymnasien, Realgymnasien und Oberrealschulen. 24ste Auflage. Leipzig, Teubner, 1899. 8vo. 14 + 330 pp. Cloth. M. 3.20

BARTRINA Y CAPELLA (J. M.). Elementos de geometria pura y de trigonometria rectilinea. Barcelona, 1898. 4to. 397 pp. Fr. 10.00

BEMAN (W. W.) and SMITH (D. E.). New plane and solid geometry. [Revised edition.] Boston, Ginn & Co., 1899. 12mo. 9 + 382 pp. Half leather. $1.25

BOLTE (F.). Leitfaden für den Unterricht in der Planimetrie, zum Gebrauche an Navigationsschulen bearbeitet. 2te Auflage. Hamburg, Peuser, 1899. 8vo. 48 pp. Boards. M. 1.20

BREMIKER (C.). Logarithmisch-trigonometrische Tafeln mit 5 Decimalstellen. 8te Auflage, besorgt von A. Kallius. Berlin, 1899. 8vo. 192 pp. Cloth. M. 1.50

BRENNERT (E.). Geometrische Konstruktionsaufgaben mit vollständiger Auflösung. Ein Hilfsbuch für Lehrer. In übersichtlicher und methodischer Form bearbeitet. 2te Auflage. Berlin, Nicolai, 1899. 8vo. 4 + 100 pp. M. 1.50

BRESINA. Die Kegelschnitte für die Prima des Gymnasiums. Soest, 1898. 4to. 22 pp. M. 1.50

BREUER (P. J.). Völliger Abschluss der Lehre über die gemeinen Logarithmen im Diesseits der Gymnasial-Mathematik. Wipperfürth, 1898. 4to. 28 pp. M. 1.80

BRIGGS (W.). Preceptors' trigonometry. London, Clive, 1899. 12mo. 268 pp. (Preceptors' series.) 2s. 6d.

Brooksmith (E. J.). See Woolwich.

Bryant (Sophie). See Euclid.

Diekmann (J.). See Koppe (K.).

Euclid. The Elements of Euclid; for the use of schools and colleges; with notes, an appendix and exercises, by I. Todhunter. New revised and enlarged edition, by S. L. Loney. London, Macmillan, 1899. 12mo. 8 + 332 + 132 pp. Cloth. 4s. 6d.

——. Elements of geometry, books 3 and 4; edited for the use of schools, by C. Smith and Sophie Bryant. New York, The Macmillan Co., 1899. 16mo. 6 + 161 + 288 pp. Cloth. $0.60.

Fisher (G. E.) and Schwatt (I. J.). Text-book of algebra, with exercises, for secondary schools and colleges. Part I. Philadelphia, Fisher & Schwatt, 1898. 8vo. 14 + 683 pp.

Fourrey (E.). Récréations arithmétiques. Paris, Nony, 1899. 8vo. 8 + 263 pp.

Glaser. Stereometrie. Leipzig, Göschen, 1899. 12mo. 126 pp. (Sammlung Göschen, No. 97.) M. 0.80

Hall (H. S.) and Stevens (F. H.). An elementary course of mathematics; comprising arithmetic, algebra and Euclid. London, Macmillan, 1899. 16mo. 9 + 110 pp. Cloth. 2s. 6d.

Henrici (J.) und Treutlein (P.). Lehrbuch der Elementar-Geometrie. In 3 Teilen. Teil I: Gleichheit der Gebilde in einer Ebene. Abbildung ohne Massänderung. 3te Auflage. Leipzig, Teubner, 1898. 8vo. 8 + 144 pp. M. 2.00

Hervé (L.). Eléments d'algèbre, conformes aux programmes de l'enseignement classique et moderne (troisième et seconde modernes, troisième, seconde et rhétorique classiques). Paris, Carré et Naud, 1899. 8vo. 199 pp.

Hessenberg (G.). Ebene und sphärische Trigonometrie Leipzig, Göschen, 1899. 12mo. 165 pp. (Sammlung Göschen, No. 99.) M. 0.80

Holzmüller (G.). Elemente der Stereometrie. Teil I: Die Lehrsätze und Konstruktionen. Leipzig, Göschen, 1899. 8vo. 11 + 383 pp. (Sammlung Schubert, No. IV.) Cloth. M. 5.40

Kallius (A.). See Bremiker (C.).

Koppe (K.). Geometrie zum Gebrauche an höheren Unterrichtsanstalten, neu bearbeitet von J. Diekmann. 19te Auflage (3te Auflage der neuen Bearbeitung). Mit zahlreichen Uebungen und Aufgaben. Teil 1 der Planimetrie, Stereometrie und Trigonometrie. Ausgabe für Reallehranstalten. Essen, Baedeker, 1899. 8vo. 10 + 227 pp. Cloth. M. 2.40

Lichtblau (W.). See Wiese (B.).

Loney (S. L.). See Euclid.

Milne (W. J.). Grammar school algebra; a course for grammar schools and beginners in public and private schools. New York, American Book Co. [1899]. 12mo. 154 pp. Half leather. $0.50

——. Plane geometry. New York, American Book Co. [1899]. 12mo. 242 pp. Half leather. $0.75

——. Plane and solid geometry. New York, American Book Co. [1899]. 12mo. 384 pp. Cloth. $1.25

MÜLLER (H.). Die Elemente der Planimetrie. Ein Beitrag zur Methode des geometrischen Unterrichts. 7te Auflage. Metz, Scriba, 1899. 8vo. 4 + 84 pp. M. 1.20

NASSÒ (M.). Algebra elementare ad uso dei licei e degl istituti tecnici (1° biennio) secondo i programmi governativi. Con copiose note storiche, molti consigli pratici per indirizzare l'alunno alla risoluzione degli esercizi, più di 2000 esercizi e problemi graduati da risolvere e circa 400 esercizi e problemi minutamente risolti. Torino, Libreria Salesiana, 1898. 8vo. 426 pp. Cloth.

NEUMANN (K. W.). Lehrbuch der allgemeinen Arithmetik und Algebra. 7te Auflage. Leipzig, 1899. 8vo. 8 + 210 pp. M. 2.80

PAHL (F.). Die Entwickelung des mathematischen Unterrichts an unseren höheren Schulen. Teil II. Charlottenburg, 1899. 4to. 37 pp. M. 1.50

PRIOLO (G.). Tavole complete dei valori dei segmenti circolari. Sciacca, 1898. 8vo. 22 pp. Fr. 2.00

PUND (O.). Algebra, mit Einschluss der elementaren Zahlentheorie. Leipzig, Göschen, 1899. 8vo. 8 + 345 pp. (Sammlung Schubert, No. VI.) Cloth. M. 4.40

RAYDT (H.). Lehrbuch der Elementar-Mathematik. Planimetrie, Arithmetik, Trigonometrie und Stereometrie. Leipzig, 1899. 8vo. 8 + 253 pp. M. 2.70

SCHOTTEN (H.). Mathematischer Unterricht. Halle, 1899. 4to. 19 pp. M. 1.20

SCHRÖN (L.). Seven-place logarithmic tables. Hungarian edition, by J. Sztoczek. 5th edition. Braunschweig, 1899. 8vo. 588 pp. M. 6.00

SCHUBERT (H.). Beispiel-Sammlung zur Arithmetik und Algebra. 2765 Aufgaben, systematisch geordnet. 2te Auflage. Leipzig, Göschen 1899. 12mo. 134 pp. (Sammlung Göschen, No. 48.) M. 0.80

——. Elementare Arithmetik und Algebra. Leipzig, Göschen, 1899. 8vo 6 + 230 pp. (Sammlung Schubert, No. I.) Cloth. M. 2.80

SCHWATT (I. J.). See FISHER (G. E.).

SCHWERING (K.). Arithmetik und Algebra für höhere Lehranstalten. 2te Auflage. Freiburg, Herder, 1899. 8vo. 7 + 80 pp. M. 1.00

——. 100 Aufgaben aus der niederen Geometrie nebst vollständigen Lösungen. 2te Auflage. Freiburg, Herder, 1899. 8vo. 11 + 168 pp. M. 2.00

——. Raumlehre für sechsstufige Schulen und Lehrerseminare, nach den neuen Lehrplänen bearbeitet. 2te Auflage. Freiburg, Herder, 1899. 8vo. 16 pp. M. 0.25

SICKENBERGER (A.). Uebungsbuch zur Algebra. 1te Abteilung. 1te und 2te Stufe der Rechnungsarten, einschliesslich der linearen Gleichungen mit einer und mehreren Unbekannten. 3te Auflage. München, Ackermann, 1899. 8vo. 5 + 106 pp. M. 1.20

SMITH (C.). See EUCLID.

SMITH (D. E.). See BEMAN (W. W.).

STEVENS (F. H.). See HALL (H. S.).

SZTOCZEK (J.). See SCHRÖN (L.).

TIMPE (W.). Mathematische Aufgaben. Berlin, 1899. 4to. 39 pp. M. 1.20

TODHUNTER (I.). See EUCLID.

TREUTLEIN (P.). See HENRICI (J.).

ULRICH (G.). Ausführliches Lehrbuch der Geometrie sowie der ebenen und sphärischen Trigonometrie für den Selbstunterricht. Mit zahlreichen Aufgaben zur Konstruktion und Berechnung, nebst Anleitung zu deren Lösung. Berlin, Schultze, 1898. 8vo. 6 + 520 pp. M. 4.00

WENTWORTH (G. A.). Plane geometry. New revised edition. Boston, Ginn & Co., 1899. 12mo. 8 + 256 pp. Half leather. $0.85

WELLS (W.). The essentials of geometry. Boston, Heath & Co., 1899. 12mo. 6 + 399 pp. Half leather. $1.25

WIESE (B.), LICHTBLAU (W.) und BACKHAUS (K.). Raumlehre für Lehrerseminare. Teil I: Die Flächenlehre (Planimetrie). 3te Auflage. Breslau, Hirt, 1899. 8vo. 190 pp. Cloth. M. 2.25

WITTING (A.). Geometrische Constructionen, insbesondere in begrenzter Ebene. Dresden, 1899. 4to. 18 pp. M. 1.20

WOOLWICH mathematical papers, for admission into the Royal Military Academy, 1889–99, edited by E. J. Brooksmith. London, Macmillan, 1899. 12mo. 6s.

ZEISZIG (E.). Algebraische Aufgaben für die Volksschule. Für die Hand des Lehrers bearbeitet. 2te Auflage. Leipzig, Wunderlich, 1899. 8vo. 5 + 46 pp. M. 0.60

III. APPLIED MATHEMATICS.

ALBRECHT (T.). Bericht über den Stand der Erforschung der Breitenvariation am Schlusse des Jahres 1898. Herausgegeben vom Centralbureau der internationalen Erdmessung. Berlin, Reimer, 1899. 4to. 22 pp., 1 plate. M. 3.00

D'ALEMBERT. Abhandlung über Dynamik, in welcher die Gesetze des Gleichgewichtes und der Bewegung der Körper auf die kleinmöglichste Zahl zurückgeführt und in neuer Weise abgeleitet werden, und in der ein allgemeines Princip zur Auffindung der Bewegung mehrerer Körper, die in beliebiger Weise auf einander wirken, gegeben wird. (1743.) Uebersetzt und herausgegeben von A. Korn. Leipzig, Engelmann, 1899. 12mo. 210 pp., 4 plates. (Ostwald's Klassiker der exakten Wissenschaften, No. 106.) M. 3.60

AUBUSSON DE CAVARLAY (E.). Cours d'électricité, professé à l'Ecole d'application du génie maritime. Vol. I and Vol. II, fascicule 1: Lois et théories usuelles; unités et mesures électriques; dynamos à courant continu. Dynamos à courant alternatif; transformateurs; commutatrices; piles et accumulateurs. Paris, 1899. 8vo. Fr. 22.00

AUERBACH (F.). Kanon der Physik. Die Begriffe, Principien, Sätze, Formeln, Dimensionsformeln und Konstanten der Physik, nach dem neuesten Stande der Wissenschaft systematisch dargestellt. Leipzig, Veit, 1899. 8vo. 12 + 522 pp. M. 11.00

BATTERMANN (H.). Resultate aus den Polhöhenbestimmungen in Berlin, ausgeführt in den Jahren 1891 und 1892 am Universal-Transit der Königl. Sternwarte. Herausgegeben vom Centralbureau der internationalen Erdmessung. Berlin, Reimer, 1899. 4to. 45 pp. M. 3.00

BELKNAP (C.). See COFFIN (J. H. C.).

BERGET (A.). See CHAPPUIS (J.).

BERTIN (L. E.). Position d' équilibre des navires sur la houle. Paris, Gauthier-Villars, 1898. 8vo. Fr. 3.00

BLUDAN (A.). See ZÖPPRITZ (K.).

BÖRNER (H.). Leitfaden der Experimental-Physik für Realschulen, sowie für den Anfangsunterricht in Oberrealschulen. 4te Auflage. Berlin, 1899. 8vo. 12 + 183 pp. Cloth. M. 2.20

BOLTE (F.). Neues Handbuch der Schiffahrtskunde. Mit einer Vorrede von Neumayer. Hamburg, Eckardt & Messtorff, 1899. 4to. 20 + 274 pp. Cloth. M. 11.00

—— Nautische Tafelsammlung. Nebst 4 magnetischen Karten, entworfen von Neumayer. Hamburg, Eckardt & Messtorff, 1899. 4to. 3 + 162 pp. Cloth. M. 7.00

BOUCHÉ-LECLERQ (A.). L'astrologie grecque. Paris, 1899. 8vo. 20 + 663 pp. Fr. 20.00

BOURLET (C.). Nouveau traité des bicycles et bicyclettes. 2e édition. IIe partie: Le travail. Paris, Gauthier-Villars, 1898. (Encyclopédie scientifique des aide-mémoire.) Fr. 2.50

BOUSSINESQ (J.). Aperçu sur la théorie de la bicyclette. Paris, Gauthier-Villars, 1899. 4to. Fr. 1.00

——. Complément à une étude récente concernant la théorie de la bicyclette. Influence, sur l'équilibre, des mouvements latéraux spontanés du cavalier. Paris, Gauthier-Villars, 1899. Fr. 1.00

BOUTY. See JAMIN (J.).

BURBURY (S. H.). A treatise on the kinetic theory of gases. London, Clay, 1899. 8vo. 8 + 157 pp. (Cambridge University Press series.) Cloth. 8s.

BRAUER (E. A.). Grundriss der Turbinen-Theorie. Leipzig, Hirzel, 1899. 8vo. 4 + 124 pp. Cloth. M. 4.00

BROUGH (B.). Treatise on mine-surveying. 7th edition, revised. London, Griffin, 1899. 12mo. 378 pp. (Mining series.) 7s. 6d.

BRUNO (K.). Der Stoss elastischer Kugeln. (Progr.) Klagenfurt, Kleinmayr, 1899. 8vo. 22 pp., 1 plate. M. 1.00

CEDERBOM (J. E.). Problemer till mekaniska värmeläran jämte deres uträkning. Stockholm, 1899. 8vo. 243 pp. M. 6.00

CHAPPUIS (J.) et BERGET (A.). Leçons de physique générale. Cours professé à l'Ecole centrale des arts et manufactures et complété suivant le programme de la licence ès sciences physiques. (En 3 volumes.) Vol. II: Électricité et magnétisme. 2e édition, entièrement refondue. Paris, Gauthier-Villars, 1899. 8vo. Fr. 13.00

CHOMÉ (F.). Cours de géométrie descriptive de l' Ecole militaire. Ire partie, livre I, à l'usage des candidats à l' Ecole militaire et aux Ecoles spéciales des Universités. 3e édition, conforme à la 2e édition, entièrement revue, corrigée et augmentée, contenant les prescriptions à observer pour l'exécution des épures. Paris, Gauthier-Villars, 1898. 4to, avec atlas de 37 planches. Fr. 10.00

——. Ire partie, livre II, à l'usage des élèves de l'enseignement supérieur. Paris, Gauthier-Villars, 1899. 4to, avec atlas de 48 planches. Fr. 15.00

COFFIN (J. H. C.). Navigation and nautical astronomy. 7th edition, revised by C. Belknap. New York, Van Nostrand Co., 1899. 12mo. 221 pp. Cloth. $3.50

CRANZ (C.) und KOCH (K. R.). Untersuchungen über Vibration des Gewehrlaufs. I. Schwingungen in verticaler Ebene bei horizontal gehaltenem Gewehr. A. Gewehre vom Typus des Mausergewehrs Modell 71. München, Franz, 1899. 4to. 31 pp., 6 plates. M. 2.00

DARIÈS (G.). Calcul des canaux et aqueducs. Paris, Gauthier-Villars [1899]. 16mo. 180 pp. (Encyclopédie scientifique des aide-mémoire, section de l'ingénieur, No. 235 B.) Fr. 2.50

DÉTREZ (G.). Applications de mathématiques. Lille, Détrez, 1896. 8vo. 32 pp.

——. Distributions d'eau. Paris, Dunod, 1899. 16mo. 12 + 567 pp. (Bibliothèque du conducteur des travaux publiques.)

DOMOGAROF (A.). Elements of mechanics. Part 2: Kinematics. (Russian.) St. Petersburg, 1899. 8vo. 212 pp.

DONATI (L.). Lezioni sulla fisica matematica, dettate nella R. Università di Bologna. Parte II. Bologna, 1899. 8vo. 92 pp. Fr. 6.50

DUHEM (P.). Traité élémentaire de mécanique chimique fondée sur la thermodynamique. Vol. 4: Les mélanges doubles ; statique chimique générale des systèmes hétérogènes. Paris, Hermann, 1899. 8vo. 387 pp.

ERNST (A.). Die Hebezeuge. Theorie und Kritik ausgeführter Konstruktionen mit besonderer Berücksichtigung der elektrischen Anlagen. 3te Auflage, unter Mitwirkung von W. Maier und R. Rau. Berlin, Springer, 1899. Vol. I: 26+808 pp.; Vol. II: 15+783 pp.; Vol. III, 7 pp., 85 Plates. Cloth. M. 60.00

FISCHER (O.). Der Gang des Menschen. Teil II: Die Bewegung des Gesammtschwerpunktes und die äusseren Kräfte. Leipzig, Teubner, 1899. 8vo. 2+130 pp, 12 plates. (*Abhandlungen der mathematisch-physischen Classe der K. Sächsischen Gesellschaft der Wissenschaften*, Vol. 25, No. I.)

FISCHER-HINNEN (J.). Die Wirkungsweise, Berechnung und Konstruktion elektrischer Gleichstrom-Maschinen. 4te, vollständig umgearbeitete und stark vermehrte Auflage. Zürich, Raustein, 1899. 8vo. 10+410 pp., 3 plates. Cloth. M. 13.20

FÖPPL (A.). Vorlesungen über technische Mechanik. (In 4 Bänden.) Vol. IV: Dynamik. Leipzig, Teubner, 1899. 8vo. 14+456 pp. Cloth. M. 12.00

FREYBERGER (H.). Perspektive, nebst einem Anhang über Schattenkonstruktion und Parallelperspektive. 2te Auflage. Leipzig, Göschen, 1899. 12mo. 127 pp. (Sammlung Göschen, No. 57.) M. 0.80

FRITSCH (H.). Die Newtonsche Gravitation abgeleitet von Aetherstössen. Königsberg, 1899. 4to. 22 pp. M. 1.50

FÜRLE (H.). Zur Theorie des Rechenschieber. Berlin, 1899. 4to. 22 pp. M. 1.00

FUHRMANN (A.). Bauwissenschaftliche Anwendungen der Differentialrechnung. Lehrbuch und Aufgabensammlung 2te Hälfte. Berlin, Ernst, 1899. 8vo. Pp. 181–348. (Teil III der Anwendungen der Infinitesimalrechnung in den Naturwissenschaften, im Hochbau und in der Technik.) M. 5.50

GÉRARD (ERIC). Leçons sur l'électricité, professées à l'Institut électrotechnique 6e édition, refondue et complétée. (En 2 volumes.) Vol. I: Théorie de l'électricité et du magnétisme; électrométrie; théorie et construction des générateurs et des transformateurs électriques. Paris, Gauthier-Villars, 1899. 8vo. Fr. 12.00

GOLDSCHEIDER (F.). Ueber die Gauss'sche Osterformel. Teil II. Berlin, 1899. 4to. 30 pp. M. 1.00

GOSSIN (H.). See JUDE (R. H.).

GOULIER (C.-M.). Etudes sur les méthodes et les instruments des nivellements de précision, revues, annotées et accompagnées d'une étude sur les variations de longueur des mires, d'après les expériences du colonel Goulier, par C. Lallemand. Paris, Gauthier-Villars, 1898. 4to., avec 8 planches et un portrait. Fr. 20.00

GREENE (C. E.). The action of materials under stress or structural mechanics, comprising the strength and resistance of materials and elements of structural design. With examples and problems. Ann Arbor, Greene, 1897. 8vo. 6 + 271 pp., 3 plates. $3.00

GUILLET (A.). See POINCARÉ (H.).

HAASE (H.). Theorie der parabolischen Brückengewölbe, oder das Grundgesetz des Horizontalschubs in seiner Anwendung auf Brückengewölbe unter der ausschliesslichen Wirkung vertikaler Aussenkräfte, entwickelt an dem Beispiel einer gewölbten Bahnüberbrückung. Regensburg, Nationale Verlagsanstalt, 1899. 8vo. 7 + 110 pp., 5 plates. M. 4.00

HAEDER (H.). Konstruieren und Rechnen, für Schule und Praxis bearbeitet. Duisburg und Düsseldorf, Schwann, 1899. Vol. I: Text. 8vo. 8 + 191 pp. Vol. II: 4to. 4 pp., 115 plates. Cloth. M. 10.00

HAUGHTON (S.). Manual of optics. New edition, enlarged by J. Warren. London, 1899. 8vo. 116 pp. Cloth. 2s. 6d.

HELMERT (F. R.). Beiträge zur Theorie des Reversionspendels. Leipzig, Teubner, 1899. 4to. 92 pp. (Veröffentlichung des K. Preussischen Geodätischen Institutes und Centralbureaus der internationalen Erdmessung.) M. 7.60

HIRSCH (K.). Urkunden zur Geschichte der Mechanik. Schwäbisch-Hall, 1898. 4to. 41 pp. M. 2.00

HOLMAN (S. W.). Matter, energy, force, and work. A plain presentation of fundamental physical concepts and of the vortex-atom and other theories. New York, The Macmillan Company, 1899. 12mo. 14 + 257 pp. Cloth. $2.50

ISAACHSEN (J.). Die Bedingungen für eine gute Regulirung. Eine Untersuchung der Regulirungsvorgänge bei Dampfmaschinen und Turbinen. Berlin, Springer, 1899. 8vo. 3 + 76 pp. M. 2.00

JACQUEMART (P.). See PÉCHEUX (H.).

JAMIN (J.). Cours de physique de l'Ecole polytechnique. Deuxième supplément par M. Bouty : Progrès de l'électricité: oscillations hertziennes, rayons cathodiques et rayons X. Paris, Gauthier-Villars, 1899. 8vo. 217 pp. Fr. 3.50

JANET (P.). Premiers principes d'électricité industrielle. Piles ; accumulateurs ; dynamos ; transformateurs. 3e édition, entièrement refondue. (Ouvrage couronné par l'Académie des sciences.) Paris, Gauthier-Villars, 1899. 8vo. 10 + 280 pp. Fr. 6.00

JENSEN (J. A. D.). Grundrids af laeren om ebbe og flod. Kjöbenhavn, 1899. 8vo. 48 pp. M. 1.20

JOUBERT (J.). Cours élémentaire d'électricité. 3e édition, revue et augmentée. Paris, 1899. 8vo. 201 pp.

JUDE (R. H.). Elementary treatise on physics, experimental and theoretical. Partly from the French of H. Gossin. Vol. I : Mechanics, hydrostatics, pneumatics, heat, and acoustics. London, 1899. 8vo. 926 pp. Cloth. 12s. 6d.

KERBER (A.). Beiträge zur Dioptrik. Heft 5. Leipzig, 1899. 8vo. 16 pp. M. 0.50

KESSLER (J.). Berechnung und Konstruktion der Turbinen. Eine kurzgefasste Theorie in elementarer Darstellung mit erläuternden Rechnungsbeispielen. 2te Auflage. Hildburghausen, Pezoldt, 1899. 8vo. 3 + 50 pp. (Technische Lehrhefte, Maschinenbau, No. 9.) M. 1.40

KHVOLSON (O. D.). Text-book of physics. Vol. III : Heat. (Russian.) St. Petersburg, 1899. 8vo. 682 pp. M. 15.00

KOCH (K. R.). See CRANZ (C.).

KÖTTER (F.). Bemerkungen zu F. Kleins und A. Sommerfelds Buch über die Theorie des Kreisels. Berlin, Mayer & Müller, 1899. 8vo. 26 pp. M. 0.80

KOPPE (M.). Die Ausbreitung der Erschütterung an der Wellenmaschine, darstellbar durch einen neuen Grenzfall der Bessel'schen Funktionen. Berlin, 1899. 4to. 28 pp., 1 plate. M. 1.00

KORN (A.). Lehrbuch der Potentialtheorie. Allgemeine Theorie des Potentials und der Potentialfunktionen in Raume. Berlin, Dümmler, 1899. 8vo. 14 + 417 pp. M. 9.00

—— See D'ALEMBERT.

KOVALEVSKY (S.). Theory and application of the double vernier. (Russian.) St. Petersburg, 1899. 8vo. 20 pp., 3 plates.

KRÜGER (L.). Beiträge zur Berechnung von Lotabweichungssystemen Leipzig, Teubner, 1899. 4to. 106 pp. (Veröffentlichung des K. Preussischen Geodätischen Institutes und Centralbureaus der internationalen Erdmessung.) M. 8.40

LALLEMAND (C.). See GOULIER (C. M.).

LAUENSTEIN (R.). Die Festigkeitslehre. Elementares Lehrbuch für den Schul- und Selbstunterricht, sowie zum Gebrauch in der Praxis, nebst einem Anhang, enthaltend Tabellen der Potenzen, Wurzeln, Kreisumfänge und Kreisinhalte. 5te Auflage. Stuttgart, Bergsträsser, 1899. 8vo. 6 + 157 pp. M. 4.00

LAURENT (P.). Résistance des bouches à feu. Paris, Gauthier-Villars, 1898. 12mo. (Encyclopédie scientifique des aide-mémoire.) Fr. 2.50

LAUSSEDAT (A.). Recherches sur les instruments, les méthodes et le dessin topographiques. (En 2 volumes.) Vol. I: Aperçu historique sur les instruments et les méthodes; la topographie dans tous les temps. Paris, Gauthier-Villars, 1899. 8vo. 11 + 450 pp., 14 planches. Fr. 15.00

LAVERGNE (G.). Les turbines. 2e édition. Paris, Gauthier-Villars, 1898. (Encyclopédie scientifique des aide-mémoire.) Fr. 2.50

LEBON (E.). Géométrie descriptive, pour l'enseignement secondaire moderne. Vol. I: Classe de seconde. Paris, Delalain, 1899. 8vo. 150 pp. Fr. 3.00

LECORNU (L.). Régularisation du mouvement dans les machines. Paris, Gauthier-Villars, 1898. (Encyclopédie scientifique des aide-mémoire.) Fr. 2.50

LERSCH (B. M.). Einleitung in die Chronologie. 2te, umgearbeitete und vermehrte Auflage. Teil II: Der christliche Kalender, seine Einrichtung, Geschichte und chronologische Verwertung. Freiburg, 1899. 8vo. 8 + 189 pp. M. 4.00

LINEHAM (W. J.). A text-book of mechanical engineering. 3d revised and enlarged edition. Philadelphia, Lippincott Co, 1899. 8vo. Cloth. $5.00

LUCCIARDI (J. S.). Traité sur la balance, ou théorie des instruments de pesage, à l'usage des vérificateurs-adjoints des poids et mesures et des aspirants à la vérification. 1re édition. Annecy, Dépollier, 1899. 8vo. 301 pp.

MAIER (W.). See ERNST (A.).

MÉCANIQUE. Eléments de statique, à l'usage des candidats au baccalauréat classique et au baccalauréat de l'enseignement moderne (3e série). Besançon, Dodivers, 1899. 8vo. 157 pp.

NEWSHOLME (A.). Elements of vital statistics. 3d edition, re-written, London, 1899. 8vo. 366 pp. Cloth. 7s. 6d.

PÉCHEUX (H.). Précis de physique industrielle. Avec une introduction par P. Jacquemart. Paris, 1899. 8vo. 7 + 577 pp.

PEIRCE (B. O.) and WILLSON (R. W.). On the thermal conductivities of certain poor conductors. I. 8vo. 56 pp. (*Proceedings of the American Academy of Arts and Sciences*, Vol. 34, No. 1, August, 1898.)

PERRY (J.). The steam engine and gas and oil engines; a book for the use of students who have time to make experiments and calculations. London, Macmillan, 1899. 8vo. 8 + 646 pp. Cloth. 7s. 6d.

PLANT (E. C.). Geometrical drawing for army and navy candidates and public school classes. Vol. I: Practical plane geometry. With an appendix on the construction of scales. New York, The Macmillan Co., 1899. 4to. 14 + 185 pp. $1.90

PILLING (O.). Ueber die Bedeutung der Energie in der Naturwissenschaft. Schleusingen, 1899. 4to. 16 pp. M. 1.50

POINCARÉ (H.). Cinématique et mécanismes; potentiel et mécanique des fluides. Cours professé à la Sorbonne par H. Poincaré, rédigé par A. Guillet. Paris, Carré et Naud, 1899. 8vo. 391 pp. (Cours de la Faculté des sciences de Paris.)

POTERIN DU MOTEL (H.). Théorie des assurances sur la vie. Paris. Gauthier-Villars, 1899. 8vo. Fr. 25.00

RAU (R.). See ERNST (A.).

Robinson (J. L.). First book in dynamics. London, Longmans, 1899. 12mo. 98 pp. 2s.

—— First book in statics. London, Longmans, 1899. 12mo. 124 pp. 2s.

—— First book in statics, with examples. London, Longmans, 1899. 12mo. 3s.

Salazar (A. E.). Kalkulos sobre las kañerias de agua. Ensayo de unifikacion de las formulas usuales i de simplifikacion de los kalkulos basada en la nozion de zirkuito idrauliko ; kondukcion de agua, kalkulos sobre la potenzia. Santiago de Chile, 1898. 4to. 17 + 246 pp. Cloth. Fr. 16.00

Scheibner (W.). Ueber die Differentialgleichungen der Mondbewegung. Leipzig, Teubner, 1899. 8vo. 2 + 26 pp. (*Abhandlungen der mathematisch-physischen Classe der K. Sächsischen Gesellschaft der Wissenschaften*, Vol. 25, No. II.) M. 1.50

Schiller (N.). Meaning of the ideas of force and mass in the theory of knowledge and in mechanics. (Russian.) Kief, University Press, 1898. 8vo. 2 + 91 pp.

Schoentjes (H.). Cours de physique expérimentale de l'Université de Gand. 2e édition. Partie II : Chaleur, magnétisme, électricité, lumière et chaleur rayonnante. Paris, 1899. 8vo. 480 pp. Fr. 10.00

Schubert (F.). Die darstellende Geometrie an maschinentechnischen Lehranstalten, Gewerbe- und Fachschulen. Als Wegweiser für Lehrende und Lernende nach den Formalstufen bearbeitet. In 3 Teilen. Teil I : Das geometrische Zeichnen. Mittweida, Polytechnische Buchhandlung, 1899. 8vo. 3 + 176 pp. Cloth. M. 4.00

Schupmann (L.). Die Medial-Fernrohre. Eine neue Konstruktion für grosse astronomische Instrumente. Leipzig, Teubner, 1898. 8vo. 6 + 146 pp. M. 4.80

Seyrig (T.). Statique graphique des systèmes triangulés. (En 2 volumes.) Vol. I : Exposés théoriques. Vol. II : Exemples d'applications. Paris, Gauthier-Villars, 1898. 12mo. (Encyclopédie scientifique des aide-mémoire.) Fr. 2.50

Tinsley (G. W.). The mechanics of a cyclone. Columbus, Ind., 1899. 8vo. 23 pp. $0.25

Walter (A.). Theorie der atmosphärischen Strahlenbrechung. Leipzig, Teubner, 1899. 8vo. 4 + 74 pp. (Veröffentlicht mit Unterstützung der K. Akademie der Wissenschaften in Wien.) M. 2.80

Warren (J.). See Haughton (S.).

Willson (F. N.). Practical engineering drawing and third-angle projection ; for students in scientific, technical, and manual training schools, and for engineering and architectural draughtsmen, sheet metal workers, etc. New York, The Macmillan Company, 1899. 4to. 178 pp. Cloth. $2.80

Willson (R. W.). See Peirce (B. O.).

Witz (A.). Thermodynamique, à l'usage des ingénieurs. 2e édition. Paris, Gauthier-Villars [1899]. 16mo. 203 pp. (Encyclopédie scientifique des aide-mémoire, section de l'ingénieur, No. 20 A_2.) Fr. 2.50

Wüllner (A.). Lehrbuch der Experimentalphysik. 5te, vielfach umgearbeitete und verbesserte Auflage. (In 4 Bänden.) Vol. IV : Die Lehre von der Strahlung. Halbband 1. Leipzig, Teubner, 1899. 8vo. Pp. 1–512. M. 7.00

ZEUNER (G.). Vorlesungen über Theorie der Turbinen, Ventilatoren und Centrifugalpumpen, mit vorbereitenden Untersuchungen aus der technischen Hydraulik. Leipzig, Felix, 1899. 8vo. 11 + 372 pp. M. 10.00

ZÖPPRITZ (K.). Leitfaden der Kartenentwurfslehre. In 2ter, neu bearbeiteter und erweiterter Auflage herausgegeben von A. Bludau. (In 2 Teilen.) Teil I: Projectionslehre. Leipzig, 1899. 8vo. 10 + 178 pp. M. 4.80

THE FORTY-EIGHTH ANNUAL MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

The forty-eighth annual meeting of the American Association for the Advancement of Science was held in Columbus, Ohio, August 21–26. There were 352 members and associates in attendance and 273 papers were presented before the sections. The members came from twenty-eight different states, the District of Columbia, and Canada. A considerable number of members of affiliated societies were also in attendance, many of whom took part in the proceedings of the sections in which they were interested. Their presence increased the general scientific interest; and in view of the desire of the various sections to have still closer relations with the respective affiliated societies, authority was given to the officers of the Association whereby any section may arrange for a joint meeting with societies of a similar scope. It is hoped that the AMERICAN MATHEMATICAL SOCIETY and the Society for the Promotion of Engineering Education may meet next year with Sections A and D respectively. Probably several other societies will hold joint meetings with the corresponding sections of the Association.

The next meeting of the Association will be held in New York City during the last week of June, 1900, under the presidency of Professor R. S. Woodward, Columbia University. Professor Asaph Hall, Jr., Michigan University, will be vice-president of Section A, and Dr. W. M. Strong, Yale University, will be secretary. It is hoped that the change in the time of these meetings from the latter part of August to the last week in June will tend to increase the attendance. This change seemed especially desirable for next year since it will enable many to attend the meetings of the Association on their way to the Paris exposition.

The meetings of the section of mathematics and astronomy were well attended. The officers of this section were: vice-president, Alexander Macfarlane; secretary, J. F. Hayford; councilor, C. L. Doolittle; sectional committee, Alexander Macfarlane, J. F. Hayford, G. B. Halsted, R. D. Bohannan, W. M. Strong. The Council elected the following mathematicians and astronomers to fellowship in the Association: S. M. Barton, W. P. Durfee, O. L. Fassig, R. A. Fessenden, C. E. Furness, R. A. Harris, G. A. Hill,

J. E. Keeler, W. H. Metzler, G. W. Meyers, D. A. Murray, W. M. Strong, J. H. Tanner, A. G. Webster.

The opening address of the Vice-President of Section A was on "The fundamental principles of algebra." It has been published in *Science*, September 15, 1899. The following papers were presented at the meetings of this section :

(1) Professor G. B. Halsted : "Report on progress in non-euclidean geometry."

(2) Professor L. E., Dickson : "Report on progress in the theory of linear groups."

(3) Professor Asaph Hall, Jr.: "The aberration constant from observations of Polaris.".

(4) Professor R. W. McFarlane : "Ancient eclipses and chronology."

(5) Professor H. C. Lord : "Some points in the design of a spectroscope."

(6) Professor J. V. Collins : "Note on Grassmann's proof that there can be but two kinds of lineal multiplication of two factors."

(7) Professor G. J. Stokes : "The theory of mathematical inference."

(8) Mr. J. W. Davis : "Internal forces that generate stellar atmospheres."

(9) Professor R. A. Fessenden : "The determination of the nature of electricity and magnetism, including a determination of the density of the ether."

(10) Dr. G. A. Miller : "On the commutators of a group."

(11) Dr. S. Kimura : "Linear vector functions."

Three of the titles that appeared on the programme are not reproduced here on account of the fact that these papers did not reach the secretary in time to be presented at the meeting. In the absence of the authors the papers by Professor Stokes and Dr. Kimura were presented by Professors Macfarlane and Shaw respectively. The paper by Professor Fessenden was read before a joint session of Sections A and B and will be reviewed in connection with the papers of Section B. Professor Dickson's report has already appeared in the October number of the Bulletin. Dr. Miller's paper will appear in a later number. Abstracts of the other papers are given below.

Professor Halsted contrasted the neglect of Lobachévski and Bolyai with the present interest in non-euclidean geometry as exhibited by the appearance in the *Monist* of a long article on this subject by Poincaré. The investigations of Sophus Lie and the contributions of Gérard were discussed. Burnside's articles in the *Proceedings of the London Mathematical Society* were interpreted. The new book, Universal algebra, by Whitehead, and the important discoveries freeing us from the misleading conceptions of curvature of space in non-euclidean geometry were elucidated. The philosophic bearings of the new space ideas were illustrated with special reference to B. A. W. Russell's book. Killing and Clifford spaces were noted. The reproduction of the non-euclidean classics, especially by Engel and Stäckel, was described, also the work of the Hungarian academy of sciences. The history of the subject (Loria) and finally the bibliography closed the report, which has recently appeared in *Science*, October 20, 1899.

Professor Hall stated that observations of the zenith distances of Polaris for the determination of the aberration constant have been made with the meridian circle of the Detroit Observatory since about May 1, 1898. The direct observations made during the interval from May, 1898 to July, 1899, have been roughly reduced and give the values 20.″60 and 20.″58, from pointings above and below the pole respectively. These seem rather large, the correct value being supposed to be about 20.″50. By observing above and below the pole the latitude variation is eliminated, but the effect of temperature on the instrumental readings needs to be followed very carefully. It is to be hoped that this work can be kept up through the year 1900. It has been suggested by several astronomers and physicists that a correct theoretical treatment of aberration has not yet been obtained. The values of the aberration constant recently found vary a great deal, ranging from about 20.″31 to 20.″66. An increase in the value of this constant would indicate, of course, a decrease in the usually adopted value of the solar parallax.

The burden of the argument in Professor McFarlane's paper was that eclipses have not aided in fixing any historical event in its proper place in history; that the difficulty is not mathematical, but arises from want of exactness in historical records; and that this want of exactness arose from the want of a fixed time from which to reckon, because the writers of ancient histories, so-called, gave, in general, no dates at all. Moreover, all references to eclipses are vague in the extreme. Taking the most celebrated of all the cases—the eclipse of Thales—it was shown that Sir George Airy and Mr. Baily differed about 25 years in the date of the eclipse, and that other astronomers, and celebrated scholars, in England, France, Germany, and elsewhere, have set the time in nine different years scattered over a period of more than forty-six years. If the most celebrated case thus falls, there is no standing for the others.

At one place in the demonstration given on page 28 of the *Ausdehnungslehre* of 1862, an equation can be derived from the preceding in a somewhat different manner than that given. As this theorem is fundamental in Grassman's system, Dr. Collins worked out the following alternative proof:

In the derivation of equation (*c*) from

$$(b) \qquad \Sigma x_{r,u}x_{s,v}(a_{r,s}[e_ue_v] + a_{s,r}[e_ve_u]) = 0$$

by successive substitution of $2x_{a,c}$ and $x_{a,c}$ instead of $+1$ and -1 as values of a particular $x_{r,u}$, we get

$$\Sigma x_{s,v}(a_{a,s}[e_ce_v] + a_{s,a}[e_ve_c]) + 3x_{a,c}a_{a,a}[e_ae_a] = 0,$$

in which the summation does not extend to $s = a$, $v = c$ at the same time. Inserting the missing term,

$$\Sigma x_{s,v}(a_{a,s}[e_ce_v] + a_{s,a}[e_ve_c]) + 2x_{a,c}a_{a,a}[e_ae_a] = 0.$$

By putting a particular $x_{s,v}$, as $x_{b,d}$ first equal to 2 and then equal to 1, we have finally

$$(c) \qquad a_{a,b}[e_ce_d] + a_{b,a}[e_de_c] = 0.$$

The derivation, by symmetry, of equation (*b*) from the equation which precedes it

$$\Sigma x_{r,u}x_{s,v}a_{r,s}[e_ue_v] = 0$$

is very interesting on account of its simplicity, but suggests logical doubts. Mr. Collins points out that the terms of

$$\Sigma x_{r,u}x_{s,v}a_{r,s}[e_ue_v]$$

which have the same coefficients (products of x's) are of two types, viz.,

$$x_{r,u}x_{s,v}(a_{r,s}[e_ue_v] + a_{s,r}[e_vc_u]), \qquad x_{r,u}x_{r,u}a_{r,r}[e_ue_u],$$

whose combination, in view of

$$\Sigma x_{r,u}x_{s,v}a_{r,s}[e_ue_v] = 0,$$

gives (b) at once. Thus the direct derivation gives the same result as symmetry.

The paper of Professor Stokes, who occupies the chair of logic in Queens College, Cork, compared the fundamental equations $x^2 = x$ and $x\ (1 - x) = 0$ of Boole's calculus with the fundamental equations $l_r^2 = 0$ and $l_r l_s = - l_s l_r$ of Grassmann's calculus of extension, and stated :—"Whereas the first set of equations belong to an algebra of self-identical unrelated units, the second set belong to an algebra in which relations, synthesis, reference beyond self, is essential."

In a paper on the "Imaginary of logic" before the British Association in 1898 the author put forward the view that as the square root of a positive quantity is + or —, the square root of a negative quantity may be expected to be + and —, in view of the logical relation between *and* and *or* pointed out originally by DeMorgan, and subsequently by Schroeder. He considers that this view receives support from the paper of Kempe on the theory of mathematical form. He concludes as follows:—"the fertile proportions of mathematics from which its wealth of content and the treasures of mathematical knowledge are drawn, are not synthetical in the sense in which Kant and the empiricists alike maintain them to be, viz : that the truths preëxist and are then seen to be synthetical, the synthetical character being as it were something subsequent to the content of the proportions and attached to it as it were adjectively ; but in this sense that those proportions are themselves the product of pure synthesis, that the very possibility of advance from entity to entity, unit to unit, or relation to correlate, determines all those laws which mathematics is employed in exploring and tracing into all their consequences, and which are infinitely more fruitful than the analytical laws of formal logic or the calculus of classes and statements. Pure synthesis generally is that 'necessity of matter' of which Hamilton spoke; the principle of material consequences which characterizes every genuine department of mathematics and defies further logical analysis."

The conclusion reached in the paper by Mr. Davis is that every celestial body wholly gaseous, large or small, composed of a mixture in which the proportion of heavier gases is considerable, necessarily generates an outrushing atmosphere, and that this atmosphere ceases to exist only upon the exhaustion of the lighter gases or the liquefaction of the central core. The atmosphere is exceedingly tenuous.

In case of the sun the yearly shrinkage of the nucleus is known to be only a few hundred feet. The consequent extrusion of the lighter gases annually adds to the atmosphere a few thousand feet, and this in the course of the year is spread throughout the solar system.

Dr. Kimura's paper is a development of many formulæ for the application of the nonion operator φ and its differential nabla ${}_{\phi}\nabla$ to mathematical physics. Expressions were given for the moduli of the characteristic equations of powers of φ; expressions for powers of φ in terms of any three other powers of φ; expressions for the reduction of products such as $\varphi'^p\varphi^q$ or $\varphi^q\varphi'^p$, where φ' is the *conjugate* of φ; expressions involving the self-conjugate part of φ, and the rotation-vector of φ; and finally a considerable study of the operator

$${}_{\phi}\nabla = \begin{vmatrix} \frac{\partial}{\partial a_1} & \frac{\partial}{\partial b_1} & \frac{\partial}{\partial c_1} \\ \frac{\partial}{\partial a_2} & \frac{\partial}{\partial b_2} & \frac{\partial}{\partial c_2} \\ \frac{\partial}{\partial a_3} & \frac{\partial}{\partial b_3} & \frac{\partial}{\partial c_3} \end{vmatrix} \varphi, \quad \text{where } \varphi = \begin{pmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{pmatrix}$$

It is proper to add that the formulæ in the main are applicable only to linear vector operators which have: (*a*) distinct roots. (*b*) two equal roots, but reducible to the form

$$\varphi = -\frac{g_1\rho_1 s\rho_2\rho_3() - g_1\rho_2 s\rho_3\rho_1() - g_2\rho_3 s\rho_1\rho_2()}{s_1\rho_1\rho_2\rho_3}.$$

(*c*) three equal roots, but reducible to the form $\varphi = g$.

G. A. Miller.

Cornell University.

THE THEORY OF FUNCTIONS.

Introduction to the Theory of Analytic Functions. By J. HARKNESS and F. MORLEY. London, Macmillan and Co., 1898. 8vo, xv + 336 pp.

THE authors, already well known through their Treatise on the theory of functions, have laid the mathematical public under a new obligation by their "Introduction to the theory of analytic functions." The name which the authors have chosen for their new book gives only an inadequate idea of its scope and object; on the one hand it is not an introduction in the sense of a *first* introduction to the theory of functions; on the other hand it is much more than an introduction; it may be shortly described as a very complete *treatise on Weierstrass's general theory of functions* with applications to elliptic and algebraic functions (Chapters VI., XIX., XXI.), preceded by an introduction devoted to the number concept and the geometric interpretation of complex quantities (Chapters I.–V.), and followed by a short account of some of the leading ideas of Riemann and Cauchy (Chapters XX., XXII.).

I. *The Introductory and Concluding Chapters.*

The book begins with a sketch of the system of real numbers considered as *ordinal numbers* (Chapter I.). A row of objects is considered with regard to their order, say from left to right. To count the objects means to label them, not primarily to say how many there are; and the integers are, primarily, mere labels. The object after which we begin to count, is labelled 0; if there are also objects to the left of it, they are labelled -1, -2, etc. Fractions are introduced by a process of relabelling: Pick out of the original row the objects p, $2p$, $3p \cdots$, and relabel them $1, 2, 3, \cdots$; if μ is not divisible by p, the object originally labelled μ is relabelled $\frac{\mu}{p}$. At the same time rules are given to decide whether two rational numbers are equal, and if not, which is the greater. A different kind of relabelling, viz., a change of origin, is used to define addition without reference to quantity. Also special irrationals like $\sqrt{2}$ can be introduced by a similar process of relabelling; but for the general definition of irrationals Dedekind's idea of a cut in the totality of rational numbers is adopted and it is shown how every decimal, finite or infinite, defines a cut.

Chapter II. gives the *geometric representation of complex numbers* and of the elementary algebraic operations with them, with numerous applications to geometrical problems. Along with the usual representation of a complex number by a point or by a stroke, the interpretation as an operator is emphasized, and, as in quaternions, the consideration of the quotient precedes that of the product.

Chapter III. contains a detailed study of the *bilinear transformation* interpreted as a one-to-one correspondence between two planes. The starting point is the geometric interpretation of the two equations

$$am\left(\frac{x-x_1}{x-x_2}\right)=\alpha \text{ and } \left|\frac{x-x_1}{x-x_2}\right|=\rho\,;$$

the former represents an arc of a circle through x_1, x_2, the latter a complete circle "about x_1, x_2," that is a circle with respect to which x_1, x_2 are inverse points; the two circles are orthogonal.

By this means a very elegant treatment of the general bilinear transformation is obtained; and at the same time a simple geometric interpretation of two harmonic pairs.

Further the anharmonic ratio of four points and the isogonality of the transformation are considered and finally the subject is connected with the theory of absolute inversion in space. An interesting—apparently new—theorem is given, viz., that *every bilinear transformation between two planes is equivalent to two inversions in space.*

In Chapter IV. "*Geometric theory of the logarithm and exponential*" the authors intend to give an elementary geometric treatment of log ξ, analogous to the way in which sin ξ is treated in trigonometry. For this purpose, the existence of the equiangular spiral is assumed, "which may be constructed by placing a right cone with axis vertical and attaching a thread AP to a point P of it. The point P is held in the horizontal plane through the vertex O; the thread is then wound round the cone, without being allowed to slip. The curve described by the point P in its plane is the spiral in question, and it can be proved in an elementary way that it cuts all rays from O at a constant angle. Choose this angle to be $\pi/4$; and then measuring the angle θ from the point where $OP=\rho$ has the value 1, define

$$\theta=\log\rho."$$

It follows then easily that

$$D \log \rho = \frac{1}{\rho}$$

$$\log \frac{\rho_1}{\rho} = \log \rho_1 - \log \rho.$$

The logarithm of a complex quantity x_1 is defined by

$$\int_1^{x_1} \frac{dx}{x},$$

and the mapping by means of $\log x$ is discussed with application to Mercator's projection.

The exponential is defined as the inverse of the logarithm. The mapping by means of $x = \exp y$ leads in a natural way to the "Napierian motion" defined by $x_t = x_0 \exp bt$, which plays an important part in the following Chapter V., in which the *bilinear transformation* is taken up again and interpreted as a *transformation of a plane into itself*, produced by a motion of all the points of the plane.

The same Chapter V. deals also with periodic transformations, in particular those of period 2 and 3. I call special attention to the elegant construction of the double points of an involution which is given by two of its pairs.

These are some of the principal features of the introductory chapters. The presentation is elegant, suggestive, and original, excellently adapted for an intelligent student who has already gone through a first introductory course in the theory of functions, such as given, for instance, in Durège, or Burkhardt, or in the function theoretical chapters of Chrystal.

But these chapters would be open to serious objections if they should be intended—as some passages in the Preface seem to indicate—for a beginner, that is a student who is introduced for the first time to the theory of functions. To such a student the many new ideas and new methods of reasoning of the theory of functions, together with the mystery still surrounding the conception of imaginaries, with which he has made only an occasional and mostly very unsatisfactory acquaintance in algebra and calculus, all these things combined usually offer considerable difficulties, and a book intended for the use of beginners ought to make the entrance as easy as possible.

A beginner will hardly be benefited by the delicate and half-philosophical developments of Chapter I. on ordinal numbers; the more so, as they do not give a complete account of the matter. In Chapter II. a more systematic, even dry and school-like treatment would better meet his needs, and, for a beginner, the elegant treatment of the bilinear transformation in Chapters III. and V. should be preceded by a more elementary chapter preparing by a study of the simplest special cases: $x + b$, ax, $1/x$ for the most general case.

In Chapter IV. the highly original construction of log ξ by means of the cone and the thread will puzzle most readers, unless a further explanation be added of what is meant by "winding the thread round the cone without allowing it to slip." But there is a more serious objection to this chapter: Its proofs have only a preliminary character and lay no claim to rigor; and yet some of its results are presupposed in Chapter XIII., where a rigorous analytical theory of the exponential and logarithm is given; so in particular in § 96 where the tacit assumption is made that the function exp x defined in § 95 is identical with the function exp x defined in § 32.

We now turn to the *concluding chapters:*

Chapter XX. illustrates the construction of *Riemann surfaces* for algebraic functions by a number of simple examples, which furnish at the same time an opportunity for the distinction between simply and multiply connected surfaces. One paragraph gives an elementary explanation of the fundamental concepts of the theory of groups of linear transformations.

The final Chapter XXII. is devoted to *Cauchy's definition of a monogenic function.* The necessary and sufficient conditions for the existence of a derivative of a function $u + iv$ of $x + iy$ are given, one of the necessary conditions (which had been overlooked until very recently) being that u and v must have total differentials. A special paragraph gives an excellent critical discussion of the difficulties underlying Cauchy's definition. One of these is of particular interest: the question is raised whether the continuity of the derivative is a necessary consequence of its existence, a question which has in the meantime been settled by Goursat's discovery that Cauchy's theorem can be proved without presupposing the continuity of the derivative.

In connection with the conditions for monogeneity, the potential is introduced, the equipotential problem is explained and solved for some simple cases, and the solution applied to the problem of the conform representation of a straight line upon a polygon. Finally Green's theorem is given and applied to Cauchy's theorem.

The authors say in the preface: "We had to make a choice between the methods of Cauchy and those of Weierstrass." I must confess that I cannot see the necessity of such exclusiveness. On the contrary, I believe the authors could materially increase the usefulness of their book by working over, in a new edition, the Chapters II.–V., XX., XXII., into an introductory part, adapted in the first place to the needs of the beginner, giving an elementary exposition of Cauchy's theory with emphasis upon conform representation, accepting to a certain extent intuitional proofs. Such an introduction would be the best preparation for Weierstrass's theory; it would culminate in Cauchy's theorem on Taylor's series, and here the point of junction with Weierstrass would be gained. A critical chapter showing the hidden difficulties of Cauchy's theory and the insufficiency of geometrical proofs would be an excellent link and would prepare for the standpoint of absolute rigor to be adhered to afterwards. For a student will only properly appreciate the stern beauty of Weierstrass's theory, if it has come to him as a salvation out of the doubts and difficulties of the geometrical methods.

II. *Weierstrass's Theory.*

The rest of the book gives a systematic exposition of Weierstrass's theory of functions, excellent in its general arrangement, clear and rigorous in the details, with the exception of some minor flaws,* too few in number and too insignificant to mar the general impression of excellency. We make the following subdivisions:

A. The *Preparatory Chapters* (VI.–IX.).

B. The *General Theory of Analytic Functions* (X.–XII., XIV., XVI.).

---

* For the benefit of the readers of the "Introduction" I call attention to the following corrigenda and omissions:

(1) P. 91, the quantity γ must be assumed to be some value *actually taken by* $|f(x)|$.

(2) P. 173, the condition for the edges should be

$$\xi = +\frac{\pi}{2},\ \eta \leqq 0;\ \ \xi = -\frac{\pi}{2},\ \eta \geqq 0.$$

(3) P. 179, replace γ by "the maximum value of $|f(x)|$ in the closed region $(B' + D - \delta)$."

(4) P. 253, line 4, $\frac{w}{2}$ must not be a period.

(5) P. 270, line 6, $4\omega_\mu$, $4\omega_\nu$ are not primitive.

(6) P. 295, line 14, a_n is an integral function of $a_1, a_2, \cdots, a_{n-1}$ and of the $c\alpha$'s *with real positive coefficients.*

C. Applications to *Special Classes of Functions* (XIII., XV., XVII.–XIX., XXI.).

A. The Preparatory Chapters.

They lay the foundation of the theory by an account of the general theory of limits and infinite series and a chapter on rational functions.

In Chapter VI. the principal theorems concerning *limits* of infinite sequences and *continuous functions* of one or two real variables are given on the basis of Dedekind's definition of irrationals; the theorem on uniform continuity leads to a discussion of the general problem of uniform convergence to a limit.

The following chapter (VII.) develops the fundamental properties of *rational functions of a complex variable* and gives the definitions of "limit, continuity, derivative, zero, pole," for the special case of these functions, but in such a manner that they can at once be generalized.

Chapter VIII. treats of the *convergence of infinite series with complex terms.* Simple tests of convergence are given and the question of association of terms, the distinction between absolutely and conditionally convergent series, and the conversion of a single series into a double series and *vice versa,* are carefully discussed.

Next follows (Chapter IX.) the subject of uniform convergence of real series, its connection with continuity and an application to real power series, including their behaviour in the frontier points.

Two desiderata must be mentioned with respect to these preparatory chapters: (1) An exhaustive theory of irrationals should be given, and (2) a scientific theory of complex numbers from the standpoint of linear associative algebra. They are two of the cornerstones of Weierstrass's theory and should not be omitted in an account of it otherwise so detailed and complete.

B. The General Theory of Analytic Functions.

The reader is now thoroughly prepared for the general theory of analytic functions which is developed in the following chapters along the line of Weierstrass's lectures, still with many minor deviations.

Chapter X. treats of the fundamental properties of *power-series,* their domain of convergence, the uniformity of their convergence and the criteria of identity of two power series.

The next Chapter XI. on *operations with power series* is based upon Weierstrass's theorem on series of power series. From this theorem (for which in many cases Cauchy's theorem on double sums may be substituted) follow easily the rules for the multiplication and division of power series, Taylor's theorem for power series along with the rules for differentiation of power series and series of power series.

The *continuation of a power series* (Chapter XII.) is only another aspect of Taylor's theorem, and thus we have reached Weierstrass's definition of an analytic function, viz. a power series together with the totality of its continuations, direct or indirect. Two theorems are necessary to complete the definition : (1) A power series $P(x - x_0)$ and any one of its direct continuations $P(x - x_0 | x_1)$ coincide throughout their common domain of convergence, (2) The series $P(x - x_0)$ is always a direct or indirect continuation of $P(x - x_0 | x_1)$. From the latter theorem follows that, no matter what element of an analytic function we take as starting point, we always obtain the same totality of power series.

Instead of this second theorem, Harkness and Morley use for the same purpose what they call "*standard chains*" for the continuation of a power series from a point a to a point b, that is, a series of points

$$a, x_1, x_2, \cdots, x_n, b$$

such that each point is in the domain of the *following* as well as the preceding point. The same series of points may then be used for the continuation from b to a. But it should be stated and proved that if $P(x - b)$ is a direct or indirect continuation of $P(x - a)$ reached by a non-reversible chain, the same continuation can also be reached by a reversible ("standard") chain.

Frequently it is convenient to think of the *continuation* of a power series as taking place *along a continuous path* from a to b. The path must be such that it is possible to choose on it a series of points $x_1, x_2, \cdots, x_n$ between a and b such that each point x_{i+1} together with the arc $x_i x_{i+1}$ of the curve is contained in the domain of the point x_i, and it can be proved* that the value reached in b depends only on the path and not on the choice of the intermediate points. Thus a one valued function is defined along the path.

The authors use such paths of continuation later on to prove the theorem that an analytic function is monodromic in a simply connected region which contains no singular point.

---

* See C. Jordan, Cours d' analyse, 2d ed., I., no. 343.

But these geometrical considerations are rather out of harmony with their strictly Weierstrassian surroundings; to make them rigorous it would be necessary to give purely arithmetical definitions of a "continuous path," "circuit," "simply connected region."

The general theory is completed in Chapter XIV., which is devoted to the *singular points*. The existence of at least one singular point on the circumference of the circle of convergence of a power series is proved and the characteristic difference between essential and non-essential singular points of one valued functions discussed and it is shown how certain classes of one valued functions are characterized by the nature of their singularities: the rational integral, the rational fractional, the transcendental integral, and the transcendental fractional functions.

It is to be regretted that the authors have helped further to increase the lamentable confusion which already exists in our *terminology* by the introduction of a new term for the concept which occurs most frequently in the theory of functions, viz., the aggregate of properties of a function, necessary and sufficient for its expansion into a power series. It seemed for a while that the term *holomorphic*, introduced by Briot and Bouquet, had gained general recognition; it was adopted by C. Jordan (1st ed.), by Forsyth, and by Harkness and Morley (Treatise). And now the latest three text-books on the theory of functions abandon it again: Jordan (2d ed.) returns to Cauchy's original "*synectic*," Burkhardt uses the term "*regular*," occasionally used by Weierstrass, and Harkness and Morley introduce the term "*analytic about a point*," which moreover has the disadvantage that the word analytic is already in use—aside from its general meaning—in a different technical sense in Weierstrass's theory.

Chapter XVI. represents a new departure from the ordinary lines of Weierstrass's theory; it develops *Cauchy's theorems on definite integrals on the basis of Weierstrass's concept of an analytic function.* For this purpose it is first shown* that the definite integral—considered as a limit of a sum—

---

* A difficulty of uniform convergence is overlooked in the proof: It is tacitly assumed that for every ε a δ exists such that

$$\left| \frac{(x_\lambda + h_\lambda)^{n+1} - x_\lambda^{n+1}}{h_\lambda} - (n+1)x_\lambda^n \right| < \varepsilon$$

as soon as all $|h_\lambda| < \delta$, *no matter where x_λ is taken on the path*; that is, uniform differentiability of x^n along the path is assumed. It could easily be proved that this condition is always satisfied.

of the function x^n, taken from x_0 to x_1 along any path L of finite length, has the value

$$\frac{x_1^{n+1} - x_0^{n+1}}{n+1}.$$

Next it is proved that for the integration of a power series $P(x) = \Sigma a_n x^n$ integration by terms is allowed, provided the path lies wholly within the circle of convergence. For such a path we have therefore the result

$$\int_{x_0}^{x_1} (\Sigma a_n x^n)\, dx = \Sigma a_n \frac{x_1^{n+1} - x_0^{n+1}}{n+1} = \overline{P}(x_1) - \overline{P}(x_0),$$

if we denote by $\overline{P}(x)$ the indefinite integral of the series $P(x)$. But this difference is Weierstrass's definition of a definite integral for a power series, and thus the two definitions are reconciled.

To extend the result to the case where the path of integration is not wholly contained in the domain of one element, but still a "path of continuation" for the analytic function under consideration, interpolate on the path a finite number of points chosen, as explained above, and apply the previous theorem to each pair of successive points. The element $P(x)$ continued along L furnishes, along L, a one valued function $f(x)$; the indefinite integral $\overline{P}(x)$ can be continued along the same path and furnishes another one valued function $F(x)$, and all along L: $\dfrac{dF(x)}{dx} = f(x)$. The final result is then

$$\int_L f(x)\, dx = F(x_1) - F(x_0).$$

From this point of view, Cauchy's theorem dwindles down almost to a triviality: it is self-evident if the region around which we integrate is contained in the domain of an element of the integrand, and in the general case it is an immediate consequence of the theorem of Article 107. But we must remember that Cauchy's original theorem states much more than in the present connection, since it does not presuppose the expansibility of the function into a power series.

Cauchy's integral and the theorems on residues follow now easily, and a collection of interesting applications to the computation of real definite integrals concludes the chapter.

C. Special Classes of Functions.

The principles of the preceding general theory are illustrated by a series of applications to special classes of functions, which occupy the remaining chapters of the book.

Chapter XIII. gives an analytic theory of the *Exponential and Logarithm.* The exponential is defined by the series, and the logarithm as its inverse. Special attention is paid to the "chief values" of the logarithm and the allied many valued functions; they are distinguished by capital initials from the general values, and defined as follows:

$$-\pi < Am\, x \leqq \pi$$

$$\operatorname{Log} x = \operatorname{Log} \ |x| \ + iAmx.$$

The trigonometric functions $\cos x$, $\sin x$ are defined by the series and their properties are derived from those of the exponential. The corresponding conform representation problems are studied, and utilized for the investigation of the chief values of the inverse function.

In Chapter XV., after a succinct account of the theory of infinite products based upon the theorem that

$$\lim_{n = \infty} \exp x_1 \cdot \exp x_2 \cdots \exp x_n = \exp \ (\lim_{n = \infty} \ (x_1 + x_2 + x_n)),$$

Weierstrass's *construction of the most general transcendental integral function with given zeros* is developed, with applications to the trigonometric functions and to the Γ-functions.

In Chapter XVII. *Laurent's theorem* is proved by means of Cauchy's integral, and applied to a discussion of the isolated singularities of one valued functions, to the expansion of a periodic function into a Fourier series, and to a succinct and elegant treatment of the *ϑ-functions:* If $g(x)$ denotes the infinite product

$$g(x) = (1 - x)(1 - rx)(1 - r^2x) \cdots, \qquad (\ |r| < 1),$$

then

$$f(x) = g(x)\, g\left(\frac{r}{x}\right)$$

is a one valued analytic function with the zeros

$$\cdots, \frac{1}{r^2}, \frac{1}{r}, 1, r, r^2, \cdots$$

and the only singular points $x = 0$ and $x = \infty$. It can be expanded into a Laurent series, and the substitution $r = q^2$, $-x = q \exp 2\pi iv$ transforms $f(x)$ into the fundamental ϑ-function, and the principal properties of the ϑ-functions flow easily from the double representation of $f(x)$ by an infinite product and by an infinite series.

The following two chapters (XVIII., XIX.) develop *the theory of elliptic functions*, on the whole along the lines of Weierstrass's lectures but with many deviations and simplifications in the detail arrangements. Thus instead of Weierstrass's half-period $\omega_2 = \omega_1 + \omega_3$, its negative is used so that

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

This change which, it seems, has been simultaneously and independently introduced by Harkness and Morley,* by Study, and by C. Jordan, is an important improvement, since it leads to a complete symmetry with respect to a cyclic permutation of the indices 1, 2, 3.

Considerable simplifications are obtained by an extensive use of integration round a period parallelogram ; I mention in particular the proof of the theorem that for every elliptic function $f(u)$ the sum of the zeros is congruent to the sum of the poles, by means of the integral

$$\int \frac{uf'(u)}{f(u)}\, du$$

taken round the period parallelogram.

In Chapter XXI. a sketch of the elementary properties of algebraic functions from the standpoint of Weierstrass is given. It is proved that an irreducible algebraic equation $F(x, y) = 0$ defines a single analytic function y of x; its singular points are determined, the Puiseux expansions in the vicinity of ordinary branch points are derived and the problems of multiple points of the curve $F = 0$ is solved for the simplest case of ordinary double points.——

The presentation is throughout clear, and numerous and well selected examples and exercises illustrate the general theorems.

The authors have adopted, for their exposition of the theory of functions, Weierstrass's standard of absolute arithmetical rigor. They insist that the intuitional method is not in itself sufficient for the superstructure of the theory ; and systematically exclude it from all *proofs*. At the same time a happy conciseness of expression prevents the clumsiness which so frequently accompanies rigorous deductions.

---

* Treatise (1893), p. 301; Study, Sphaerische Trigonometrie and Elliptische Functionen (1893), p. 190; C. Jordan, Cours d'Analyse (1894), II., p. 337.

A wise moderation has been exercised in the selection of topics, and the arrangement of the rich material is pedagogical as well as logical. The concept of the analytic function takes the central place, everything else is subservient to it, either by preparing for it or by illustrating it. By this means, a harmony and equilibrium between the different parts is attained, which impart to the whole in a remarkable degree the character of an organic unity.

Thus the authors have succeeded in producing not only a work of high scientific and pedagogical value but at the same time of a singular beauty and elegance. But there are numerous beauties of detail as well, for which, however, the reader must be referred to the book itself. A certain freshness and originality pervade the whole, even in places where the authors follow along beaten tracks, and give at every turn evidence of the complete mastery of the subject with which the book is written.

OSKAR BOLZA.

UNIVERSITY OF CHICAGO,
June 30, 1899.

McAULAY'S OCTONIONS.

Octonions. By A. McAULAY, M.A. Cambridge, The University Press, 1898. 8vo, 253 pp.

THIS treatise is a development of Clifford's biquaternions with applications to ordinary space. The starting point of the analysis is quaternions, combined later with methods from Grassman's Ausdehnungslehre.

The development is open to criticism, as a work for beginners, because of its extremely refined formal character. This is perhaps unavoidable because the book was first compiled as a scientific paper, and, as the author says, he did not feel justified in recasting the whole appropriately. Discussions of formal laws and expert reasoning upon terms that are imperfectly defined, relying upon subsequent developments to bring out their full meaning, are not conducive to clearness of apprehension on the part of a learner. Perhaps as good a review, therefore, as can be made of the book, which is indeed an extensive and thorough development of what must prove to be a valuable analysis, is to give a brief and clear exposition of the octonion system.

The octonion is a quantity defined by four numbers called its *tensor*, *scalar*, *convert*, and *pitch*, and an axis having a fixed position in space. This makes in all eight numbers, upon which the octonion depends, and hence its name. Let q, r be two quaternions, and O a given point. Then q, r, O determine the octonion as follows: the four numbers, as named above, are Tq, Sq, Sr, and $S{\cdot}r/q$; the axis is a line parallel to Vq, whose vector distance from O is $V{\cdot}Vr/Vq$. McAulay expresses this octonion by $Q = q + r\Omega$, where Ω is a unit of separation; but it is better, because different points may be employed in the definition of the same octonion, to exhibit the point of reference, say $Q = qO + rO\Omega$; and we shall presently see that we may write $O\Omega = \Omega$.

If instead of O, we employ another point P, at a vector distance ρ from O, certain changes must be made in r to give the same octonion; q evidently will remain unchanged since the numbers Tq, Sq, and the direction of the axis (parallel to Vq), completely determine q. Since Sr remains unchanged, r can change only by a vector σ, and since $S{\cdot}r/q$ remains unchanged, therefore $S{\cdot}\sigma/q = 0$, or σ is a vector perpendicular to Vq. Finally, $\sigma/Vq =$ increase of perpendicular distance of the axis $=$ component of PO perpendicular to $Vq = -V\rho Vq{\cdot}/Vq$, and hence $\sigma = -V\rho Vq$. Thus,

$$Q = qO + rO\Omega = qP + (r - V\rho Vq)P\Omega.$$

When $Vq = 0$, the axis is either at infinity ($Vr \neq 0$) or indeterminate ($Vr = 0$); in either case the quaternion components of Q are independent of the reference point, so that it may be omitted. In particular, $rO\Omega = rP\Omega = r\Omega$ say; so that in effect $O\Omega = P\Omega = \Omega$.

Addition is defined by adding coefficients of corresponding units, *viz.*, if

$$Q = qO + r\Omega = qP + (r - V\rho Vq)\Omega,\ Q' = q'O + r'\Omega = \text{etc.},$$

$$\text{then,}\quad Q + Q' = (q + q')O + (r + r')\Omega = (q + q')P \\ + [r + r' - V\rho V(q + q')]\Omega.$$

Here we add Q and Q' as to the reference point O, and also, as to the reference point P, and find the two sums to be equal. Addition is thus seen to be independent of the reference point O or P, and this means, of course, that the sum is determined wholly by the numbers and axis that define each term of the sum. Addition is also, from quaternion laws, associative and commutative.

Multiplication is defined by making it distributive and the units O, Ω commutative with everything, with the table $O^2 = O$, $\Omega^2 = 0$, and $O\Omega = \Omega O = \Omega$ as found above. It follows that multiplication is also independent of the reference point, viz.:

$$QQ' = qq'O + (qr' + rq')\Omega = qq'P + (qr' + rq' - V\rho Vqq')\Omega,$$

$$= qq'P + (qr' + rq' - qV\rho Vq' - V\rho Vq \cdot q')\Omega,$$

since $$V \cdot \rho Vqq' = q\, V\rho Vq' + V\rho Vq \cdot q'.$$

Multiplication is also associative, but not in general commutative. An octonion scalar $x + y\Omega$ is commutative with everything; nonaxised octonions are commutative with each other; a nonaxised and an axised octonion are commutative only when the (secondary) direction (Vr) of the first is parallel to the axis of the second; axised octonions are commutative only when their axes coincide.

Any octonion may be resolved into four commutative factors in one and only one way. viz.:

A *tensor* Tq, an *additor* $1 + a\Omega$, a *versor* $Uq \cdot A$, and a *translator* $1 + b\,Vq \cdot \Omega$. The product of these four is, $Q = qA + r\Omega$, where A is a point of the axis, since $Vr \| Vq$. The above factors are denoted by T_1Q, T_2Q, U_1Q, U_2Q, respectively; also

$$TQ = T_1QT_2Q = \textit{augmentor of } Q = \textit{octonion tensor},$$

$$UQ = U_1Q \cdot U_2Q = \textit{twister of } Q = \textit{unit octonion}..$$

In terms of q, r, A, we have

$$TQ = Tq(1 + S \cdot r/q \cdot \Omega),\ UQ = Uq \cdot A(1 + V \cdot r/q \cdot \Omega).$$

The above names are derived from the effects of these factors on a vector octonion or *motor*, whose axis intersects the axis of the multiplier at right angles. Any motor $\alpha A + \beta\Omega$ is characterized by a tensor ($T\alpha$), pitch ($S \cdot \beta/\alpha$), and axis ($V\beta/\alpha$ *from* A); of these Tq affects only $T\alpha$ (the tensor of a product = product of the tensors of the factors); $1 + a\Omega$ affects only the pitch, adding to it a, the pitch of the multiplier (pitch of product = sum of pitches of the factors); $1 + V \cdot r/q \cdot \Omega$ translates the axis of the motor the distance $V \cdot r/q$; $Uq \cdot A$ turns the axis of the motor round the axis of Q through the angle of q. Conversely, any octonion is the ratio of two motors, and in fact, octonion and motor stand in very much the relationship of quaternion and vector.

The symbols S, V, K have application as invariant operations, viz.:

$$SQ = Sq + Sr\Omega,\ VQ = Vq \cdot O + Vr \cdot \Omega,\ KQ = Kq \cdot O + Kr \cdot \Omega.$$

A complex scalar (SQ) is self-conjugate, and its square is a positive scalar (if the first term (Sq) determines the sign).

A complex vector, *i. e.*, a motor, is conjugate to its negative, and its square is a negative scalar. McAulay changes VQ to MQ because V does not connote with motor, but such change is unnecessary, and is not given weight in the equally important cases of T for augmentor, U for twister, s for convert, t for pitch, etc.

We have $Q = SQ + VQ$, $KQ = SQ - VQ$, $QKQ = KQ \cdot Q = (SQ)^2 - (VQ)^2 = (TQ)^2$ $K \cdot QQ' = KQ' \cdot KQ$, etc. If $QQ' = 0$, then either $Q = 0$ or $Q' = 0$ or $Q = r\Omega$, $Q' = r'\Omega$.

All results not depending upon divisions by a nil octonion $r\Omega$, are therefore identical in form with quaternion results, but with wider geometrical and physical meanings. For example, a motor $\alpha O + \beta\Omega = \alpha P + \beta'\Omega$, may be represented by the velocity system of a rigid body in which α is the vector angular velocity and β, β' the linear velocities at O, P. The axis is the instantaneous axis of rotation. This is what Ball calls a twist about a screw; the pitch of the motor is the pitch of the screw, which is right or left handed according as the pitch is positive or negative. If A be a point of the axis the motor takes the form $\alpha A + c\alpha\Omega$ where c is the pitch and $c\alpha$ the translation vector. The motor is also represented by a system of forces acting on a rigid body, which reduces to what Ball calls a wrench about a screw; α is the resultant force, and β, β' the moments of the system about O, P. It is also represented by a system of impulses, whose momentum is α, and moments of momentum about O, P are β, β'.

It will now be seen in what respect this analysis can prove valuable, and McAulay has not only developed this analysis very thoroughly in respect to linear functions of octonions, differentiations, nabla, and McAulayan differentiators, but he has applied it to develop and extend the results of Ball's theory of screws. It would seem that the "A process" which was presented by the reviewer before the Indiana academy of science in 1897, would prove a valuable addition to this theory. It is also possible that like quaternions (BULLETIN, November, 1897) the system may be extended to fourfold space with complete representation of the octonion.

ARTHUR S. HATHAWAY.

ROSE POLYTECHNIC INSTITUTE.

THESES IN MATHEMATICS AT THE UNIVERSITY OF PARIS.

1. *Sur quelques points de la théorie des fonctions,* par M. L. Desaint. Paris, Gauthier-Villars, 1897. 4to, 74 pp.
2. *Sur une classe particulière de groupes hyperabéliens,* par M. Henry Bourget. Ibid., 1898. 4to, 90 pp.
3. *Sur l'intégration des équations de la chaleur,* par M. Édouard LeRoy. Ibid., 1898. 4to, 258 pp.
4. *Les équations différentielles linéaires et la théorie des groupes,* par M. F. Marotte. Ibid., 1898. 4to, 92 pp.
5. *Essai sur une théorie générale de l'intégration et sur la classification des transcendantes,* par M. Jules Drach. Ibid., 1898. 4to, 144 pp.*

These five theses were presented to the faculty of sciences of the University of Paris for the degree of doctor in mathematical science; they were sustained in order of time as placed above during the academic year from November, 1897, to June, 1898, inclusive.

Before proceeding to a brief review of each it may be interesting to observe that the creation of the faculties of mathematical and physical science in France dates from March 17, 1808.† The first degree of this kind conferred by the faculty of sciences at Paris was given to Pierre-Louise-Marie Bourdon in 1811; his two theses maintained respectively on the 19th and 23d of March of that year were 1° the motion of a solid body solicited by any accelerative forces and turning about a fixed point, and 2° the theory of the elliptic movement of the planets according to the principle of universal gravitation and its application to the determination of the masses of some planets.

From 1811 to 1890 inclusive there were conferred one hundred and eighty-four such degrees at Paris, and forty-four by the other universities of France.‡ The average age of the candidates is thirty years and a fraction; the majority lie in the range from twenty-three to thirty-four years, the average for each of these years being twelve; there were thirteen at twenty-three, eighteen (the maximum) at twenty-eight, and ten at thirty-four ; the ages range from seventeen years to fifty-five ; Joseph Bertrand was the youngest candidate ; the next age is twenty, and with but four exceptions, every age from twenty to fifty-five is represented ; five candidates attained the degree at twenty-one and it was conferred on five who had passed the fifty-year mark ; the average age of the first ten of this period (1810–1890) was twenty-six years, that of the last ten is also twenty-six, the fraction being slightly larger for the former.

---

* The memoirs of Desaint, LeRoy, and Drach are also published in the *Annales de l'Ec. norm. sup.*, for the respective years 1897, 1897–8, 1898; those of Bourget and Marotte appeared in the *Annales de la Faculté des Sciences de Toulouse* for 1898.

† See Maire's Catalogue des thèses de sciences soutenues in France de 1810 à 1890 inclusivement, Paris, H. Welter, 1892.

‡ For the same period the respective numbers in the physical sciences were 281 and 61; in the natural sciences, 236 and 67.

1. In the first part of his memoir Desaint studies the distribution of the zeros of a function. This problem is one way of approaching the general problem of the resolution of equations ; he does not propose to solve the very difficult problem of determining their roots point by point, but to limit the regions of the plane of the complex variable within which a function can vanish. The case of a polynomial with real roots is one of the rare examples in which the zeros of a function can be found point by point ; if certain roots of a polynomial are commensurable these are obtained immediately ; as to incommensurable roots the author proceeds thus : the value of the root sought is placed between two commensurable numbers, one, a, greater, and the other, b, less ; consider the segment ab of the axis of reals and to each zero make correspond an analogous segment ; the ensemble E of these segments represents the precision with which the equation has been resolved ; from the author's point of view E limits the region of the plane where the roots of the polynomial are found.

The first chapter presents the geometric method of which Desaint makes use ; it rests upon the following simple principle : consider an ensemble of segments F' starting from a point z_1; if all these segments are situated below a straight line D, their resultant is essentially different from zero and situated below D. Applying this to rational fractions the author is led to a fundamental theorem which he applies to the roots of a polynomial, to algebraic functions, and to the zeros of uniform functions with polar discontinuities. Certain propositions found complete the investigations of Lucas, Berloty, and Cesàro.

Desaint is occupied in the second chapter with definite integrals studied by Hermite. Theorems on uniform functions are arrived at relative to the distribution of the values of the variable which make these functions assume a given value u. A theorem is presented which is applied to elliptic and hyperelliptic integrals, and especially to hypergeometric integrals.

In the second part of the memoir the theorem of Cauchy is used to turn the preceding theorems to a solution of the problem of the distribution of the values of the variable which assign a given value to a uniform function. The problem of finding the value of a function at any point when its values on a contour are given has received a great deal of attention; the author proposes the inverse problem of finding the values of the variable which assign the value u to a function, when the function is given by its values along a contour. He calls particular attention to this proposition: Let $F(z)$ be a uniform function for which infinity is an ordinary point; trace a circle C enclosing all the discontinuities of $F(z)$ and as near as we wish to the convex contour of a minimal surface enclosing these points; designate by M the maximum modulus of $F(z)$ upon the circle C of radius R; the values of z for which $F(z)$ takes the value u are on the interior of the circle T concentric with C and having as radius $R\sqrt{2}\left(1+\frac{M}{|A-u|}\right)$. From this theorem we deduce with facility a general theorem on the continuity of functions. The second part closes with a study of integral functions and of integrals of differential equations.

M. Desaint's second thesis consisted of general theorems of Galois upon algebraic equations; application to resolution by radicals of equations of first degree and to the construction of regular polygons.

2. M. Bourget, at the suggestion of Professor Picard, studies the particular hyperabelian group pointed out by the latter geometer in his memoir in the first volume of the fourth series of *Liouville's Journal* and previously in the *Comptes Rendus* for 17 March, 1884.

This particular group has a double origin. We can derive it from the transformation of the first order of the abelian functions of the second kind;* or it can be considered as isomorphic to the group of transformations similar arithmetically of the quaternary form

$$u_1^2 - Du_2^2 + u_3u_4.$$

The thesis falls into three parts. In the first of these, after generalities on similar substitutions of quaternary quadratic forms and on the arithmetic study of such forms, Bourget shows how the group which he studies is situated relative to the analogous groups considered by Goursat and Bianchi.

---

* Hermite, "Sur la transformation des fonctions abéliennes," *Comptes rendus*, vol. 40 (1855).

The second part is devoted to the study of the group. Seeking first the explicit form of the substitutions, the discontinuity for the imaginary values $\xi = \xi_1 + i\xi_2$, $\eta = \eta_1 + i\eta_2$ appertaining to the domain δ ($\xi_2 > 0$, $\eta_2 > 0$) is then demonstrated; the substitutions of the group are reduced to five of them, which fundamental substitutions present a remarkable analogy to the fundamental substitutions of the modular group; the author also brings to light an infinite number of subgroups analogous to the congruence subgroups of the modular group.

Bourget inquires in the third part how the ten functions

$$\vartheta(0,0/\tau_{11}, \tau_{12}, \tau_{22})$$

even with zero arguments, are affected by the transformations of the group, and from this study deduces a process which, by the aid of the theta functions, constructs an infinitude of functions which are reproduced by all the substitutions of the group. The moduli of Borchardt and Richelot are invariant under certain subgroups. In conclusion the general properties, according to Picard, of the fonctions of the group are given, and it is shown that, notwithstanding the cusp situated on the limit of the domain δ (whose existence was established in the second part), any three of these functions are connected by an algebraic relation.

M. Bourget's second thesis was the exposition of the method of Gauss for the calculation of secular variations.

3. The object of LeRoy's memoir is the resolution of certain problems of the integral calculus which support the mathematical theory of heat. These problems are referred to certain partial differential equations, of which Laplace's equation is the type. Each of them consists in establishing a principle analogous to that of Dirichlet.

The author proposes 1° to construct an integral $V(x, y, z)$ of the equation

$$\Delta V + aV_x + bV_y + cV_z = f(x, y, z, V) + \varphi(x, y, z),$$

which shall be definite and continuous in every point of a closed domain, on the boundary of which the function sought will be constrained to take values given in advance;

2° to design an integral $V(x, y, z, t)$ of the more general equation which he calls the equation of Fourier

$$\Delta V + a(x, y, z)V_x + b(x, y, z)V_y + c(x, y, z)V_z$$
$$= f(x, y, z, V) + \varphi(x, y, z, t) + \psi(x, y, z)V_t,$$

where $adx + bdy + cdz$ is an exact integral, which integral shall be definite and continuous for every point of a closed domain and for every positive value of the time, reducing for $t = 0$ to a function (x, y, z) given in advance and taking at the surface of the body values assigned *a priori*.

The plan followed in the paper may be briefly sketched thus: A remark made by M. Paraf[*] permits the author to extend the principal propositions relative to Laplace's equation to equations of a general type whatever be the number of variables; then having reduced the method called by Poincaré the méthode du balayage to a canonical form, he deduces by a uniform process the complete resolution of the problems of the theory of heat, at least so far as existence theorems are concerned.

The equations of thermic equilibrium are considered from the point of view of the generalization of Dirichlet's principle; a point of view taken by Picard in several well-known memoirs.[†] The calculations are made in three point variables. The méthode du balayage employed is independent of any preliminary theory of Laplace's equation; for linear equations the author demonstrates the principles of Dirichlet and its generalization at one stroke applicable alike to plane and space; as to non-linear equations a new method is proposed, which succeeds for all such equations as are suggested by the theory of heat.

In the second part the author studies the properties of functions which he calls the fundamental harmonic functions attached to a closed surface; his guide here is a memoir of Poincaré.[‡] The existence theorems demonstrated lead to developments in series by which, after Lamé, the construction of the solution of Dirichlet's problem would be sought. The author is limited here to Laplace's equation.

He arrives finally at the equations of cooling of solid bodies and to Fourier's problem. For the case of a homogeneous body an imitation of the méthode du balayage is employed. The author shows that it is again possible to find, *a posteriori*, solutions under the form of series to which he gives the name of Lamé; the process throws new light on several questions of physics; for example, the problem of vibrating membranes. In conclusion LeRoy considers the possibility of applying to the general equations of a variable régime the process imagined by Picard for the equations of permanent régime.

---

[*] A. Paraf, "Sur le problème de Dirichlet et son extension au cas de l'équation linéaire générale," *Toulouse Annals*, vol. 6 (1892).

[†] Picard, *Acta Math.*, vol. 12; *Liouville's Journal*, 1890, 1896; *Jour. de l'Éc. poly.*, cahier 60.

[‡] Poincaré, *Rend. del circ. mat. di Palermo*, 1894; *Acta Math.*, 1896.

M. LeRoy's second thesis was on the integrals of total differentials and on the double integrals of the first kind in the theory of algebraic surfaces.

4. Marotte's memoir has to do with the question of extending to linear differential equations the ideas introduced by Galois into the theory of algebraic equations. The point of view was discovered by Picard* in the following theorem: To every *n*th order linear differential equation there corresponds an algebraic group of linear transformations in *n* variables, which possesses properties analogous to those of the Galois group of an algebraic equation.

Vessiot† demonstrated completely the double property of this group of rationality and made evident its close relation to the problem of integration. Drach ‡ extended the theory of Galois to partial differential equations of the first order.

The first part of Marotte's paper is occupied with an analytical study of the singularities of linear differential equations having rational coefficients and the classification of the transcendents which integrate them. The author shows that to each singular point *a* of such an equation there is attached a group of linear transformations which plays the same rôle in the study of a singularity as does the group of Galois in the resolution of an algebraic equation, or the group of rationality in the integration of a linear differential equation. This group is called the group of meromorphism, because its differential invariants are expressed by functions of *x* meromorphic at point *a*; the group characterizes the nature of the singular point. For an *n*th order equation there are as many classes of singular points as there are subgroups in the linear group of *n* variables.

After having established the relations which exist between the group of rationality, the group of monodromism and the groups of meromorphism, the author enlarges the field of application of the methods of Galois and shows how they lead to the notion of group of monodromism. The group of rationality gives the position of the integrals with respect to the ensemble of rational functions exactly as the group of monodromism gives their position relative to the ensemble of uniform functions. The preceding results are then applied to the classification of the transcendents which integrate linear differential equations with rational coefficients; the notion species of Riemann is the ultimate classifying element. A method is presented by which it can be recognized when two given equations are of the same species.

---

* *Comptes rendus*, 1883; *Toulouse Annals*, 1887.

† *Ann. de l'Ec. norm.*, 1892.

‡ *Comptes rendus*, 1893, 1895.

The second part of the memoir is devoted to the determination of the group of rationality of a differential equation with rational (or algebraic) coefficients. A new classification of homogeneous linear groups enables the author to refer the determination of the group of rationality to the resolution of the problem of determining whether a linear differential equation with rational coefficients admits of an integral whose logarithmic derivative is rational or algebraic; hence the conclusion that we can always determine the group of rationality of a linear equation of order two, three or four, or refer the determination to the study of an abelian integral. The author thus finds all the possible cases of reduction of a linear equation of the fourth order. He refers the integration problem to its canonic form. The method used extends itself immediately to equations of higher order.

In the concluding chapter the work of von Koch* is applied to the study of the question of finding whether a linear equation with rational coefficients admits of an integral whose logarithmic derivative is meromorphic (normal integral), to which problem Marotte refers the determination of the group of monodromism attached to a singular point of a linear equation.

M. Marotte's second thesis was on the general principles of dynamics.

5. The remarkable thesis of Drach calls for more extended notice than can be given to it in the space at command for this article. A suitable review of the memoir will appear in a subsequent number of the BULLETIN.

E. O. LOVETT.

PRINCETON UNIVERSITY.

NOTES.

AT the annual meeting of the Society, December 28, 1899, President R. S. WOODWARD will deliver a presidential address on "The century's progress in applied mathematics."

THE president (Lord KELVIN), the vice-presidents and the secretaries of the London mathematical society have been renominated to serve in the same capacity on the council for the ensuing year. Professor W. BURNSIDE,

---

* *Acta Math.*, vol. 18 ; *Comptes rendus*, 1893.

Mr. H. M. MACDONALD and Mr. E. T. WHITTAKER, have been nominated to fill the vacancies on the council. At the annual meeting which was held on November 9th, the DeMorgan medal was presented to the sixth medalist, Professor BURNSIDE. The Council have sanctioned the issue by the secretaries of an index to the first thirty volumes of the *Proceedings*, which will be drawn up on the same lines as that of the first fifty volumes of the *Mathematische Annalen.* They have also authorized Mr. R. TUCKER to draw up a list of all the members elected since the foundation of the society in 1865.

THE American physical society held its first regular meeting at Columbia University, October 28th. It is proposed to hold four meetings annually, simultaneously with the AMERICAN MATHEMATICAL SOCIETY. The present officers are Professors H. A. ROWLAND, president; A. A. MICHELSON, vice-president; E. MERRITT, secretary; W. HALLOCK, treasurer.

THE recently organized Astronomical and astrophysical society of America has elected the following officers: president, Professor SIMON NEWCOMB, of Johns Hopkins University; first vice-president, Professor CHARLES A. YOUNG, of Princeton University; second vice-president, Professor GEORGE E. HALE, of Yerkes Observatory; secretary, Professor GEORGE C. COMSTOCK, of Washburn Observatory; treasurer, Professor C. L. DOOLITTLE, of the University of Pennsylvania.

The Festschrift prepared in honor of the seventieth birthday, August 23, 1899, of Professor MORITZ CANTOR, of Heidelberg, the well-known historian of mathematical science, has just appeared as a supplement to the forty-fourth volume of the *Zeitschrift für Mathematik und Physik*, from the press of Teubner, under the editorship of Professors M. CURTZE, of Thorn, and S. GÜNTHER, of Munich. The volume of more than six hundred pages contains a heliogravure portrait of Professor Cantor, a catalogue of his scientific work, and the following thirty-two memoirs devoted to historical subjects:

"Développement des procédés servants à décomposer le quotient en quantièmes," by V. V. BOBYNEN, of Moscow; "Zur Geschichte der prosthaphaeretischen Methode in der Trigonometrie," by A. v. BRAUNMÜHL, of Munich; "Notes on the history of logarithms," by FLORIAN CAJORI, of Colorado Springs; "Der Tractatus Quadrantis des Robertus

Anglicus in deutscher Uebersetzung aus dem Jahre 1477," by M. CURTZE, of Thorn; "Zur Geschichte der Prinzipien der Infinitesimalrechnung. Die Kritiker der 'Théorie des fonctions analytiques' von Lagrange," by S. DICKSTEIN, of Warsaw; "P. W. Wargentin und die sogenannte Halley'sche Methode. Ein Beitrag zur Geschichte der mathematischen Statistik," by G. ENESTRÖM, of Stockholm; "Intorno ad un inedito e sconosciuto Trattato di Mechaniche di Galileo-Galilei nell'Archivo di S. A. il Principe di Thurn-Taxis in Ratisbona," by A. FAVARO, of Padua; "Zur Geschichte der Längenbestimmung zur See," by E. GELCICH, of Triest; "Die Geometrie von Le Clerc und Ozanam, ein interessantes mathematisches Plagiat aus dem Ende des XVII. Jahrhunderts," by J. H. GRAF, of Bern; "Nikolaus von Cusa und seine Beziehungen zur mathematischen und physikalischen Geographie," by S. GÜNTHER, of Munich; "On an allusion in Aristotle to a construction for parallels," by T. S. HEATH, of Cambridge, England; "Byzantinische Analekten," by J. L. HEIBERG, of Copenhagen; "Ueber die Aufgaben einer Geschichte der Physik," by A. HELLER, of Budapest; "Winkelmessungen durch die Hipparchische Dioptra," by F. HULTSCH, of Dresden; "Des Rheticus Canon doctrinæ triangulorum und Vieta's Canon mathematicus," by K. HUNRATH, of Rendsburg; "Il 'Giornale de Letterati d'Italia' di Venezia e la 'Raccotta Calogerà' come fonti per la storia delle matematiche nel secolo XVIII.," by G. LORIA, of Genoa; "Notes sur le caractère géométrique de l'ancienne astronomie," by P. MANSION, of Gand; "Ueber die Encyclopädie der mathematischen Wissenchaften," by W. F. MEYER, of Königsberg; "Zur Terminologie der ältesten mathematischen Schriften in deutscher Sprache," by F. MÜLLER, of Loschwitz; "Die Rechenmethoden auf dem griechischen Abakus," by A. NAGL, of Vienna; "Die Geschichte der exakten Wissenschaften und der Nutzen ihres Studiums," by F. ROSENBERGER, of Frankfort; "Die Unverzagt'schen Linienkoordinaten. Ein Beitrag zur Geschichte der analytischen Geometrie," by F. RUDIO, of Zurich; "Franz Adolph Taurinus. Ein Beitrag zur Vorgeschichte der nichteuklidischen Geometrie," by PAUL STÄCKEL, of Kiel; Johann Scheubel, ein deutscher Algebraiker des XVI. Jahrhunderts, by H. STAIGMÜLLER, of Stuttgart; "Mathematik bei den Juden (1501–1550)," by M. STEINSCHNEIDER, of Berlin; "Bemerkungen zur Geschichte der altgriechischen Mathematik," by A. STURM, of Seitenstetten; "Der Loculus Archimedius oder das Syntemachion des Archimedes. Zum ersten Male nach zwei Manuscripten der Kgl. Bibliothek zu Berlin herausgegeben und übersetzt,'' by H. SUTER, of Zurich; "Les 'Excerpta ex M. SS. R. Des-Cartes,' '' by PAUL TANNERY, of Pantin; "Einige Additionsmaschinen,'' by F. A. UNGER, of Leipzig; "Zur Geschichte der deutschen Algebra,'' by E. WAPPLER, of Zwickau; "Pierre Fermat's Streit mit John Wallis. Ein Beitrag zur Geschichte der Zahlentheorie,'' by G. WERTHEIM, of Frankfort; "Die Entdeckung der Parabelform der Wurflinie,'' by E. WOHLWILL, of Hamburg.

THE following recently issued catalogues of new and second hand scientific publications are more or less rich in mathematical works :—catalogue 40, Karl Krebs, Giessen, Germany; catalogue 276, Macmillan and Bowes, Cambridge, England; catalogue 173, Mayer and Müller, Berlin, Germany; catalogue 92, Oswald Weigèl, Leipsic, Germany. M. A. Hermann, of Paris, will issue a new catalogue this month (November).

OXFORD UNIVERSITY. The courses of lectures announced for the fall term of 1899 include the following on mathematical subjects :—By Professor W. ESSON : Analytic theory of place curves; Synthetic theory of place curves.—By Professor H. TURNER : Mathematical astronomy.—By Professor A. E. H. LOVE : Electricity and magnetism.—By Professor E. ELLIOTT : Theory of numbers.

CAMBRIDGE UNIVERSITY. Mathematical courses are announced as follows for the current academic year : *Michaelmas term,* 1899 :—By Professor G. G. STOKES : Hydrodynamics, three hours.—By Professor A. R. FORSYTH : Theory of differential equations, three hours; Calculus of variations, two hours.—By Professor G. H. DARWIN : Lunar theory, three hours.—By Mr. R. PENDLEBURY : Theory of equations, three hours; Spherical astronomy, three hours.—By Dr. E. W. HOBSON : Spherical and cylindrical harmonics, three hours.—By Mr. J. LARMOR : Electricity and magnetism, three hours.—By Mr. H. F. BAKER : Theory of functions, three hours.—By Mr. H. M. MACDONALD : Waves (especially waves of light), three hours —By Mr. H. W. RICHMOND : Plane analytical geometry, three hours.—By Mr. G. T. WALKER: The electromagnetic field, three hours.—By Mr. E. T. WHITTAKER : The problem of three bodies, three hours.—By Mr. J. H. GRACE : Invariants and geometrical applications, three hours.—*Lent term,* 1900 :—By Professor G. G. STOKES : Physical optics.

—By Professor A. R. Forsyth: The differential equations of theoretical dynamics.—By Professor R. S. Ball: Planetary theory.—By Mr. R. Pendlebury: Theory of numbers.—By Dr. E. W. Hobson: Vibrations and sound.—By Mr. J. Larmor: Electricity and magnetism (continuation).—By Mr. H. F. Baker: Theory of functions (continuation).—By Mr. H. M. Macdonald: Elasticity.—By Dr. J. W. L. Glaisher: Elliptic functions.—By Mr. R. A. Herman: Hydrodynamics.—By Mr. A. N. Whitehead: Universal algebra.—By Mr. A. Berry: Elliptic functions.—By Mr. G. T. Walker: Physical optics.—By Mr. G. T. Bennett: Linear and quadratic complexes.—*Easter term*, 1900:—By Mr. W. L. Mollison: Theory of potential and electrostatics.—By Mr. E. T. Whittaker: Analysis.—*Long vacation*, 1900:—By Professor G. H. Darwin: Attraction and potential.

During the winter semester 1899–1900 the several universities mentioned below offer the following mathematical courses:

University of Giessen. By Professor M. Pasch: Functions of complex variables, four hours; Seminar, two hours.—By Professor E. Netto: Analytical mechanics, four hours; Seminar, two hours.—By Professor R. Haussner: Differential and integral calculus, four hours; Perspective, two hours; Theory of probabilities, one hour; Exercises in the elements of higher mathematics, one hour.

University of Marburg. By Professor F. Schottky: Integral calculus, four hours; General theory of functions, five hours; Seminar, two hours.—By Professor A. E. Hess: Geometry of space, synthetic and analytic, four hours; Theory of algebraic equations and determinants, four hours; Seminar, two hours.—By Dr. F. Dalwigk: Introduction to higher mathematics and exercises, for hours; Selected subjects of the theory of functions, two hours; Exercises in the construction of mathematical models, one hour.

University of Prague. By Professor F. Lippich: Theoretical mechanics, three hours; Theory of capillarity, two hours; Seminar, two hours.—By Professor G. Pick: Differential and integral calculus, three hours; Algebraic analysis, two hours; Seminar, two hours.—By Professor K. Bobek: Curves and surfaces in space, two hours; Descriptive geometry and constructions, four hours.

UNIVERSITY OF STRASSBURG. By Professor T. REYE: Analytical geometry of space (new methods), three hours; Mathematical theory of elasticity of solid bodies, two hours; Seminar, two hours.—By Professor H. WEBER: Integration of the partial differential equations of mathematical physics, four hours; Introduction to higher mathematics, two hours; Exercises in mathematical seminar, three hours.—By Professor G. ROTH: Algebraic analysis and determinants, three hours; Analytical geometry of space, two hours; Ordinary differential equations, two hours.—By Professor A. KRAZER: Differential and integral calculus, four hours; Analytical geometry of the plane, three hours; Exercises in infinitesimal calculus, two hours.—By Dr. E. TIMERDING: Vector analysis, two hours.—By Dr. J. WELLSTEIN: Descriptive geometry and exercises, four hours; Seminar in coöperation with Professor WEBER, two hours.

UNIVERSITY OF TÜBINGEN. By Professor A. VON BRILL: Introduction to higher mathematics, four hours; Theory of algebraic curves, three hours; Seminar, two hours.—By Professor H. STAHL: Higher analysis, three hours; Partial differential equations, three hours; Seminar, two hours.—By Professor L. MAURER: Descriptive geometry and exercises, three hours; Elements of differential and integral calculus, two hours; Exercises in algebra and geometry for students of natural science, two hours.

UNIVERSITY OF VIENNA. By Professor G. VON ESCHERICH: Elements of differential and integral calculus, five hours; Exercises on the preceding course, two hours; Proseminar, one hour; Seminar, two hours.—By Professor L. GEGENBAUER: Algebra, four hours; Spherical and cylindrical harmonics, one hour; Proseminar, one hour; Seminar, two hours.—By Professor F. MERTENS: On the laws of reciprocity, five hours; Seminar, two hours; Proseminar, one hour.—By Professor G. KOHN: Analytical geometry and exercises, four hours.—By Dr. V. SERSAWY: The mathematics of life insurance, two parts, of three and four hours, respectively.—By Dr. A. TAUBER: Insurance mathematics and exercises, six hours.—By Dr. K. ZINDLER: Differential equations, two hours.—By Dr. E. BLASCHKE: Introduction to mathematical statistics, three hours.

UNIVERSITY OF WURZBURG. By Professor F. PRYM: Differential calculus, four hours; Higher theory of functions, four hours; Seminar in differential calculus, two hours; Seminar in selected sections of higher mathematics, two hours.—By Professor A. Voss: Analytical mechanics, four hours; Exercises in analytical mechanics, two hours; Algebra, three hours; Seminar in synthetic and analytic geometry, one hour.

UNIVERSITY OF ZURICH. By Professor H. BURKHARDT: Differential and integral calculus, four hours; mechanics, four hours; Seminar, two hours.—By Dr. A. WEILER: Descriptive geometry and exercises, four hours; Analytical geometry and exercises, four hours.—By Dr. F. KRAFT: Differential equations, three hours; Analytical geometry, three hours; Theory of probabilities, two hours; Seminar, two hours.—By Dr. E. GUBLER, algebra, three hours; spherical trigonometry, one hour; methods of mathematical instruction, two hours.

ANDREW GRAY, F.R.S., professor of physics in the University of North Wales since 1884, has been appointed professor of natural philosophy in Glasgow University, to succeed Lord KELVIN, who recently resigned.

A STATUE erected in memory of the late Professor F. TISSERAND was unveiled at Nuits-Saint-Georges on the 15th of October.

MR. LINDSAY DUNCAN has been appointed instructor in mathematics at Union College, and MR. EDWIN HAVILAND has been made assistant in mathematics in Swarthmore College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BURKHARDT (H.). See ENCYKLOPÄDIE.

BURNSIDE (W. S.) and PANTON (A. W.). Introduction to determinants; being a chapter from the theory of equations. London, Longmans, 1899. 8vo. 2s. 6d.

——. Theory of equations. 4th edition. Vol. I. London, Longmans, 1899. 8vo. 9s. 6d.

CAPELLI (A.). Lezioni di algebra complementare, ad uso degli aspiranti alla licenza universitaria in scienze fisiche e matematiche. 2a edizione, con aggiunte. Napoli, Pellerano, 1898. 8vo. 16 + 680 pp. Fr. 8.50

CORDARA (S.). Nota sulla impossibilità della quadratura del cerchio. Bologna, Azzoguidi, 1899. 8vo. 11 pp.

Deter (J.). Repetitorium der Differential- und Integralrechnung. 3te Auflage. Berlin, Rockenstein, 1899. 8vo. 119 pp. M. 1.60

Dickstein (S.). See Klein (F.).

Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Mit Unterstützung der Akademien der Wissenschaften zu München und Wien und der Gesellschaft der Wissenschaften zu Göttingen, sowie unter Mitwirkung zahlreicher Fachgenossen. [In 2 Teilen.] Teil I. Reine Mathematik, herausgegeben von H. Burkhardt und W. F. Meyer Vol. I. Arithmetik und Algebra, redigiert von W. F. Meyer. Heft 3, enthaltend : E. Netto, Rationale Funktionen einer Veränderlichen ; ihre Nullstellen —E. Netto, Rationale Funktionen mehrerer Veränderlichen.—G. Landsberg, Algebraische Gebilde ; arithmetische Theorie algebraischer Grössen.—W. F. Meyer, Invariantentheorie.—Leipzig, Teubner, 1899. 8vo. Pp. 225–352. M. 3.80

Graham (J.). Elementary treatise on practical mathematics for technical colleges and schools. London, Arnold, 1899. 8vo. 284 pp. (Mathematical series.) 3s. 6d.

Guldberg (A.). Sur la théorie des solutions singulières des équations aux différentielles totales du premier ordre. Christiania, Dybwad, 1899. 8vo. 26 pp. (*Videnskabsselskabets Skrifter*, I. Math.-naturv. Klasse, 1899, No. 4).

Hessenberg (G.). Ueber die Invarianten linearer und quadratischer binärer Differentialformen und ihre Anwendung auf die Deformation der Flächen. Berlin, 1899. 4to. 50 pp. M. 2.50

Huygens (C.). Oeuvres complètes, publiées par la Société hollandaise des sciences. Vol. VIII : Correspondance 1676–84. La Haye. 1899. 4to. 631 pp., 7 plates, 1 portrait. Fr. 30.00

Klein (F.). Odczyty o matematyce miane w Evanston od 28 sierpnia do 9 września 1893 r. dla czlonków kongresu matematycznego odbytego w czasie wystawy wszechświatowej w Chicago. Spisane przez A. Ziweta, Przelożyl za upoważnieniem autora S. Dickstein. Warszawa, 1899. 8vo. 6 + 111 pp., with Klein's portrait.

Krauss (E.). See Pinet (H.).

Landau (E.). Neuer Beweis der Gleichung $\sum_{k=1}^{\infty}\frac{\mu(k)}{k}=0$. Berlin, 1899. 8vo. 16 pp. M. 1.00

Landsberg (G.). See Encyklopädie.

Lindelöf (E.). Remarques sur un principe général de la théorie des fonctions analytiques. Helsingfors, 1898. 4to. 39 pp. M. 2.00

——. Recherches sur les polyèdres maxima. Helsingfors, 1898. 4to. 47 pp. M. 2.00

Meyer (W. F.). See Encyklopädie.

Miller (G. A.). Memoir on the substitution-groups whose degree does not exceed eight. 4to. (*American Journal of Mathematics*, Vol. 21, pp. 287–338.)

Müller (H.). Die Lehre von den Coordinaten und Kegelschnitten, für den Unterricht dargestellt. Berlin, Moeser, 1899. 8vo. [From

Müller, Die Mathematik auf den Gymnasien und Realschulen.] Pp. 165–216. Boards. M. 1.00

NETTO (E.). See ENCYKLOPÄDIE.

PANTON (A. W.). See BURNSIDE (W. S.).

PASCAL (E.). Repertorium der höheren Mathematik (Definitionen, Formeln, Theoreme, Literaturnachweise). Autorisierte deutsche Ausgabe von A. Schepp. (In 2 Teilen.) Teil I: Die Analysis. Leipzig, Teubner, 1899. 8vo. Cloth.

PINET (H.). Mémoire sur une nouvelle méthode pour la résolution des équations numériques. Suivi d'un appendice donnant le détail des opérations, par E. Krauss. Paris, Nony, 1899. 4to. Pp. 47.

RAZZABONI (A.). Le formole di Frenet in geometria iperbolica e loro principali applicazioni. Bologna, Gamberini e Parmeggiani, 1899. 4to. 22 pp. Fr. 2.50

SCHEPP (A.). See PASCAL (E.).

SEVERINI (C.). Sull'integrazione approssimata delle equazioni differenziali ordinarie. Bologna, Zanichelli, 1899. 4to. 27 pp. Fr. 3.00

ZIGNAGO (I.). Osservazioni sulla convergenza delle serie. Genova, Instituto Sordomuti, 1899. 16mo. 10 pp.

ZIWET (A.). See KLEIN (F.).

II. ELEMENTARY MATHEMATICS.

BELLACCHI (G.). Lezioni ed esercizî di algebra complementare. Fascicolo 2. Firenze, Barbèra, 1899. 16mo. 143 pp. Fr. 2.00

DETER (J.). Mathematisches Formelbuch für höhere Unterrichtsanstalten. Neu herausgegeben von E. Arndt. 4te Auflage. Berlin, Rockenstein, 1899. 8vo. 5 + 58 pp. M. 0.90

FAIFOFER (A.). Trattato di geometria intuitiva, ad uso delle scuole tecniche e normali. 31a edizione. Venezia, Tipografia Emiliana, 1899. 16mo. 165 pp. Fr. 2.00

——. Elementi di trigonometria piana e tavole logaritmico-trigonometriche ad uso dei licei. 11a edizione. Venezia, Tipografia Emiliana, 1899. 16mo. 92 + 69 pp. Fr. 2.00.

GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 58ste Auflage. Halle, Strien, 1899. 8vo. 166 + 35 pp. Half leather. M. 2.50

——. Vierstellige logarithmisch-trigonometrische Handtafel. Stereotyp-Druck. 3te Auflage. Halle, Strien, 1899. Folio. M. 0.60

——. Dieselbe, für Decimalteilung des Quadranten. Stereotyp-Druck. 2te Auflage. Halle, Strien, 1899. Folio. M. 0.80

GOSETTI (L.). Corso di aritmetica e di algebra. Parte I. Venezia, Ferrari, 1899. 8vo. 201 pp. Fr. 2.00

HUC (E.). Algèbre, à l'usage des candidats à la première partie des baccalauréats. Bar-le-Duc, Comte-Jacquet, 1899. 16mo. 104 pp.

LEONARDI (O.). Elementi di algebra. Foligno, Salvati, 1899. 16mo. 47 pp.

MACKAY (J. S.). Arithmetic theoretical and practical. London and Edinburgh, W. and R. Chambers, 1899. 8vo. 11 + 472 pp.

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. (In 2 Teilen.), Teil I: Die Unterstufe: Lehraufgabe der Klassen Quarta bis Unter-Sekunda. 8 + 152 pp.—Teil II: Die Oberstufe: Lehraufgabe der Klassen Ober-Sekunda und Prima. 10 + 216 pp. Berlin, Moeser, 1899. 8vo. Cloth. M. 5.70

NEOVIUS (L.). Elementarkurs i algebra. Del I: Rationella expressioner. Helsingfors, 1898. 8vo. 10 + 111 pp. M. 2.00

——. Lärobok i plan-trigonometri. Helsingfors, 1899. 8vo. 10 + 119 pp. M. 2.00

PADOA (A.). Note critiche agli Elementi di geometria, di G. Veronese. Pinerolo, Chiantore-Mascarelli, 1899. 8vo. 22 pp.

——. Note critiche al Libro di aritmetica e di algebra elementare, di P. Cazzaniga. Pinerolo, Chiantore-Mascarelli, 1899. 8vo. 17 pp.

PUCHETTI (N.). Aritmetica e geometria per la quarta classe elementare. Torino, Paravia, 1899. 16mo. 78 pp. Fr. 0.60

——. Aritmetica e geometria per la quinta classe elementare. Torino, Paravia, 1899. 16mo. 84 pp. Fr. 0.60

RIDDEL (J.). Practical plane and solid geometry. With exercises and examination questions. Edinburgh, Oliver & Boyd, 1899. 8vo. 336 pp. 2s.

SCHLÖMILCH (O.). Fünfstellige logarithmische und trigonometrische Tafeln. Wohlfeile Schulausgabe. 15te Auflage. Braunschweig, 1899. 8vo. 151 pp. M. 1.00

SCHUSTER (M.). Geometrische Aufgaben. Ein Lehr- und Uebungsbuch zum Gebrauch beim Unterricht an höheren Schulen. Ausgabe A: für Vollanstalten. Leipzig, Teubner, 1899. 8vo. 8 + 147 pp., 2 plates. Cloth. M. 2.00

STEREOSCOPIC VIEWS of solid geometry figures; with references to Wells' "Essentials of solid geometry." Boston, Heath, 1899. 93 cards in a pasteboard box. $0.60

VALLE (G.). Funzioni ad una variabile e loro limiti; un capitolo di algebra elementare. Noto, Zammit, 1899. 8vo. 38 pp.

WELLS (W.). New higher algebra. Boston, Heath, 1899. 12mo. 8 + 446 pp. Half-leather. $1.25

——. See STEREOSCOPIC VIEWS.

III. APPLIED MATHEMATICS.

BRYAN (G. H.) and ROSENBERG (F.). First stage mechanics of fluids. For elementary examination of Science and art department. 2d edition. London, Clive, 1899. 12mo. 224 pp. (Organic science series.) 2s.

CLERICI (V.). Considerazioni intorno alle formole per la misura delle acque correnti. Firenze, Ricci, 1899. 8vo. 18 pp.

CREW (H.). The elements of physics, for use in high schools. New York, The Macmillan Company, 1899. 12mo. 14 + 347 pp. $1.10

Eddy (H. T.). The graphical treatment of alternating currents in branching circuits. With special reference to the case of variable frequency. 8vo. (A paper to be presented at the fifteenth general meeting of the American Institute of electrical engineers, Omaha, June 30, 1898.) Pp. 511-554.

Eddy (H. T.), Morley (E. W.), and Miller (D. C.). The velocity of light in the magnetic field. 8vo. (*Physical Review*, Vol. VII., December, 1898. Pp. 283-295.)

Guidi (C.). Lezioni sulla scienza delle costruzioni. Parte II: Teoria dell' elasticità e resistenza dei materiali. 3a edizione. Torino, Bertolero, 1899. 8vo. 7 + 240 pp., 7 plates. Fr. 6.00

Hutton (F. R.). Heat and heat engines; a study of the principles which underlie the mechanical engineering of a power plant. New York, Wiley, 1899. 8vo. 21 + 553 pp. Cloth. $5.00

Jordan (W.). Hülfstafeln für Tachymetrie. 2te Auflage. Stuttgart, Metzler, 1899. 8vo. 15 + 246 pp. M. 8.00

Madamet (A.). Tiroirs et distributeurs de vapeur; appareils de mise en marche et de changement de marche. 2me édition. Paris, Gauthier-Villars, 1899. 16mo. 151 pp. (Encyclopédie des aide-mémoire, section de l'ingénieur.)

Mazzotto (D.). Sugli armonici delle vibrazioni elettriche; nota. Pisa, Pieraccini, 1899. 8vo. 8 pp.

Merriman (M.). Elements of precise surveying and geodesy. New York, Wiley, 1899. 8vo. 261 pp. Cloth. $2.50

Miller (D. C.). See Eddy (H.T.).

Morley (E. W.). See Eddy (H. T.).

Portugal de Faria (A. de). See Soares de Barros.

Reinhertz (C.). Geodäsie. Einführung in die wesentlichsten Aufgaben der Erdmessung und Landesvermessung. Leipzig, 1899. 12mo. 181 pp. Cloth. M. 0.80

Reuter (J.). Geometriska uppgifter fran det praktiska. omrade Heltsingfors, 1898. 4to. 65 pp M. 2.00

Rey-Pailhade (J. de). Extension du système métrique à la mesure du temps et des angles; conférence faite sous les auspices de la chambre syndicale d'horlogerie, le 23 juin 1898, à l'Ecole d'horlogerie de Paris. Montdidier, Carpentier, 1899. 8vo. 40 pp. 5 plates.

Rosenberg (F.). See Bryan (G. H.).

Savander (O.). Détermination relative de la pesanteur à Helsingfors précédée d'un aperçu sur les formules de réduction. Helsingfors, 1898. 8vo. 3 + 195 pp. 2 plates. M. 4.00

Soares de Barros e Vasconcellos (J. J.). Oeuvres, reproduites par A. de Portugal de Faria. I: Nouvelles équations pour la perfection de la théorie des satellites de Jupiter et pour la correction des longitudes terrestres déterminées par les observations des mêmes satellites.—II: Observations et explications de quelques phénomènes vus dans le passage de Mercure au devant du disque du soleil, observé à l'hotel de Clugny à Paris le 6 may 1753, et leur application pour la perfection de l'astronomie. Livourne, Giusti, 1899. 8vo. 73 pp.

Ser. No. ....
Book No. ....

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 28, 1899. Thirty persons were in attendance at the two sessions, including the following members of the Society:

Professor Maxime Bôcher, Professor E. W. Brown, Professor F. N. Colé, Professor T. W. Edmondson, Professor A. M. Ely, Professor T. S. Fiske, Mr. A. S. Gale, Dr. G. B. Germann, Miss Carrie Hammerslough, Mr. S. A. Joffe, Mr. C. J. Keyser, Dr. G. H. Ling, Professor F. H. Loud, Dr. Emory McClintock, Dr. James Maclay, Dr. G. A. Miller, Professor E. H. Moore, Professor F. Morley, Professor B. O. Peirce, Professor A. W. Phillips, Professor James Pierpont, Professor M. I. Pupin, Miss Amy Rayson, Professor C. A. Scott, Professor H. D. Thompson, Professor E. B. Van Vleck, Professor A. G. Webster, Miss E. C. Williams, Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair. At the opening of the morning session a recess was taken to enable the members to attend the presidential address of Professor H. A. Rowland before the American Physical Society, which was in session in the same building. The simultaneous meeting of the two Societies, which is intended to be a permanent feature, naturally added to the interest and activity on both sides. The two organizations are modelled on a common plan; they have many members in common; and their hearty coöperation, which seems assured, cannot fail to be an important factor in the advancement of science.

The Council announced the election of the following persons to membership in the Society: Professor M. E. Bogarte, Northern Indiana Normal School, Valparaiso, Ind.; Mr. A. S. Gale, Yale University, New Haven, Conn.; Mr. B. F. Groat, University of Minnesota, Minneapolis, Minn.; Dr. Edward Kasner, Columbia University, New York, N. Y.; Professor J. A. Miller, University of Indiana, Bloomington, Ind.; Professor A. M. Sawin, Clark University, Atlanta, Ga.; Professor S. A. Singer, Capital University, Columbus, Ohio; Dr. H. E. Slaught, University of Chicago, Chicago, Ill.; Professor E. P. Thompson, Miami University, Oxford, Ohio. Seven applications for membership were received.

With a view to economy of time, the Council voted that hereafter the morning session of the meetings of the Society should open at 11 a. m. and the afternoon session at 2 p. m. The Council will meet at 10.15 a. m.

The next meeting of the Society, Thursday, December 28th, will be the annual meeting for the election of officers and three other members of the Council. At this meeting President R. S. Woodward will deliver a presidential address on "The century's progress in applied mathematics."

The following papers were presented at this meeting:

(1) Professor PAUL GORDAN: "Formentheoretische Entwickelung der in Herrn White's Abhandlung über Curven dritter Ordnung enthaltenen Sätze."

(2) Professor E. O. LOVETT: "The transformation of straight lines into spheres."

(3) Dr. G. A. MILLER: "On the simply transitive primitive groups."

(4) Professor C. A. SCOTT: "The conditions imposed on a curve by assigned multiple points."

(5) Professor E. H. MOORE: "On the generational determination of abstract groups" (preliminary communication).

(6) Professor C. A. SCOTT: "The status of imaginaries in pure geometry."

(7) Professor MAXIME BÔCHER: "On Sturm's theorem of comparison" (preliminary communication).

(8) Professor F. MORLEY: "On a fundamental geometric construction."

(9) Mr. E. B. WILSON: "The decomposition of the general collineation of space into three skew reflections."

(10) Dr. G. A. MILLER: On the order of the product of two substitutions."

(11) Mr. J. K. WHITTEMORE: "On a generalization of the fundamental problem of the calculus of variations."

(12) Mr. J. L. COOLIDGE: "A projective representation of the imaginary points of a plane."

Professor Gordan's paper was presented to the Society through Professor H. S. White, and Mr. Coolidge's through Professor Bôcher. Mr. Wilson was introduced by Professor Pierpont. In the absence of the authors the paper of Professor Gordan was read by Professor Moore, that of Professor Lovett was read by title, and those of Mr. Whittemore and Mr. Coolidge were read by Professor Bôcher.

Dr. Miller's first paper appears in the present number of the BULLETIN. Miss Scott's second paper will appear in a subsequent number. Abstracts of the other papers are given below.

Professor Gordan's paper will appear in the *Transactions.* The following is a summary of its content: The theorems proven by Professor H. S. White in a paper read at the Columbus meeting of this Society in August, 1899, were established by the use of a canonical form of the ternary cubic. These theorems can be obtained quite as readily without using a canonical form. For this purpose the Hessian and Cayleyan are written for the linear combinations $\kappa f + \lambda \triangle$ and $\kappa\pi + \lambda s$, where f denotes a nonsingular ternary cubic, $\triangle$ its Hessian, π its conjugate cubic in contragredient variables, and s its Cayleyan, the Hessian of π. The forms so found are cubics in κ, λ; hence are determined the parameters of curves having the same Hessian or the same Cayleyan as either f or π. These parameters contain a square root of an invariant of f, but no other irrationality. From explicit formulæ so found it is easy to calculate the simplest invariants of each of the eight forms called A, B, F, G, φ, ψ, α, β in the paper of Professor White, and to show that for certain pairs of them a simultaneous covariant resembling θ_f must be identical with one of the pair multiplied into an invariant factor. These invariant factors are explicitly determined; and the conjugate property of A and α, B and β, F and φ, G and ψ is exhibited. Finally it is shown that a set of three invariantive equations is sufficient to characterize completely such a triple system as f, A, B. The process is in general the converse of that employed in the earlier paper cited above, and confirms its results, with the addition of important details.

Professor Lovett's paper, which will appear in the *American Journal of Mathematics*, is in abstract as follows: The transformations of the point space (x, y, z) into the point space (X, Y, Z) which are determined by the two *æquationes directrices*

$$x\varphi_1 + y\varphi_2 + z\varphi_3 + \varphi_4 = 0, \quad x\varphi_5 + y\varphi_6 + z\varphi_7 + \varphi_8 = 0, \quad (1)$$

where $\varphi_i \equiv a_i X + b_i Y + c_i Z + d_i$ $(a_i, b_i, c_i, d_i$ constants),

define a family of ∞^{30} transformations which change the straight line

$$y + kx + m = 0, \quad z + lx + n = 0, \quad (2)$$

into the quadric

$$\begin{vmatrix} k & m \\ l & n \end{vmatrix} \cdot \begin{vmatrix} \varphi_2 & \varphi_3 \\ \varphi_6 & \varphi_7 \end{vmatrix} - k \begin{vmatrix} \varphi_2 & \varphi_4 \\ \varphi_6 & \varphi_8 \end{vmatrix} - l \begin{vmatrix} \varphi_3 & \varphi_4 \\ \varphi_7 & \varphi_8 \end{vmatrix} - m \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_5 & \varphi_6 \end{vmatrix}$$

$$- n \begin{vmatrix} \varphi_1 & \varphi_3 \\ \varphi_5 & \varphi_7 \end{vmatrix} + \begin{vmatrix} \varphi_1 & \varphi_4 \\ \varphi_5 & \varphi_8 \end{vmatrix} = 0. \qquad (3)$$

This quadric reduces to a sphere, without k, l, m, n entering the equations of condition, in the following cases and in no other: 1° when any determinant of the matrix

$$\left\Vert \begin{matrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ \psi_5 & \psi_6 & \psi_7 & \psi_8 \end{matrix} \right\Vert \qquad (\psi_i \equiv a_i X + b_i Y + c_i Z \equiv \varphi_i - d_i), \quad (4)$$

reduces to the form $\quad \text{Const.}\ (X^2 + Y^2 + Z^2), \qquad (5)$

and at the same time the functions φ_i corresponding to the ψ_i remaining in the matrix reduce to constants; for example, $\varphi_2, \varphi_3, \varphi_6, \varphi_7$ constants and the determinant $\psi_1\psi_8 - \psi_4\psi_5$ of the form (5); 2° when all six determinants of the matrix (4) are of the form (5).

The first case gives six families each of ∞^{12} line-sphere contact transformations determined by two linear equations of the form

$$\begin{aligned} (a_1X + b_1Y + c_1Z + d_1)x + d_2y + d_3z + a_4X \\ + b_4Y + c_4Z + d_4 = 0, \\ (a_5X + b_5Y + c_5Z + d_5)x + d_6y + d_7z + a_8X \\ + b_8Y + c_8Z + d_8 = 0, \end{aligned} \qquad (6)$$

where the constants are subject to the conditions

$$\begin{gathered} a_1a_8 - a_4a_5 = b_1b_8 - b_4b_5 = c_1c_8 - c_4c_5, \\ a_1b_8 + b_1a_8 - a_4b_5 - b_4a_5 = 0, \ b_1c_8 + c_1b_8 - b_4c_5 - c_4b_5 = 0, \\ c_1a_8 + a_1c_8 - c_4a_5 - a_4c_5 = 0. \end{gathered} \qquad (7)$$

Assigning the following particular values to the above constants

$$\begin{gathered} a_1 = a_8 = b_1 = b_8 = c_4 = c_5 = d_1 = d_2 = d_4 \\ = d_5 = d_7 = d_8 = 0, \\ a_4 = a_5 = c_1 = -c_8 = d_3 = d_6 = 1, \\ b_4 = -b_5 = i = \sqrt{-1}, \end{gathered} \qquad (8)$$

we have the celebrated correspondence of Lie determined by the equations

$$Zx + z + X + iY = 0, \quad (X - iY)x + y - Z = 0. \qquad (9)$$

The above transformations change right lines to points if

$$d_1 : d_2 : d_3 : d_4 = d_5 : d_6 : d_7 : d_8. \qquad (10)$$

The second class of conditions demands that the functions ψ_i have the forms

$$\begin{aligned} \psi_j &\equiv a_j X + i a_j Y + c_j Z, \\ \psi_{j+4} &\equiv c_j X - i c_j Y - a_j Z \end{aligned} \quad (j = 1,2,3,4); \qquad (11)$$

the corresponding equations (1) define a fifteen-parameter family of line-sphere contact transformations whose transformations may be derived by operating on the transformations of the ordinary projective group with the transformation (9) of Lie.

In conclusion the note points out the corresponding generalizations for spaces of higher dimensions which have been incorporated in another paper; sets up a six-parameter family of line-sphere contact transformations by generalizing the form given by Darboux* to Lie's transformation; and finally calls attention to the fact that infinite families† of line-sphere contact transformations have been constructed.

If the nominal conditions imposed on a curve by points of assigned order are in number N, and the actual conditions are in number D, the m-ic excess e_m is $N - D$. The variation in e_m for different values of m has been studied in detail by Dr. Macaulay (*Proceedings of the London Mathematical Society*) for the case where all given points are simple. The object of Professor Scott's paper was to give independent proofs for the case of multiple points. The theorems proved are: (1) if m be sufficiently great, $e_m = 0$; (2) $e_m - e_{m+1} \leq m$, where the sign of equality can occur only in a particular case; (3) $e_m - e_{m+1} \geq 0$, where the sign of equality occurs only if $e_m = 0$.

On the basis of the current definition of abstract groups (discontinuous or continuous, of order finite or infinite) Professor Moore's paper investigates under what conditions and in what sense a set of generator symbols subject to certain generational relations serves to determine a definite abstract group. This paper will be published in the *Transactions*.

---

*Darboux, Théorie des surfaces, vol. 1, §157.

†*Comptes rendus*, August 21, 1899.

It was the object of Professor Bôcher's communication to show how some of Sturm's theorems concerning linear differential equations of the second order can be proved rigorously and at the same time simply. The analogy of a part of the method here used with a method suggested by Mr. Porter (Cf. BULLETIN, March, 1897, p. 210) should be noticed. Rigorous but complicated proofs had previously been given by the author in the BULLETIN for April, 1898.

Sturm's first "theorem of comparison"* will evidently be proved if it can be shown that y_2 has at least one root in each of the intervals $a<x<x_1$, $x_1<x<x_2$, $\cdots$. By combining equations numbered (1) and (2) we readily get

$$y_2(x_1)y_1'(x_1) = \int_a^{x_1}(\varphi_1 - \varphi_2)y_1y_2dx.$$

If y_2 had no root in the interval $a<x<x_1$, y_1 and y_2 would have the same sign throughout this interval. The right side of the above equation would therefore be greater than zero. The left side is, however, obviously either negative or zero. In either case we have a contradiction, and we thus see that y_2 has at least one root in the interval $a<x<x_1$. Consider now the solution $\bar{y}_2$ of (2) which vanishes at x_1 and whose derivative there has the same value as the derivative of y_1 at that point. By a proof exactly like the one just given we see that $\bar{y}_2$ has at least one root in the interval $x_1<x<x_2$, and, therefore, by a well known and easily proved theorem of Sturm,* the same is true of y_2. In the same way each of the other intervals $x_2<x<x_3$, $x_3<x<x_4$, $\cdots$ is treated. The second theorem of comparison (*loc. cit.*, p. 300) admits an analogous treatment; and the same is true of certain generalized forms of these theorems.

Professor Morley proved the following theorem: Given three points and four planes, take the polar of any two planes as to the points, and let this polar point be joined to the intersection of the other two planes. We thus have six planes. These meet in a point; and the third polar of this point as to the four planes is the polar of the original three points as to the original four planes. A precisely similar theorem holds for $n+1$ spaces S_{n-1} and n points in a space S_n of n dimensions. That is, the polar of any $n-1$ spaces

---

* Loc. cit., p. 298. The notation there explained is used here without change.

* Cf. BULLETIN, March, 1897, p. 209.

S_{n-1} as to the points is joined to the intersection of the other two S_{n-1}'s. We have thus $n(n+1)/2$ spaces S_{n-1} which meet in a point; the n^{th} polar of this point as to all the S_{n-1}'s is the polar of the n given points as to all the S_{n-1}'s. This yields, for points and lines lying in a plane, the proof of some statements made by Caporali (Works, Pellerano, Naples, 1888; pp. 262–269).

Mr. Wilson's paper showed that the collineation in three dimensional space can, in general, be resolved into three skew reflections. The method pursued was to give a geometrical construction for three skew reflections whose product was the given collineation. The paper is offered for publication in the *Transactions*.

Dr. Miller showed that if any three arbitrary numbers larger than unity are given we can always find three substitutions s_1, s_2, s_3 whose orders are these three numbers respectively, in the given order, and which satisfy the equation $s_1 s_2 = s_3$. These substitutions can always be so selected that the number of the different elements involved in all of them does not exceed the order of the largest of the given numbers increased by two. In the proof of this theorem the following lemma was frequently used: If two circular substitutions have an odd number of consecutive elements in common their product is a circular substitution that involves all the elements of both.

Mr. Whittemore's paper is in abstract as follows: The fundamental problem of the calculus of variations may be said to be: To determine y, a function of x, whose values for $x = a$ and $x = b$ are given, such that

$$u = \int_a^b F(x,y,y')dx \tag{1}$$

shall be a maximum or a minimum. In (1) F is a given function, supposed analytic, of $x,y,$ and $y' = dy/dx$. This problem is identical with the following: To determine y, a function of x, whose values for $x = a$ and $x = b$ are given, such that a solution of the differential equation

$$u' = F(x,y,y') \tag{2}$$

which vanishes for $x = a$, shall have a maximum or a minimum for $x = b$. The second statement leads naturally to the posing of the same problem for the more general differential equation

$$u' = F(x,u,y,y'). \tag{3}$$

If we denote by η the variation of y and by π the corresponding "first variation" of u, and if we suppose η to be a "regular" variation, that is such a variation that η'^2 may be neglected as well as η^2, it follows from (3) that π is determined by the linear differential equation of the first order

$$\pi' - \frac{\partial F}{\partial u}\pi - \frac{\partial F}{\partial y}\eta - \frac{\partial F}{\partial y'}\eta' = 0, \tag{4}$$

where we have further the initial condition $\pi(a) = 0$. A necessary condition that u shall have for $x = b$ a maximum or a minimum is $\pi(b) = 0$ for all regular variations η where $\eta(a) = \eta(b) = 0$. If now in (4) we regard u, y, and η as known functions of x, we can find π by quadratures, and we may then see by the usual method of the calculus of variations that we must necessarily have

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial u} \cdot \frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{5}$$

This condition corresponds to Lagrange's necessary condition for a maximum or a minimum in the calculus of variations, and reduces to it when $\frac{\partial F}{\partial u} = 0$.

If now we consider the differential equation

$$u'' = F(x,u,u',y,y') \tag{6}$$

we have for π, the "first variation" of u, a linear differential equation of the second order

$$\pi'' - \frac{\partial F}{\partial u'}\pi' - \frac{\partial F}{\partial u}\pi - \frac{\partial F}{\partial y}\eta - \frac{\partial F}{\partial y'}\eta' = 0, \tag{7}$$

which cannot be solved by quadratures. A method is therefore necessary which does not depend upon the integration of (7). Such a method is discussed in the present paper, and applied to the equation (3) leads again to (5). Applied to (6) the method leads to the proposition: There is in general no function y, having given values for $x = a$ and $x = b$ such that u, determined by (6) and the two initial conditions $u(a) = u_0$, $u'(a) = u_0'$, shall have a maximum or a minimum for $x = b$.

It seems probable that the same method would lead to a similar statement for all differential equations of an order greater than one in u. The presence in the differential equations of derivatives of y of order greater than one does not affect the result in the latter case, while in the case of equation (3), as in the calculus of variations, we are led to a resulting differential equation more complicated than (5).

Mr. Coolidge's paper began with a definition of imaginary geometrical elements after von Staudt and August. The usual method of representing imaginary points in a real line by real points in a plane was then mentioned, with an explanation why this proceeding could not be so extended as to include all imaginary points in a plane. It was then shown that every imaginary point in a real plane might be represented by the real base of its projection from a real point on an imaginary plane, and the necessary construction was explained. The assemblages of lines representing points on real and imaginary lines were discussed, as well as the relations of those lines which represented chains of imaginary points. The paper closed with a brief reference to the congruences of lines representing points lying on coinc sections, and a discussion of the advisability of finding a method of representing all the imaginary points in space.

F. N. COLE.

COLUMBIA UNIVERSITY.

NOTE ON THE SIMPLY TRANSITIVE PRIMITIVE GROUPS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society, October 28, 1899.)

THE simply transitive primitive groups present many difficulties which have not yet been solved. In his "Traité des substitutions" Jordan proved some fundamental theorems in regard to these groups.* Several additional theorems appeared in the "*Proceedings of the London Mathematical Society,*" 1897, pp. 534–536. The following theorems and corollaries are closely connected with the results of this article.

Every primitive group G of degree n contains a maximal subgroup G_1 of degree $n-1$. When G is simply transitive

---

*P. 284.

G_1 is intransitive and the order of each of its transitive constituents is divisible by the same prime numbers.†

THEOREM I. *When G_1 is of order p^m, p being any prime number, it cannot contain as a transitive constituent any regular group whose order is less than the order of G_1.*

The subgroup of G_1 which would correspond to identity in this transitive constituent would be self-conjugate and it would have to occur in each of the conjugates of G_1 which do not involve all the elements of the given transitive constituent. In each of these conjugates it would be transformed into itself by substitutions that are not found in G_1, since each subgroup of any group of order p^m is transformed into itself by substitutions that are not contained in it. These conditions are impossible, since G_1 is a maximal subgroup of G.

COROLLARY I. *When G_1 is of order p^m it cannot contain any self-conjugate subgroup of degree less than $n - 1$ and of order greater than p^{m-3}, $m > 2$.*

COROLLARY II. *When one of the transitive constituents of G_1 is of order p^2, G_1 must be of this order and G must be of class $n - 1$.*

THEOREM II. *If the order of G_1 is p^5 and if G_1 contains a transitive constituent of order p^3, then any two of the p^2 conjugates of G_1 which do not involve all the elements of the given transitive constituent contain a common abelian group of order p^3.*

It follows from theorem I that the given transitive constituent must be of degree p^2. Each of the given p^2 conjugates contains a subgroup of order p whose substitutions are commutative to every substitution of the group and whose degree does not exceed $n - 1 - p^2$. This subgroup must occur in G_1 since the self-conjugate subgroup S of G_1 which corresponds to identity in the given constituent of order p^3 cannot be transformed into itself by any substitution of G except those that are contained in G_1. It cannot occur in S since each subgroup of S is transformed into itself by at least p^4 substitutions of G_1. The given subgroup of order p and S must therefore generate an abelian group of order p^3 which includes all the substitutions of G_1 that are common to it and the corresponding conjugate to G_1.

CORNELL UNIVERSITY,
July, 1899.

---

† Jordan, loc. cit.

ON THE COMMUTATORS OF A GIVEN GROUP.

BY DR. G. A. MILLER.

LET $s_1, s_2, s_3, \ldots, s_g$ represent all the operators of a given group G, and let t represent any operator whatsoever. From the identities

$$s_\beta^{-1}s_\alpha^{-1}t^{-1}s_\alpha t s_\beta \equiv (s_\alpha s_\beta)^{-1}t^{-1}s_\alpha s_\beta t\cdot t^{-1}s_\beta^{-}t s_\beta, \quad (\alpha, \beta < g+1)$$
$$s_\beta^{-1}t^{-1}s_\alpha^{-1}t s_\alpha s_\beta \equiv (t s_\beta)^{-1}s_\alpha^{-1}t s_\beta s_\alpha \cdot s_\alpha^{-1}s_\beta^{-1}s_\alpha s_\beta$$

we observe that the transform, with respect to any operator of G, of the commutators formed with t and the operators of G is the product of two such commutators. All of these commutators must, therefore, form a group which is transformed into itself by G. When t transforms G into itself the given commutators generate the smallest self-conjugate subgroup of G which has the property that all of the operators of the corresponding quotient group are commutative to t; *i. e.*, t transforms each of the divisions of G with respect to this self-conjugate subgroup into itself. By letting t represent, in succession, all the operators of G we arrive at the known theorem that the commutator subgroup of a group is the smallest self-conjugate subgroup with respect to which the group is isomorphic to an abelian group.*

From the fact that the commutator subgroup of G is a characteristic subgroup it follows that it is self-conjugate in every group that contains G self-conjugately. In particular, if G is of order p^α and is not the abelian group of the type (1, 1, 1, ..., 1), then every group that contains G self-conjugately must also contain a self-conjugate subgroup of order p^β, $\alpha > \beta > 0$. When G is the abelian group of the type (1, 1, 1, ..., 1), it is clearly always possible to construct a group which contains G self-conjugately and which transforms none of the subgroups of G besides identity into itself.†

If t remains fixed in the commutator $C \equiv s^{-1}t^{-1}st$ while s is multiplied on the left by all the operators of G we observe that C remains unchanged when this multiplier is commutative to t and that it is changed for every other multiplier. Arranging all the operators of G in the following manner, the first row being composed of the operators that are commutative to t,

---

* *Quar. Jour. of Math.*, vol. 28 (1896), p. 268.

† Moore, BULLETIN, vol. 2 (1896), p. 33.

$$\begin{array}{lllll}
1, & s_2, & s_3, & \cdots & s_\gamma, \\
r_2, & s_2 r_2, & s_3 r_2, & \cdots & s_\gamma r_2, \\
r_3, & s_2 r_3, & s_3 r_3. & \cdots & s_\gamma r_3, \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
r_\lambda, & s_2 r_\lambda, & s_3 r_\lambda, & \cdots & s_\gamma r_\lambda,
\end{array}$$

C has the same value when s is multiplied on the left by each one of the γ operators of a row, and if s is multiplied by two operators from different rows the corresponding values of C must evidently be different. Hence C has just $\lambda = g \div \gamma$ distinct values when s is multiplied by all the operators of G and each value of C corresponds to the same number of operators of G. This result is independent of whether s or t belong to G or not. When s belongs to G we may say that C has λ distinct values when s is successively replaced by all the operators of G and that each value of C corresponds to the same number of the operators of G.

While C has the same number of different values as t has conjugates when it is transformed by all the operators of G, yet these different values of C need not include any of these conjugates of t nor is it necessary that these form a complete system of conjugates with respect to the operators of G, *e. g.*, when t is one of the operators of order three in the alternating group of order twelve while s assumes successively all the values of the operators of this group, the four values of C are the operators of order two and identity. If s had assumed all the values of the operators of the symmetric group of order twenty-four, C would have assumed the values of four operators of order three in addition to the given four values.

The following two equations

$$a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot \cdots \cdot l_1l_2 \cdot m_1m_2 \times a_2b_1 \cdot b_2c_1 \cdot \cdots \cdot l_2m_1$$
$$= a_1b_1c_1 \cdots m_1m_2l_2 \cdots c_2b_2a_2,$$
$$a_1a_2 \cdot b_1b_2 \cdot c_1c_2 \cdot \cdots \cdot k_1k_2 \cdot l_1l_2 \times a_2b_1 \cdot b_2c_1 \cdot \cdots \cdot k_2l_1 \cdot l_2m$$
$$= a_1b_1c_1 \cdots k_1l_1ml_2k_2 \cdots c_2b_2a_2$$

show that every substitution is the product of two substitutions of order two that do not involve any element except those involved in the given substitution.* The given substitutions of order two are similar when the given cyclical product involves an odd number of letters; when this product involves an even number of letters, one of the given substitutions of order two contains one more transposition than the other. Hence every positive substitution is the product of two similar substitutions of order two. Since these two similar substitutions can be transformed into each other by some substitution we have the theorem:

---

* It may be observed that each of these substitutions transforms the given product into its inverse.

Every positive substitution is the commutator of two substitutions involving only elements that are contained in the given positive substitution.

From the following examples we see that any circular substitution of an odd order is the product of two similar circular substitutions which may have either one or three common elements,

$$a_1a_3 \cdot a_1a_2 = a_1a_3a_2$$

$$a_1a_2a_3 \cdot a_1a_2a_3 = a_1a_3a_2$$

$$a_1a_3a_4 \cdot a_1a_2a_5 = a_1a_3a_4a_2a_5$$

$$a_1a_2a_3a_4 \cdot a_1a_2a_3a_5 = a_1a_3a_4a_2a_5$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$a_1a_3a_4 \cdots a_{n+2} \cdot a_1a_2a_{n+3}a_{n+4} \cdots a_{2n+1}$$
$$= a_1a_3a_4 \cdots a_{n+2}a_2a_{n+3}a_{n+4} \cdots a_{2n+1}$$

$$a_1a_2a_3 \cdots a_{n+2} \cdot a_1a_2a_3a_{n+3}a_{n+4} \cdots a_{2n+1}$$
$$= a_1a_3a_4 \cdots a_{n+2}a_2a_{n+3}a_{n+4} \cdots a_{2n+1}$$

Hence we observe that any cycle of an odd order greater than three is the product of two smaller similar cycles each of which is of an odd order. It is evident that the two given similar cycles can always be transformed into the inverse of each other by some even substitution; *i. e., every positive cycle whose degree exceeds three is a commutator of two positive substitutions which do not involve any elements except those which are contained in the given cycle.*

From the following equations, in which $0 < 2a < n + 1$,

$$a_1a_2 \cdots a_{n+1} \cdot a_1\,a_{n+2}\,a_{n+3} \cdots a_{2n}\,a_{2\alpha+1}$$
$$= a_1a_2 \cdots a_{2\alpha} \cdot a_{2\alpha+1}\,a_{2\alpha+2} \cdots a_{2n}\,, \qquad n > 1$$

$$a_1a_2 \cdots a_{n+2} \cdot a_1a_2a_3a_{n+3}\,a_{n+4} \cdots a_{2n}\,a_{2\alpha+2}$$
$$= a_1a_3a_4 \cdots a_{2\alpha+1} \cdot a_{2\alpha+2}\,a_{2\alpha+3} \cdots a_{n+2}\,a_2a_{n+3} \cdots a_{2n}\,, \quad (n > 2),$$

we observe that two negative cycles of degrees $2a$ and $2(n - a)$ are always the product of two positive cycles involving the same elements and of degree $n+1$ if n is even but of degree $n+2$ if n is odd. Since not all the elements of these positive cycles are common, one of them may always be transformed into the inverse of the other by some positive substitution involving the same elements.* We have now proved that every positive substitution which does not include a cycle of order three is the commutator of two positive substitutions that do not involve any element except those of the given positive substitution. If a positive substitution consists of a cycle of order three it is evidently a commutator of positive substitutions with five but no smaller number of elements. If it contains a cycle of order three and some other elements it can clearly be expressed as the product of two positive substitutions such that the one can be transformed into the inverse of the other by a positive substitution in the same elements. Hence

THEOREM I. *Every substitution of the alternating group of degree* n $(n > 4)$ *is a commutator of two substitutions of the same group.*

The inverse of each of the two similar cycles having one common element whose product may be made any desired odd cycle can evidently be transformed into the other by a substitution of any arbitrary order. Hence any odd cycle is the product of two similar substitutions whose order is entirely arbitrary. Similarly we observe that any two negative cycles are the product of two similar substitutions whose order is arbitrary. In particular, any given operator is the product of two operators of the same arbitrary order. Hence the commutators of two operators may be employed to give a very simple proof of this special case of the general theorem. If l, m, n are any integers greater than unity it is always possible to find three operators L, M, N whose orders are l, m, n and which satisfy the relation $L = MN$.

The holomorph of a cyclical group is evidently isomorphic to an abelian group with respect to this cyclical subgroup.† When the order of the given cyclical group is odd the given holomorph must contain systems of conjugate operators each of which includes as many operators as the order of the given cyclical subgroup. When this order is even the number of conjugates in a system must be half as large. Hence

THEOREM II. *If the order of a cyclical group is odd, it is the commutator subgroup of its holomorph and all its operators are*

---

*Since $ab \cdot cd$ and $abcd \cdot ef$ are the commutators of abc, $ad \cdot bc$ and $bfdec$, $ac \cdot ef$ respectively, we do not need to consider the cases when $n = 2$ or 3. In fact, the general method is not directly applicable to these cases.

† Cf. Burnside, Theory of Groups, 1897, p. 240.

commutators of this holomorph. When this order is even, the commutator subgroup of the holomorph includes half of the operators of this cyclical group and all of these operators are commutators of this holomorph.

Since $s^{-1}t^{-1}$ is similar to ts and this is similar to st, we observe that the commutator of two operators is similar to the commutator formed by means of one of these operators and the inverse of the other. The preceding results are, in part, supplementary to those contained in the paper "On the commutator groups," BULLETIN, Vol. IV., pp. 135–139.

CORNELL UNIVERSITY.

THE CALCULUS OF GENERALIZATION.

Calcul de Généralisation. Par G. OLTRAMARE, Doyen de la Faculté des Sciences de l'Université de Genève. Paris, A. Hermann, 1899. 8vo, viii + 191 pp.

THIS work is the magnum opus of the venerable dean of the faculty of sciences, of Geneva, who is probably the oldest living pupil of Cauchy. The volume recapitulates and completes the works of the author published during the last twenty years.

Oltramare regards every function as developable in a series of exponentials; thus, a designating an independent variable, he puts

$$\varphi(a) = A_\alpha e^{\alpha a} + A_\beta e^{\beta a} + A_\gamma e^{\gamma a} + \cdots,$$

where $\alpha, \beta, \gamma, \cdots$ are any constants real or imaginary, in number finite or infinite. He adopts the shorter notation Ge^{au} for the series $\Sigma A_u e^{au}$, u taking successively the values $\alpha, \beta, \gamma, \cdots$, and the equation

$$\varphi(a) = Ge^{au}$$

then expresses that the function $\varphi(a)$ is generated from e^{au} by generalization.

This is in fact an extension of Liouville's generalized derivatives; the latter defined the derivative of index μ of the function $\varphi(a)$ as given by the equation

$$\frac{d^\mu \varphi}{da^\mu} = A_\alpha e^{a\alpha}\alpha^\mu + A_\beta e^{a\beta}\beta^\mu + A_\gamma e^{a\gamma}\gamma^\mu + \cdots;$$

and Oltramare proposed to construct a more general calculus* by considering expressions of the form

---

* See Laisant's introduction to Oltramare's lithographed essay on the calculus of generalization published previously.

$$A_\alpha e^{a\alpha}\psi(\alpha) + A_\beta e^{a\beta}\psi(\beta) + A^\gamma e^{a\gamma}\psi(\gamma) + \cdots;$$

the functions ψ not containing a; this expression he represents by the symbol $Ge^{au}\psi(u)$, which is clearly the result of an operation effected on $\varphi(a)$ if Ge^{au} represents $\varphi(a)$. Thus, for example, $Ge^{au}u^\mu$ is the μth differential coefficient of the function $\varphi(a)$. Since $e^{au}\psi(u)$ is any function of a and u, $Ge^{au}\psi(u)$ is equivalent to $Gf(u)$, and this last is the result of a certain operation on $\varphi(a)$; this operation is the object of the calculus of generalization; the inverse operation is called degeneralization.

These notions and their usefulness will perhaps be made clearer by a few examples taken from the treatise.

1° Assuming the identity

$$e^{au} = 1 + \frac{au}{1} + \frac{a^2u^2}{1.2} + \cdots + \frac{a^nu^n}{n!} + \cdots,$$

let us define φ by means of the equation

$$Ge^{au} = \varphi(x + a);$$

from this equation we deduce the following

$$G1 = \varphi(x),\ Gu = \varphi'(x),\ Gu^2 = \varphi''(x), \cdots, Gu^n = \varphi^{(n)}(x),$$

hence we have

$$\varphi(x + a) = \varphi(x) + \varphi'(x)\frac{a}{1} + \varphi''(x)\frac{a^2}{1.2} + \cdots$$
$$+ \varphi^{(n)}(x)\frac{a^n}{n!} + \cdots,$$

Taylor's series.

Conversely, putting $\varphi(x)$ equal to e^{xu} we obtain

$$\varphi(x + a) = e^{(x+a)u}, \quad \varphi(x) = ue^{xu}, \cdots, \varphi^{(n)}(x) = u^ne^{xu},$$

which, put in Taylor's series, give the development of e^{au} in series.

2° Let it be proposed to determine the function $\Psi(x, y)$ which satisfies the equation of finite differences

$$\Psi(x, y) - a\Psi(x - 1, y + 1) = 0.$$

We admit that this relation has been obtained by the generalization (using this word always in the technical sense ascribed to it above) of an identity which it is easy to determine.

As every function of two variables can be expressed by the generalizing function Ge^{xu+yv} let us put

$$\Psi(x, y) = Ge^{xu+yv}.$$

This value substituted in the proposed equation gives

$$Ge^{xu+yv}(1 - ae^{-u+v}) = 0 ;$$

whence, suppressing the factor e^{xu+yv} and degeneralizing, we have

$$1 - ae^{-u+v} = 0 ;$$

from which relation we deduce

$$e^u = ae^v ;$$

raising to the xth power and multiplying by e^{yv} we have

$$e^{xu+yv} = a^x e^{(x+y)v},$$

which generalized gives

$$\Psi(x,y) = a^x \varphi(x + y),$$

the integral of the proposed equation, φ being an arbitrary function.

3° As another example take the integration of the linear equation

$$\frac{dz}{dx} + a\frac{dz}{dy} = bz.$$

Writing $$z = Ge^{xu+yv},$$

which is permissible since every function of two variables can be put in this form, and substituting this value for z in the given equation, the latter becomes

$$Ge^{xu+yv}(u + av - b) = 0,$$

whence, degeneralizing this expression and suppressing the factor e^{xu+yv}, we have

$$u + av - b = 0 ;$$

and hence $$e^{u+av-b} = 1, \text{ or } e^u = e^{b-av} ;$$

which raised to xth power and multiplied by e^{yv} gives

$$e^{xu+yv} = e^{bx+(y-ax)v},$$

then the value of z will be expressed by generalizing the two members ; we shall have

$$z = Ge^{xu+yv} = e^{bx}Ge^{(y-ax)v} = e^{bx}\Psi(y - ax),$$

where Ψ is an arbitrary function.

4° As a last example consider the generalizing function of the numbers of Bernouilli :

$$\frac{ue^u}{e^u - 1} = e^{Bu} = 1 + B_1 u + B_2 \frac{u^2}{1.2} + B_3 \frac{u^4}{4\,!} + \cdots;$$

by replacing u by hu we deduce

$$hue^{hu} = e^{(h + Bh)u} - e^{Bhu},$$

then putting $Ge^{xu} = F(x)$ as defining equation of the function $F(x)$ we have, by generalizing both members of this identity,

$$F(x + h + Bh) - F(x + Bh) = hF'(x + h),$$

the fundamental formula of the theory of Bernouillian numbers.

By multiplying both numbers of the identity above by $e^{nu} - 1$ we have

$$\frac{ue^u(e^{nu} - 1)}{e^u - 1} = e^{(n + B)u} - e^{Bu} = u(e^u + e^{2u} + \cdots + e^{nu})\,;$$

generalizing both members and putting $Ge^{xu} = f(x)$, we find the summation formula of Maclaurin

$$f'(1) + f'(2) + \cdots + f'(n) = f(n + B) - f(B).$$

These examples show the part to be played by the calculus of generalization when the operations are linear; on the other hand, when the operations are non-linear the calculus is no longer applicable; it may then be regarded as a theory of linear operations.

Its general problem as regards its own mechanism is to determine the value of the expression $G\Psi(u, v, w, \ldots)$ represented by the development of

$$\Psi(D_x, D_y, D_z, \ldots)\,\varphi(x, y, z, \ldots),$$

the function φ being defined by the relation

$$\varphi(x + a,\ y + b,\ z + c,\ \ldots) = Ge^{au + bv + cw + \cdots}$$

The order followed in the construction of the various forms $G\Psi$ and the applications to be made of these tools appear from the chapters of Oltramare's work which are in order 1° calculus of generalization, 2° generalization (technical use of word always) of functions of a single variable, 3° generalization of functions of several variables, 4° of rational functions, 5° of exponential functions, 6° of logarithmic functions, 7° of circular functions, 8° of various forms of transcendental functions, 9° expression of the integral "$\int\varphi(x)dx$" as a definite integral, 10° differentiation and integration with fractional indices, 11° transformations of series into definite integrals and reciprocally, 12° expression of the sums of certain general series as definite integrals, 13° integration of equations, 14° determination of a particular integral of every linear differential, difference, or partial differential equation, having constant coefficients and second member, 15° inverse calculus of definite integrals, 16° integration of linear differential or difference equations with constant coefficients, 17° integration of linear partial differential or difference equations with constant coefficients, 18° integration of certain partial differential equations with variable coefficients, 19° integration of equations of mixed differences, 20° integration of simultaneous equations, 21° integration of certain equations which can be transformed into linear equations.

One of the principal advantages which the author claims for the method is the application which can be made of it to the integration of differential equations. Its rôle in higher analysis he likens to that of logarithms in numerical reckoning. It makes the integration of equations, difference and partial, single or simultaneous, depend on the solution of algebraic equations; moreover, it permits of determining in general the maximum to the number of arbitrary functions entering an integral in order that it be complete, and in particular cases the process can determine the exact number of these arbitrary functions.

The general process employed by this calculus for the integration of equations consists in representing the known and unknown functions under the generalized forms, substituting these in the equations, and deducing the integrals by the aid of generalization.

E. O. Lovett.

Princeton University.

SHORTER NOTICES.

Jacob Steiner's Vorlesungen über synthetische Geometrie. Zweiter Theil: Die Theorie der Kegelschnitte gestützt auf projective Eigenschaften. Auf Grund von Universitätsvorträgen und mit Benutzung hinterlassener Manuscripte Jacob Steiner's. Bearbeitet von Heinrich Schröter. Dritte Auflage, durchgesehen von Rudolf Sturm. Teubner, Leipzig, 1898. xvii + 537 pp. Price, 14 marks.

The fact that this new edition of Steiner's lectures, edited and published by Schröter in 1866, has been prepared by

Professor Sturm at the request of the publishers, shows that the demand which called for a second edition in 1876 has not been satisfied by the appearance of more modern books on the subject. It is a curious coincidence that Reye's "Geometrie der Lage" also made its first appearance in 1866 and went into a new edition in 1876.

To those not familiar with the scope of the work a short sketch of the contents may be of interest. It was Steiner's intention to publish his investigations in synthetic geometry in five parts, and the first, the important work: "Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander" was published in 1832. The remaining four parts never appeared, but much of the subject matter that was to have formed the fifth part was included in a course of lectures which Steiner delivered several times at the University of Berlin. In 1852 Schröter attended a course of lectures by Steiner "Ueber die neueren Methoden der synthetischen Geometrie" and the notes he took of these lectures supplemented by manuscript notes left by Steiner form the basis of the present volume. In the "Systematische Entwickelung" (1832) Steiner, to avoid being thought too radical, had followed the old method, considering a conic as a section of a right circular cone, although, as he says in 1836, he fully felt that this method was unsuitable for his purpose; in his later unwritten work, part 5, he intended to recapitulate briefly the properties of projective ranges and pencils and from these to deduce his definition of the conic and the theories of involution and reciprocal polars. This plan Schröter has carried out in the first two sections of the work under discussion, following Steiner's arrangement almost exactly. Steiner had come to the conclusion that the use of trigonometry might be avoided in the development of the projective properties of lines and pencils and that this new geometry might, with great advantage, be incorporated with the most elementary geometry. Trigonometrical expressions are used, therefore, only in so far as they are needed for the further development of the subject, but the use of metrical properties is not avoided nor is an endeavor made to build up a logical structure on the ground work of fundamental geometrical axioms. Parallel lines and circles, are introduced without definition, algebra and trigonometry are used freely, the principle of duality is adopted without discussion, and the axes, area, curvature, and other metrical properties of conics are investigated.

The third section, of 178 pages, is devoted to an elementary treatment of pencils and ranges of conics; the different kinds of conics are discriminated for cases of real and imaginary points and the pencil and range are investigated separately, little use being made of the principle of duality.

In the fourth section, of 112 pages, the polar system of the plane and the net of conics are discussed in some detail.

The book, as a text book, has the defect so often found in books compiled from lecture notes—namely, a lack of continuity in notation. This is especially noticeable in the treatment of fundamental points, which are denoted by four or five different sets of letters in as many articles. Professor Sturm has improved the edition greatly in this respect and has modernized the terminology. There are one or two misprints, for instance on page 132, and there is some slight confusion in the proof given on page 60, but the edition seems on the whole remarkably free from errors.

ISABEL MADDISON.

Compositions d'Analyse, Cinématique, Mécanique et Astronomie, données depuis 1889 à la Sorbonne pour la licence ès sciences mathématiques. Par E. VILLIÉ. Paris, Gauthier-Villars, 1898. 8vo, x + 299 pp.

AN American reader will probably peruse this volume more for the insight it gives as to the standard required from French students just finishing their courses in mathematics, than for the matter it contains. The questions are those set in the final examinations for students who have finished the training which corresponds to a college course here, and who expect to go out immediately and teach. The candidate who can successfully grapple with such problems in a limited time must have had a good preparation and be well equipped, as far as knowledge goes, for the profession of teaching mathematics.

M. Villié has previously published two similar volumes of worked-out problems; the present one contains the solution of questions set since 1889. To American and English students, these volumes will be perhaps superfluous, our text books generally doing what is needed in that direction; in France, so far as may be gathered from their mathematical publications, this part is left to the instructor and the volume before us doubtless serves a very useful purpose. In analysis we have chapters containing solved problems in integration, differential equations, orthogonal trajectories, radii and lines of curvature, asymptotic and geodesic lines, and imaginary variables. The last will be of assistance to any instructor, consisting as it does, of various well chosen examples in integration round a circuit. In the portions containing kinematics and mechanics we find problems similar to those given by, for example, Routh in his various books. The fourth section contains numerical problems in spherical astronomy.

The admirable clearness of explanation and care of details are to be highly commended.

ERNEST W. BROWN.

Keplers Traum vom Mond. Von LUDWIG GÜNTHER. Leipzig, B. G. Teubner, 1898. xxii + 185 pp.

WE should consider Herr Günther's edition of Kepler's "Dream" rather incommensurate with the importance of his text if it had not evidently been a labor of love on his part. The German translation (the Latin of Kepler is not given) occupies 19 pages, that of Kepler's notes perhaps another 50 pages, the editor being responsible for the rest. The work, or rather scientific romance, is little known and it cannot be said to be of importance except in so far as it shows what stage Kepler had reached in astronomical ideas. Its interest is, in fact, less historical than antiquarian. As a romance it would hardly find many readers at the present day, being chiefly details of the appearance of the sky as seen from the moon. The editor is scarcely fair in comparing it with the well-known tale of "Julius" (!) Verne—the first inspiration of many a schoolboy—which he dismisses with a contemptuous remark. The fullness of the notes on astronomical and other matters will satisfy the most exacting of critics.

ERNEST W. BROWN.

A Short Table of Integrals. By B. O. PEIRCE. Boston, Ginn and Co., 1899. 8vo, 134 pp.

THIS is a revised and much enlarged edition of the author's well-known table of integrals, forming a very useful handbook of formulæ which in many cases are too long and complicated to remember. It constitutes a labor saving volume of considerable value. There are 897 formulæ in all. These include the indefinite integrals of many rational and irrational algebraic and transcendental functions, formulæ of reduction, and the more important definite integrals. There are also numerous auxiliary formulæ, for example, those arising in trigonometry, the principal relations between the elliptic integrals (Jacobian notation), and series for frequently occurring functions. In the last the author has been careful to state the limits within which the expansions are valid. At the end of the book are several numerical tables which include, beside the usual logarithmic and trigonometric ones, some four place tables of the elliptic integrals and of the gamma functions. In future editions a table of contents might be added.

ERNEST W. BROWN.

NOTES.

ATTENTION is called to the change in the hours of meeting of the AMERICAN MATHEMATICAL SOCIETY. Hereafter the morning session will open at 11 o'clock and the afternoon session at 2 o'clock. The Council will meet at 10:15 a. m.

A NEW list of members of the Society will be issued in January. Forms for furnishing necessary information have been sent to each member, and a prompt response will be of great assistance to the Secretary.

A meeting of the National academy of sciences was held at Columbia University, November 14–16, pure and applied mathematics being represented by Professors Cleveland Abbe, C. S. Hastings, E. S. Holden, A. E. Michelson, Simon Newcomb, Mr. C. S. Peirce, and Professor R. S. Woodward. The following mathematical papers were presented at the meeting: Professor R. S. WOODWARD: "The statical properties of the atmosphere," "A direct proof of the effect on the eulerian cycle of an inequality in the equatorial moments of inertia of the earth;" Mr. C. S. PEIRCE: "The definition of continuity," "Topical geometry in general," "The map-coloring problem."

THE preliminary programme for the November meeting of the London mathematical society announced the following mathematical papers: "Note on Clebsch's second method for the integration of a Pfaffian equation," by MR. J. BRILL; "On the forms of lines of force near a point of equilibrium"; "The reduction of conics and quadrics to their principal axes by the Weierstrassian method of reducing quadratic forms;" and "On the reduction of a linear substitution to a canonical form, with applications to linear differential equations and quadratic forms, by Mr. T. J. I. Bromwich; "The abstract group isomorphic to the symmetric group of k letters," by Dr. L. E. Dickson; "On Ampère's equation $Rr + 2Ss + Tt + U(rt-s^2) = V$," by Professor A. C. Dixon; "Certain correspondences between spaces of n dimensions," by Professor E. O. Lovett; "The fundamental solutions of the indeterminate relation $\lambda x \geqq \mu y$," by Major P. A. MacMahon.

At the meeting of the Cambridge philosophical society held October 30th, 1899, a mathematical paper, "On semiconvergent series," was presented by Mr. W. McF. Orr.

Beginning with the year 1900, the journal *Bibliotheca Mathematica*, devoted to the history of the mathematical sciences, founded in 1887 by G. Eneström, of the library of the University of Stockholm, will be published by B. G. Teubner, of Leipzig, under the editorial supervision of M. Eneström. In the future the form of the journal will be that of the *Mathematische Annalen*, and each volume will contain about five hundred and fifty pages. The editor solicits not only historical articles, but also all items of interest relative to current mathematical activity. Manuscripts may be written in German, French, English or Italian, and acceptable contributions will be accorded honoraria of twenty marks per sixteen printed pages, together with from ten to twenty reprints. The editor's address is Brahegatan 43, Stockholm.

The publishing house of B. G. Teubner, Leipsic, announces that Vorlesungen über die Geschichte der Trigonometrie, by Dr. A. von Braunmühl; and Vielecke und Vielflache, Theorie und Geschichte, by Dr. M. Brückner, are in preparation.

Messrs. Longmans, Green and Company, of New York, are issuing the fourth edition of Burnside and Panton's Theory of equations in two volumes; the first volume is ready and contains all that is usually given in elementary works on the theory of equations, together with a short chapter on complex numbers; the second volume is to include those subjects which are more appropriately included under the title of modern higher algebra.

Among the mathematical numbers of Harper's Scientific Memoirs, published by Messrs. Harper and Brothers, of New York, are The second law of thermodynamics, edited by Professor W. F. Magie; The law of gravitation, by Professor A. S. MACKENZIE; The wave-theory of light, by Professor HENRY CREW. The first of these is already in print.

PARIS FACULTY OF SCIENCES. The following mathematical courses are offered during the first semester of the current academic year, each course occupying two hours per week:—By Professor G. DARBOUX: Fundamental principles of infinitesimal geometry.—By Professor E. GOURSAT: Definite integrals and analytic functions.—By Professor P. APPELL: General laws of equilibrium and of motion.—By Professor H. POINCARÉ: The lunar theory.—By Professor V. J. BOUSSINESQ: Mechanical properties and motions of fluids.—By Professor G. KOENIGS: Kinematics of rigid or deformable bodies, applications to the study of machines.—Unassigned: Elements of analysis and mechanics.—By M. H. ANDOYER: General theory of planetary perturbations. Conferences will be conducted by Messrs. RAFFY, HADAMARD, PUISEUX, ANDOYER, and BLUTEL.

For the second semester the following courses are announced:—By Professor É. PICARD: Partial differential equations.—By Professor E. GOURSAT: Differential equations.—By Professor P. APPELL: General laws of motion of systems; Analytic mechanics; Hydrostatics and hydrodynamics.—By Professor BOUSSINESQ: Theory of waves.—By Professor KOENIGS: Theory of machines.—By M. RAFFY: Differential equations and their applications to mechanics and physics.

CAMBRIDGE UNIVERSITY. Dr. E. J. ROUTH and Mr. A. N. WHITEHEAD are moderators and Messrs. E. G. GALLOP and P. C. GAUL examiners for part I. of the mathematical tripos 1900; the examiners for part II. of the same year are Professor W. BURNSIDE and Messrs. H. F. BAKER, G. T. WALKER and H. M. MACDONALD. At St. John's College, Mr. W. A. HOUSTON, fifth wrangler 1896 and Smith's prizeman, 1898, has been elected to a fellowship in mathematics, and Mr. F. W. B. FRANKLAND, third wrangler in 1897, has received a similar appointment at Clare College.

A LEGACY left by one of its members, S. THOMASO VALLAURI, has enabled the Turin academy of sciences to establish a prize of thirty thousand lire to be conferred on the Italian or foreign scientist who shall have published during the interval between January 1, 1899, and December 31, 1902, the most remarkable and celebrated work on any of the physical sciences, interpreting the phrase "physical sciences" in its largest sense. The prize will be awarded one year after the expiration of this interval. Neither resident nor non-resident national members of the Turin academy are eligible to the competition. No manuscripts will be received, nor will competing works be returned. The president of the academy is S. GIUSEPPE CARLE and S. A. NACCARI is secretary of the section of the physical sciences, mathematical and natural.

THE subject of Professor L. FUCHS's inaugural address as rector of the University of Berlin for the current year was "The reciprocal relations of pure science and technology."

LORD RAYLEIGH, professor of natural philosophy in the Royal Institution of Great Britain, received the Copley medal at the anniversary meeting of the London Royal Society this year for his investigations in mathematical and physical science.

MR. R. F. MUIRHEAD has been elected president of the Edinburgh mathematical society for the coming year.

DR. J. B. FAUGHT has been promoted from an instructorship to an assistant professorship at the University of Indiana.

DR. H. LIEBMANN has qualified as privat docent in mathematics at the University of Koenigsberg.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDRÉEV (K. A.). Elementary course in analytic geometry. 2nd edition. (Russian.) Kharkov, 1896. R. 3.20

BIANCHI (L.). Lezioni sulla teoria dei gruppi di sostituzioni e delle equazioni algebriche secondo Galois. *Pisa*, 1899. 8°. 283 pp. Fr. 10.00

——. Vorlesungen über Differentialgeometrie. Autorisierte deutsche Uebersetzung von G. Lukat. 3te (Schluss-) Lieferung. Leipzig, Teubner, 1898. 8vo. Pp. 1–16 and 529–659. M. 4.00

BLUMENTHAL (O.). Ueber die Entwicklung einer willkürlichen Funktion nach den Nennern des Kettenbruches für $\int_{-\infty}^{\infty}\frac{\phi(\xi)d\xi}{z-\xi}$. (Diss.) Göttingen, 1898. 8vo 57 pp.

BOBYNIN (V. V.). The section of mathematics, astronomy and physics at the first nine meetings of the Association of Russian naturalists and physicians; its objects and activity. Part I: The first, second, third and fourth meetings. (Russian.) Moscow, 1896. 8vo. 169 pp.

Boussinesq (J.). Infinitesimal analysis, translated by A. P. Nenachev. Vol. I: Differential calculus. Part 1: Elementary part. (Russian.) Moscow, 1899. 8vo. 312 pp.

Bremiker (H.). Sur la transformation des courbes algébriques en général et sur celle des courbes du troisième ordre en particulier (d'après des cours de M. Weierstrass et de M. Bruns). (Progr.) Berlin, 1899. 4to. 19 pp.

Cahen (A.). Sur la formation explicite des équations différentielles du premier ordre dont l'intégrale générale est une fonction à un nombre fini de branches. (Thèse.) Paris, Gauthier-Villars, 1899. 4to. 99 pp.

Cajori (F.). Notes on the history of logarithms. (Sonderabdruck aus *Abhandlungen zur Geschichte der Mathematik*, IX, pp. 33–39.) Leipzig, Teubner, 1899. 8vo.

Chelpanov (G.). The problem of space perception in its relation to the doctrine of à priori and innate ideas. Part I: The idea of space from the point of view of psychology. (Russian.) Kiev, 1896. 8vo. 15 + 387 pp. R. 2.50

De Morgan (A.). Elementary illustrations of the differential and integral calculus. New edition. Chicago, The Open Court Publishing Co., 1899. 12mo. 8 + 144 pp. Cloth. $1.00

Dolbnia (I.). Memoir on the theory of Abelian integrals, presented to the Council of the Mining Institute. (Russian.) St. Petersburg, 1896. 8vo. 155 pp.

Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Teil 1, Band I, Heft 4: F. W. Meyer, Invariantentheorie (Schluss); C. Runge, Gleichungen, Separation und Approximation der Wurzeln; K. T. Vahlen, Gleichungen, rationale Funktionen der Wurzeln, symmetrische und Affektfunktionen; O. Hölder, Gleichungen, Galois'sche Theorie mit Anwendungen. Leipzig, Teubner, 1899. 8vo. 353–512 pp. M. 4.50

Ermakov (V. P.). Theory of Abelian functions. (Russian.) Kiev, 1897. 8vo. 120 pp.

Filippov (M. M.). Elementary theory of probabilities. (Russian.) St. Petersburg, 1896. R. 0.40

Fleury (H.). Théorie rationnelle de l'infini mathématique et du calcul infinitésimal. 2e édition, augmentée d'une théorie nouvelle du problème analytique des tangentes. Paris, 1899. 8vo. Fr. 10.00

Frankenbach (F. W.). Die Anwendung trimetrischer Punktkoordinaten auf die merkwürdigen Punkte des Dreiecks. (Progr.) Liegnitz, 1899. 8vo. 39 pp.

Gerlach (R.). Die Metrik in projektivischen Koordinaten. Zürich, 1899. 8vo. 112 pp. M. 2.00

Glage (F.). Anwendung der Gruppentheorie auf die irreduzibeln Gleichungen vom sechsten Grade. (Diss.) Königsberg, 1899. 8vo. 96 pp.

Goldscheider (F.). Ueber die Gauss'sche Osterformel. Teil II. (Progr.). Berlin, 1899. 4to. 30 pp.

Gottschalk (A.). Beiträge zur konformen Abbildung Pascal'scher Schnecken. (Progr.) Burgsteinfurt, 1899. 4to. 16 pp., 1 plate.

Grassmann (H.). Punktrechnung und projektive Geometrie. Teil III: Die linearen Verwandtschaften in der Ebene. Festschrift. Halle, 1898. 4to. 54 pp.

Gundermann (G.). Die Zahlzeichen. Giessen, 1899. 4to. 50 pp. M. 2.00

Haas (A.). Lehrbuch der Integralrechnung. Teil II: Anwendung der bestimmten Integrale auf Quadratur, Rektifikation, Komplanation und Kubatur, sowie auf Aufgaben aus der Mechanik und Technik. Mit 246 vollständig gelösten Aufgaben, 163 Figuren und 137 Erklärungen, nebst ausführlichem Formelverzeichnis. Zum Selbststudium und zum Gebrauch an Lehranstalten bearbeitet. Stuttgart, Maier, 1899. 8vo. 8+284 pp. (Kleyers Encyklopädie der gesammten mathematischen, technischen und exakten Naturwissenschaften.) M. 9.00

Henschel (A.). Untersuchungen über Berührungskugeln. (Progr.) Weimar, 1899. 4to. 16 pp. M. 1.50

Hill (H.). South African arithmetic. London, 1899. 8vo. Cloth. 4s.

Hochheim (E.). Ueber eine Art der Erzeugung der Kurven dritter Klasse mit einer Doppeltangente. (Diss.) Halle, 1899. 8vo. 50 pp.

Hölder (O.). See Encyklopädie.

Jahrbuch über die Fortschritte der Mathematik, begründet von C. Ohrtmann. Unter besonderer Mitwirkung von F. Müller und A. Wangerin herausgegeben von E. Lampe. Vol. 28: 1897. Heft 2: Pp. 417-608. Berlin, Reimer, 1899. 8vo. M. 6.00

Juga (G.). Die cyklischen Minimalflächen. (Diss.) Strassburg, 1898. 8vo. 39 pp.

Kaplan (I. T.). On the distribution in the natural series of the numbers prime to a given one. (Russian.) Odessa, 1897. 8vo. 56 pp.

Kowalewski (G.). Die primitiven Transformationsgruppen in fünf Veränderlichen. Habilitationsschrift. Leipzig, 1899. 8vo. 76 pp.

Lampe (E.). See Jahrbuch.

Loewy (A.). Ueber bilineare Formen mit konjugirt imaginären Variablen. Habilitationsschrift. Freiburg, 1898. 4to. 68 pp.

Lovett (E. O.). The theory of perturbations and Lie's theory of contact transformations. (Diss.) Leipzig, 1898. 8vo. 193 pp.

Lukat (M.). See Bianchi (L.).

Mansion (P.). Einleitung in die Theorie der Determinanten für Gymnasien und Realschulen. Aus der 3ten französischen Auflage übersetzt. Leipzig, Teubner, 1899. 8vo. 40 pp. M. 1.00

Meyer (F. W.). See Encyklopädie.

Minin (A. P.). Collection of problems in the differential and integral calculus. (Russian.) St. Petersburg, 1899. 8vo. 78 pp. M. 2.00

Möhring (E.). Ueber die Abbildung der fünf platonischen Körper in centrierten Systemen sphärischer Flächen. (Diss.) Rostock, 1898. 8vo. 36 pp., 2 plates. M. 1.80

Müller (F.). See Jahrbuch.

Nau (F.). See Sabokt (S.).

Nekrasov (P. A.). Application of algebra to geometry. 2d edition, with the addition of a chapter on a concrete interpretation of Lobachevsky's geometry. (Russian.) Moscow. 1896. 8vo. R. 1.50

Nenachev (A. P.). See Boussinesq (J.).

Obréimov (V. I.). Mathematical sophisms. 3d edition, corrected and augmented. (Russian.) St. Petersburg. 1898. 8vo. 92 pp. R. 0.40

Ohrtmann (C.). See Jahrbuch.

Papperitz (E.). Die Mathematik an den deutschen technischen Hochschulen. Beitrag zur Beurteilung einer schwebenden Frage des höheren Unterrichtswesens. Leipzig, Veit, 1899. 8vo. 4 + 68 pp., 1 plate. M. 1.50

Pascal (E.) Repertorio di matematiche superiori (definizioni, formole, teoremi, cenni bibliografici). Vol. I.: Analisi. Milano, Hoepli, 1898. 16mo. 16 + 642 pp. (Manuali Hoepli.) Fr. 6.00

Peirce (B. O.). A short table of integrals. Revised edition. Boston, Ginn & Company, 1899. 8vo. 134 pp. $1.00

Pochinsky (N. S.). New and simple method for solving numerical algebraic equations of any integral degree. Part I. Solution of equations with one variable. (Russian.) St. Petersburg, 1896. R. 1.50

Porfiriev (N. I.). Syllabus of a course of public lectures on the calculus of probabilities. (Russian.) Kazan, 1896. 8vo. 23 pp.

Poretsky (P. S.). Modern science and the academician Imshenetsky. (Russian.) Kovno, 1897. 8vo. 19 pp.

Raffy (L.). See Robin (G.).

Reinhardt (N. V.). Non-Euclidean geometry and positivism. (Russian.) Kazan, 1897. 16mo. 56 pp. R. 0.35

Robin (G.). Oeuvres scientifiques, réunies et publiées, sous les auspices du ministère de l'instruction publique, par L. Raffy. (Physique mathématique.) Paris, Gauthier-Villars, 1899. 8vo. 6+150 pp.

Rossevich (I. N.). See Serret (J. A.).

Rudert (E.). Grundlagen zu einer Geometrie der Kugel nach Grassmann's Ausdehnungslehre. (Progr.) Leipzig, 1899. 4to. 44 pp

Runge (C.). See Encyklopädie.

Sabokt (S.). Le traité sur l'astrolabe plan de Sévère Sabokt, écrit au VII^e siècle, d'après des sources grecques, et publié pour la première fois, avec traduction française, par F. Nau. Paris, Leroux, 1899. 8vo. 116 pp. (Extrait du *Journal Asiatique.*)

Scarpis (N.). Primi elementi della teoria dei numeri. Milano, Hoepli, 1897. 16mo. 8+152 pp. (Manuali Hoepli.) Fr. 1.50

Schilling (F.). Ueber neue kinematische Modelle, sowie eine neue Einführung in die Theorie der cyklischen Kurven. Halle, Schilling, 1899. 8vo. 15 pp., 2 plates. M. 1.20

Schultz (E.). Integrationsmöglichkeiten der Hamiltonschen partiellen Differentialgleichung mit drei Variablen. (Progr.) Stettin, 1898. 4to. 16 pp.

Serret (J. A.). Higher Algebra, translated by I. N. Rossevich. Part I. General properties and numerical solution of equations. Part II: Symmetric functions. Edited by M. O. Wolf. (Russian.) St. Petersburg, 1897. 8vo. 615 pp. R. 3.00

Servant (M.). Essai sur les séries divergentes. (Thèse.) Paris, Gauthier-Villars, 1899. 4to. 65 pp.

Shaposhnikov (N. A.). Attempt at a mathematical presentation of the ideas and results of ethics. (Russian.) Moscow, 1896. 8vo. 2 + 30 pp. R. 0.20

Shidlovsky (V.). On the doctrine of the differential and integral, (Russian.) St. Petersburg, 1896.

Spindeler (K.). Ein Beitrag zur Einführung in das Gebeit der räumlichen Konfiguration. Fortsetzung. (Progr.) Diedenhofen, 1898. 8vo. 28 pp.

Stiner (G.). Die dreimal berührenden Ellipsen der Steinerschen Hypocykloide. (Progr.) St. Gallen, 1899. 4to. 17 pp., 5 plates.

Tikhomandritsky (M.). Differential and integral calculus, with exercises. 2d edition. Part 2. (Russian.) St. Petersburg, 1899. 8vo. Price of the complete work, in 2 vols. M. 8.00

Tzitzéica (G.). Sur les congruences cycliques et sur les systèmes triplement conjugués. (Thèse.) Paris, Gauthier-Villars, 1899. 4to. 57 pp.

Vahlen (K. T.). See Encyklopädie.

Vallée Poussin (Ch. I. de la). Cours d'analyse infinitésimale. Ie et IIe parties. Paris, 1899. 4to. 2 + 212 + 222 pp. Fr. 19.00

Van Vleck (E. B.). On the determination of a series of Sturm's functions by the calculation of a single determinant. (*Annals of Mathematics*, 2nd series, Vol. I.) Salem, 1899. 4to. 13 pp.

——. On certain differential equations of the second order allied to Hermite's equation. (*American Journal of Mathematics*, Vol. 21, pp. 126–167.) 1899. 4to, 9 plates.

Voronoi (G.). On a certain generalization of the algorithm of continued fractions. (Russian.) Warsaw, 1896. 8vo. 219 pp. R. 1.50

Wangerin (A.). See Jahrbuch.

Wolf (M. O.). See Serret (J. A.).

II. ELEMENTARY MATHEMATICS.

André (J. L.). See Buchanan (L. A.).

Arbuzov (V.), Minin (A.), Minin (V.), and Nazarov (D.). Collection of problems in algebra, for the use of pupils in the upper classes of secondary institutions. With a collection of examination problems proposed in various Russian educational districts. 4th edition. (Russian.) 1896. 8vo. 4 + 86 pp. R. 0.50

BERTRAND (J.). Theoretical arithmetic, adapted by N. Billbin. 4th edition. (Russian.) St. Petersburg, 1899. 8vo. 294 pp. M. 5.00

BILIBIN (N.). See BERTRAND (J.).

BREITMANN. The theory of combinations and Newton's binomial formula, for pupils in the upper classes of secondary schools. (Russian.) Odessa, 1895. 8vo. 31 pp. R. 0.40

BRIGGS (W.). Second stage mathematics. 2nd edition. London, Clive, 1899. 12mo. 474 pp. (Organized science series.) 3s. 6d.

BUCHANAN (L. A.) and ANDRÉ (J. L.). The algebraic solution of equations of any degree; a novel, simple, and direct method for the solution of equations of the *n*th degree. San Francisco, Cal., published for the authors by The Whitaker & Ray Co., 1899. 12mo. 4 + 61 pp. Limp Cloth. $0.80

BUSCH (F.). Vorbereitender Lehrgang in der Körperlehre. (Progr.) Arnsberg, 1899. 4to. 20 pp.

CASTLE (F.). Elementary practical mathematics. New York, The Macmillan Co., 1899. 16mo. 10 + 401 pp. Cloth. $0.80

CHAILAN (E.). Géométrie, à l'usage des élèves des classes de lettres, conforme au programme de la première partie du baccalauréat de l'enseignement secondaire classique. Paris, Poussielgue, 1899. 16mo. 8 + 240 pp. (Alliance des maisons d'éducation chrétienne.)

DAVIDOV (A.). Geometry for grammar schools, based on Diesterweg. 11th edition. (Russian.) Moscow, Dymnov, 1897. 8vo. 63 pp. R. 0.35

——. Elementary geometry, for use in the gymnasium. 18th edition. (Russian.) Moscow, Dymnov, 1897. 8vo. 348 pp. R. 1.35

DOLANSKY (A.). Is it possible to square the circle by geometrical construction with greater precision than heretofore attained? 1896.

EBERHARD. Ueber die im Pensum höherer Lehranstalten vorkommenden sogenannten imaginären Werte. (Progr.) Cassel, 1898. 4to. 58 pp., 1 plate.

EGOROV (F.). See MALININ (A.).

EICHHORN (A.). Sammlung von mathematischen Formeln und Regeln zum Gebrauche an höheren Lehranstalten. Lüneburg, Herold & Wahlstab, 1899. 12mo. 21 pp. M. 0.50

ESTEL (V.). See SÄRCHINGER (E.).

EVTUSHEVSKY (V.). Methodology of arithmetic, for the use of teachers' institutes, seminaries, etc. 14th edition. (Russian.) St. Petersburg, 1899. 8vo. 352 pp. M. 5.00

FRENCH (C. H.) and OSBORN (G.). Elementary algebra (to quadratics). London, Churchill, 1899. 12mo. 318 pp. 3s. 6d.

GÉRARD (L.). See NIEWENGLOWSKI (B.).

GHERSI (I.). Metodi facili per risolvere i problemi di geometria elementare. Milano, Hoepli, 1899. 16mo. 12 + 190 pp. (Manuali Hoepli.)

GOLDENBERG (A. N.). Methodology of elementary arithmetic. 13th. edition. (Russian.) St. Petersburg, 1899. 8vo. 192 pp. M. 2.00

HARRISON (J.) and BAXANDALL (G. A.). Practical plane and solid geometry for advanced students, including graphic statics. Adapted to the Kensington Syllabus. London and New York, Macmillan, 1899. 16mo. 12 + 557 pp. Cloth. $1.00

HEBEL (V.). Collection of geometrical exercises in computation, construction and proof, with articles supplementary to the course in elementary geometry. (Russian.) Published by the author, 1897. 8vo. 6 + 100 pp. R. 0.40

HECHT (W.) and LOTHER (H.). Beispiele und Aufgaben aus der Buchstabenrechnung. Resultate. Nürnberg, Korn, 1899. 8vo. 94 + 2 pp. Cloth. M. 1.25

HÖHR (D.). Lehrbuch der Arithmetik für Untergymnasien. Teil I: Für die erste und zweite Klasse. 2te Auflage. Hermannstadt, Krafft, 1899. 8vo. 8 + 142 pp. Cloth. M. 1.70

——. Teil II: Für die dritte und vierte Klasse. 2te Auflage. Hermannstadt, Krafft, 1899. 8vo. 6 + 92 pp. M. 1.36

IAKOVLEV (M. D.). The division of angles according to the rules of elementary geometry. (Russian.) St. Petersburg, 1897. 12mo. 9 pp.

IUREVICH (G. I.). Elementary geometry, with a collection of geometrical problems and a brief outline of surveying. For use in girl's gymnasia, teachers' seminaries and city schools. (Russian.) Dorpat, 1897. 8vo. 223 pp. R. 0.60

KHAILOV (N.). See MASING (K.).

KIRIANOV (I. I.). Derivation of the ratio of the circumference to the diameter, and consequently solution of the problem of the quadrature of the circle. (Russian.) Warsaw, 1896. 16mo 11 pp. R. 0.50

KISELEV (A.). Elementary geometry for secondary schools. With numerous exercises and a supplement: the principal methods of solving geometrical problems by construction. 5th edition. (Russian.) Moscow, 1897. 8vo. 311 pp. R. 1.25

KRYLOV (A. N.). Text-book of spherical trigonometry for naval schools. (Russian.) St. Petersburg, 1899. 8vo. 124 pp. M. 2.00

LOTHER (H.). See HECHT (W.).

MALININ (A.). Geometry and collection of geometrical problems. Text-book for country and city schools. 10th edition. (Russian.) Moscow, 1896. 8vo. 199 pp. R. 0.65

MALININ (A.) and EGOROV (F.). Geometry and collection of geometrical problems. Text-book for girls' schools and teachers' seminaries. 4th edition. (Russian.) Moscow, 1899. 8vo. 272 pp. R. 1.00

MASING (K.) and KHAILOV (N.). Systematic collection of algebraic problems. 3d edition. (Russian.) Moscow, 1897. 8vo. 339 pp. R. 1.25

°MATHEMATICS [for mechanics and engineers]. From *The Chicago Record*. New York, Doubleday & McClure Co., 1897–99. 12mo. 6 + 340 pp. Cloth. (Home study circle No. 2.) $1.00

MERCHINSKY (A.). The analytical method of progressions. (Russian.) Moscow, 1896. 8vo. 16 pp.

MININ (A.). See ARBUZOV (V.).

MININ (V.). Collection of geometrical problems, with a collection of problems to be solved by the simultaneous application of geometry and trigonometry. 6th edition. (Russian.) Moscow, 1896. R. 0.90

——. See ARBUZOV (V.).

MIRONOV (P.). Text-book of geometry, with the addition of: (1) questions for review, (2) geometrical exercises, and (3) developments of solids. 2d edition. (Russian.) Ufa, 1896. 8vo. Part I. 3 + 110 pp. Part II. 3 + 138 pp. R. 0.80

——. Text-book of geometry, with collection of geometrical problems. (Russian.) Ufa, 1897. 8vo. Part III. 184 pp. Part IV. 118 pp. R. 1.05

NAU (T.) Recueil de problèmes de trigonométrie rectiligne, renfermant en particulier 150 questions proposées aux baccalauréats depuis 1892. Première partie: Enoncés. Paris, Poussielgue, 1899. 8vo. 48 pp. (Alliance des maisons d'éducation chrétienne.)

NAZAROV (D.). See ARBUZOV (V.).

NEUMANN (C. E. O.). Formelbuch, enthaltend die hauptsächlichsten Formeln, Sätze und Regeln der Elementar-Mathematik, zum Gebrauche an Realschulen und Gymnasien übersichtlich zusammengestellt. 6te Auflage. Dresden, Axt, 1899. 16mo. 4 + 168 pp. Cloth. M. 1.50

NIEWENGLOWSKI (B.) et GÉRARD (L.). Cours de géométrie élémentaire, à l'usage des classes de l'enseignement moderne. Géométrie dans l'espace. Paris, Carré et Naud, 1899. 8vo. 267 pp.

——. Cours de géométrie élémentaire, à l'usage des classes de lettres. Géométrie dans l'espace. Paris, Carré et Naud, 1899. 8vo. 126 pp.

——. Cours de géométrie élémentaire, à l'usage des élèves de mathématiques élémentaires, de mathématiques spéciales, des candidats aux écoles du gouvernement et des candidats à l'agrégation. Géométrie dans l'espace. Paris, Carré et Naud, 1899. 8vo. 500 pp.

OSBORN (G.). See FRENCH (C. H.).

PRONTUARIO delle formule principali di trigonometria piana e sferica. Livorno, Fagiolini e Co., 1899. 8vo. 3 pp.

PRZHEVALSKY (E.). Collection of algebraical problems. 6th edition. (Russian.) Moscow, 1897. 8vo. 288 pp. R. 1.25

——. Plane trigonometry, with a collection of trigonometrical problems. 5th edition. (Russian.) Moscow, 1897. 8vo. 248 pp. R. 1.25

R. (A.). Syllabus of geometry, with the solution of the more difficult problems proposed in A. Davidov's Text-book of elementary geometry. (Russian.) Kiev, Johanson, 1897. 16mo. 125 pp. R. 0.40

ROHRBACH (C.). Vierstellige logarithmisch-trigonometrische Tafeln, nebst einigen physikalischen und astronomischen Tafeln, für den Gebrauch an höheren Schulen zusammengestellt. 2te Auflage. Gotha, Thienemann, 1899. 8vo. 36 pp. Boards. M. 0.60

RYBKIN (N.). Collection of trigonometric problems for secondary schools. 2nd edition. (Russian.) Moscow, 1897. 8vo. 79 pp. R. 0.40

——. Collection of stereometric problems requiring the application of trigonometry. 4th edition. (Russian.) Moscow, 1897. 8vo. 84 pp. R. 0.40

[illegible]ärchinger (E.) und Estel (V.). Resultate zur zweiten Auflage der Aufgabensammlung für den Rechenunterricht in den Unterklassen der Gymnasien. Erstes Heft: Sexta. Leipzig, Teubner, 1899. 8vo. 40 pp. Boards. M. 0.80

Sasso (M.). Formole dei quadrati e cubi dei polinomi; loro applicazioni per fare il quadrato e cubo dei numeri. Avellino, Pergola, 1899. 8vo. 22 pp. Fr. 1.00

Scharr (J.). Der erste geometrische Anschauungs-, Zeichen- und Rechenunterricht. Zum Gebrauche in einem Vorkursus des Geometrie-Unterrichts. Leipzig, Klinkhardt, 1899. 8vo. 4 + 64 pp., 4 plates. M. 0.60

Schürmann (F.). Kleine praktische Geometrie. 16te Auflage. Moers, Spaarmann, 1899. 8vo. 8 + 180 pp., 9 plates. M. 1.50

Schultz (E.). Leitfaden der Planimetrie für gewerbliche Lehranstalten. Teil I. 2te Auflage. Essen, Baedeker, 1899. 8vo. 4 + 79 pp. Cloth. M. 0.75

Shaposhnikov (N. A.). Text-book of Algebra, adapted to the programmes of the secondary schools. 3d edition. (In 2 parts.) (Russian.) Part I: Course of the third and fourth classes of the gymnasia, with supplementary articles. Moscow, 1897. 8vo. 175 pp. R. 0.70

——. Plane trigonometry, with a collection of trigonometrical problems. 7th edition. (Russian.) Moscow, 1899. 8vo. 136 pp. M. 2.70

Socci (A.) e Tolomei (G.). Elementi di matematica; libro di testo, per le scuole complementari, conforme ai programmi governativi. Vol. I: per la prima classe complementare. Firenze, Le Monnier, 1899. 8vo. 4 + 208 pp. Fr. 1.50

——. Vol. II: per la seconda classe complementare. Firenze, Le Monnier, 1899. 8vo. 156 pp. Fr. 1.50

——. Vol. III: per la terza classe complementare. Firenze, Le Monnier, 1899. 8vo. 196 pp. Fr. 1.50

Socci (A.) e Tolomei (G.). Elementi di matematica; libro di testo per le scuole normali conforme ai programmi governativi. Vol. I: per la prima classe normale. Firenze, Le Monnier, 1899. 8vo. 4 + 207 pp. Fr. 1.50

——. Vol. II: per la seconda classe normale. Firenze, Le Monnier, 1899. 8vo. 188 pp., 3 plates. Fr. 1.50

Spitz (C.). Lehrbuch der ebenen Geometrie, nebst einer Sammlung von 800 Uebungsaufgaben, zum Gebrauch an höheren Lehranstalten und beim Selbststudium. 10te Auflage. Hierzu eine Beigabe: Erläuternde Tafeln mit erklärendem Text von K. Traub. Leipzig, Winter, 1899. 8vo. 12 + 294 + 32 pp., 25 plates. M. 4.50

——. Die Resultate und Andeutungen zur Auflösung der in dem Lehrbuch [der ebenen Geometrie] befindlichen Aufgaben. 10te Auflage. Leipzig, Winter, 1899. 8vo. 4 + 113 pp. M. 1.60

Syzransky (P. I.). The solution of the quadrature of the circle with mathematical precision. (Russian.) Odessa, 1897. 8vo. 10 pp., 2 plates. R. 0.20

Tikhomirov (I. N.). Exercises and problems in elementary algebra. Systematic text-book for secondary schools. 2nd edition. (Russian.) Moscow, 1899. 8vo. 392 pp. M. 4.00

Timpenfeld (P.). Tabellen der Quadrate von 1–10 000, Kuben von 1–2500, Quadrat und Kubikwurzeln von 1–1000, Kreisumfänge und Inhalte von 1–1000. 2te vermehrte Auflage. Dortmund, 1899. 8vo. M. 3.75

Tolomei (G.). See Socci (A.).

Traub (K.). See Spitz (C.).

Vereshchagin (I.). Collection of algebraical problems for pupils of the upper classes of secondary institutions. 3d edition. (Russian.) St. Petersburg, 1896. 8vo. 180 pp. R. 0.90

——. Collection of questions and problems in plane trigonometry, for gymnasia and real schools. 4th edition. (Russian.) St. Petersburg, 1897. 8vo. 285 pp. R. 1.50

Voimov (V.). Practical methods for raising a number to a power and extracting from a number the 2d, 3d, 4th, 6th, 8th, 9th, 12th, 24th, 27th roots. For students in naval classes, technical, and other schools. 2d edition. (Russian.) Kerch, 1896. 8vo. 50 pp. R. 0.50

Volzhin (V.). The theory of limits and its applications. For use in real schools. (Russian.) Morshansk, 1896. 8vo. 19 pp.

Vulikh (Z.). Short course in geometry, with a collection of geometrical problems. 18th edition. (Russian.) St. Petersburg, 1897. 8vo. 210 pp. R. 0.80

Wentworth (G. A.). Solid geometry. Revised edition. Boston, Ginn & Co., 1899. 12mo. 16 + 469 pp. Half leather. $0.85

Witting (A.). Geometrische Konstruktionen, insbesondere in begrenzter Ebene. (Progr.) Dresden, 1899. 4to. 18 pp.

Wroblewski (W.). Solutions of the collection of exercises in F. Bychkov's Course of elementary algebra. Part III. (Russian.) St. Petersburg, 1899. 8vo. Pp. 399–640. M. 5.00

Zlotchansky (P.). Elevation to a power of polynomials of numbers; the formulae resulting from the division of $a^n \pm b^n$ by $a \pm b$. (Russian.) Odessa, 1897. 8vo. 24 pp. R. 0.30

——. Plane trigonometry, for secondary schools. 2nd edition, revised and augmented. (Russian.) Odessa, 1897. 8vo. 113 pp. R. 0.75

III. APPLIED MATHEMATICS.

Albrecht (M. F.) und Vierow (C. S.). Lehrbuch der Navigation. Für die königl. preussischen Navigations-Schulen bearbeitet. 7te Auflage. Anhang, herausgegeben im Auftrage des königl. Ministeriums für Handel und Gewerbe. Berlin, Decker, 1899. 8vo. 15 pp. M. 0.75

Appell (P.). Les mouvements de roulement en dynamique. Avec deux notes de J. Hadamard. Evreux, Hérissey, 1899. 16mo. 70 pp. (Scientia, physique mathématique, No. 4.)

Binns (W.). An elementary treatise on orthographic projection; being a new method of teaching the science of mechanical and engineering drawing. 13th edition. New York, Spon & Chamberlain, 1899. 8vo. 108 pp. Cloth. $3.50

Buchholz (H.). See Klinkerfues (W.).

Chadwick (O.). See Middleton (R. E.).

Dicknether (F.). Leitfaden der darstellenden Geometrie. 2te Auflage. München, Lindauer, 1899. 8vo. 4 + 76 pp. M. 1.50

Ekman (S. S.). Utveckling af teorien der centrifugal-regulatorer, jämte darpa grundad handledning för deras beräkning. Stockholm, 1899. 8vo. 108 pp. M. 3.80

Franklin (W. S.) and Williamson (R. B.). The elements of alternating currents. New York, The Macmillan Co., 1899. 8vo. 212 pp. Cloth. $1.75

Gibbs (J. W.). Equilibre des systèmes chimiques. Traduit par H. Le Chatelier. Paris, Carré et Naud, 1899. 8vo. 12 + 212 pp.

Grave (D. A.). On the fundamental problems of the mathematical theory of the construction of maps. (Russian.) St. Petersburg, 1896. 8vo. 4 + 197 pp., 7 plates.

Hadamard (J.). See Appell (P.).

Hadley (H. E.). Magnetism and electricity for beginners. London and New York, Macmillan, 1899. 16mo. 8 + 327 pp. Cloth. $0.60

Hemming (G. W.). Billiards mathematically treated. London and New York, Macmillan, 1899. 8vo. 45 pp. Cloth. $1.25

Jackwitz (E.). Gleichgewichtslagen und Schwingungen eines Pendelsystems. (Progr.) Schrimm, 1899. 4to. 20 pp., 1 plate.

Juppont (P.). Température et énergies. Essai sur une équation de dimensions de la température. Paris, 1899. 8vo. 94 pp. Fr. 3.00

Klinkerfues (W.). Theoretische Astronomie. 2te Auflage, herausgegeben von H. Buchholz. Braunschweig, Vieweg, 1899. 4to. 17 + 935 pp. Portrait. M. 34.00

Klinkert (G.). Die Bewegung elektro-magnetisch erregter Saiten. (Diss.) Marburg, 1898. 8vo. 26 pp., 2 plates.

Lampa (A.). Ueber Wirbelbewegung. Wien, 1899. 8vo. 13 pp. (Schriften des Vereins zur Verbreitung naturwissenschaftlicher Kenntnisse.) M. 1.50

Le Chatelier (H.). See Gibbs (J. W.).

Leutz (H.). Geschichte, Theorie und Anwendungen des Horizontalpendels. Teil I: Geschichte und elementare Theorie des Instrumentes. (Progr.) Karlsruhe, 1898. 4to. 20 pp.

Lorenz (L.). Oeuvres scientifiques, revues et annotées par H. Valentiner, publiées aux frais de la fondation Carlsberg. Vol. II, part 1. Copenhague, 1899. 8vo. Pp. 1–320. M. 6.50

Makarov (N. I.). Descriptive geometry. 4th revised edition. (Russian.) St. Petersburg, 1896. 8vo. 31 + 402 pp. With an atlas of 98 plates in 4to.

——. Linear perspective in the plane. (Russian.) St. Petersburg, 1896. 8vo. 19 + 8 + 482 pp. With an atlas of 98 plates in 4to. R. 5.00

Mezhericher (P. I.). Projective drawing; an elementary course in descriptive geometry. (Russian.) Odessa, 1896. 8vo. 6 + 94 pp., 20 plates.

Middleton (R. E.) and Chadwick (O.). Treatise on surveying. Part I. London, Spon, 1899. 8vo. 298 pp. 10s. 6d.

Müller (C. H.). Der logarithmische Rechenstab. (Progr.) Frankfurt a. M., 1899. 4to. 52 pp., 5 plates.

Müller (R.). Leitfaden für die Vorlesungen über darstellende Geometrie an der herzogl. technischen Hochschule zu Braunschweig. Braunschweig, Vieweg, 1899. 8vo. 7 + 88 pp. M. 2.50

Neumann (E.). Zur Poisson'schen Theorie der Elektrostatik. I. (Diss.) Leipzig, 1899. 4to. 39 pp.

Nickel (E.). Ueber graphische Darstellungen in den Naturwissenschaften. (Progr.) Frankfurt, 1899. 4to. 31 pp.

Nipher (F. E.). Electricity and magnetism. A mathematical treatise for advanced undergraduate students. 2nd edition, revised, with additions. St. Louis, Mo., Boland, 1898. 12mo.

Palschau (A. N.). Elements of descriptive geometry, with an addition on the drawing of curves, for the use in real schools. 3d edition. (Russian.) Moscow, 1897. 8vo. 204 pp. R. 1.35

Pasini (C.). Orologi solari; costruzioni grafiche e calcolo degli orologi solari e delle meridiane a tempo medio, loro pratica esecuzione e storia. Padova, Draghi, 1900. 8vo. 9 + 172 pp., 4 plates. Fr. 3.50

Percival (A. S.). Optics; a manual for students. London and New York, Macmillan, 1899. 12mo. 10 + 400 pp. Cloth. $3.25

Picone-Gusmano (A.). Il metodo di Gauss applicato alla compensazione degli errori nel rilevamento topografico. Vol. I: Preliminari di analisi; la teoria del metodo; le applicazioni. Melfi, Grieco, 1899. 8vo. 11 + 238 pp., 5 plates. Fr. 6.00

Recknagel (G.). Gleichgewichtsstudien. (Progr.) Augsburg, 1898. 8vo. 25 pp.

Robson (H.). Principles of mechanics. Part I: Mechanics of solids. Part II: Mechanics of fluids. London, Scientific Press, 1899. 12mo. 158 pp. 2s. 6d.

Schiller (N.). Die Bedeutung des osmotischen Druckes in der Thermodynamik der Lösungen. Leipzig, Barth, 1898. 8vo. (*Annalen der Physik und Chemie*, neue Folge, Vol. 67, pp. 291–306.)

——. On the variation of the internal energy in the dilution of solutions. (Russian.) Kiev, 1899. 8vo. 10 pp.

——. Origin and development of the ideas of "temperature" and "heat." (Russian.) Kiev, University Press, 1899. 8vo. 4 + 53 pp.

Schubert (F.). Die darstellende Geometrie an maschinentechnischen Lehranstalten, Gewerbe- und Fachschulen. Als Wegweiser für Lehrende und Lernende nach den Formalstufen bearbeitet. Teil II: Die darstellende Geometrie, einschliesslich der Elemente der Projektionslehre, Schattenlehre, Axonometrie und Perspektive. (Stereometrisches Linearzeichnen.) Mittweida, Polytechnische Buchhandlung, 1899. 8vo. 254 pp. Cloth. M. 5.50

SHAMSHINOV (A.). Short course in the rules of perspective. (Russian.) Tiflis, 1899. 8vo. 40 pp. M. 4.00

TEICHMÜLLER (J.). Die elektrischen Gleichstrommaschinen mit Rücksicht auf ihre Elastizität. In Form eines Lehrbuches dargestellt. Habilitationsschrift zur Erlangung der venia legendi für Elektrotechnik an der grossherzogl. Technischen Hochschule in Karlsruhe i. B. Stuttgart, Hoffmann, 1898. 8vo. 245 pp.

UTUSHKIN (D.). Synopsis of geometry and projective drawing for elementary schools. (Russian.) Ekaterinburg, 1896. 8vo. 87 pp. R. 0.35

VALENTINER (H.). See LORENZ (L.).

VIEROW (C. S.). See ALBRECHT (M. F.).

VOINAROVSKY (P. D.). The elements of higher mathematics, as required for the electrical engineer. Part I. of a theoretical and practical course in electrical engineering. (Russian.) St. Petersburg, Ricker, 1897. 12mo. 207 pp.

——. The foundations of the theory of electric and magnetic phenomena. Part II. of a course in electrical engineering. (Russian.) St. Petersburg, Ricker, 1899. 12mo. 270 pp. M. 6.00

WILLIAMSON (R. B.). See FRANKLIN (W. S.).

WINTER (W.). Grundriss der Mechanik und Physik. 3te Auflage. München, 1899. 8vo. 5 + 352 pp. M. 3.20

WÜLLNER (A.). Lehrbuch der Experimentalphysik. 5te Auflage. Vol. IV: Die Lehre von der Strahlung. 2ter Halbband. Pp. 513–1042. Leipzig, Teubner, 1899. 8vo. 3 plates. M. 7.00

THE CENTURY'S PROGRESS IN APPLIED MATHEMATICS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY AT ITS SIXTH ANNUAL MEETING, DECEMBER 28, 1899.

BY PRESIDENT R. S. WOODWARD.

The honor of election to the presidency of the American Mathematical Society carries with it the difficult duty of preparing an address, which may be at once interesting and instructive to a majority of the membership, and which may indicate at the same time the lines along which progress may be expected in one or more branches of our favorite science. In partial recognition of the honor you have conferred upon me it has seemed that I could do no better than to consider with you some of the principal advances that have been made in mathematical science during the past century. But here at the outset one must needs feel sharply restricted by the limitations of his knowledge and by the wide extent of the domain to be surveyed. Especially must this be the case with one who belongs to no school of mathematicians unless it be the 'old school' of inadequate opportunities and desultory training. On account of these conditions, I have found it essential to accept the ordinary division of the science into pure and applied mathematics and to confine my attention in this address wholly to applied mathematics. Here again, however, it is necessary to impose restrictions, for the domain thus divided is still far too large to be reviewed adequately in the brief interval allotted to the present occasion. I have therefore limited my considerations chiefly to those branches of applied mathematics which were already recognized as such at the beginning of the present century. The most important of these branches appear to be analytical mechanics, geodesy, dynamical astronomy, spherical or observational astronomy, the theory of elasticity, and hydromechanics. This rather arbitrary subdivision may be made to include several important branches not enumerated, while it must exclude others of equal or greater importance. Thus the theory of heat diffusion which led Fourier to the wonderful analysis which bears his name may be alluded to under physical geodesy; the theories of sound and light may be regarded as applications merely of the theories of elasticity and hydromechanics; while the theories of electricity, magnetism, and thermodynamics, which are the peculiar and perhaps most important developments of the present century, must be excluded almost altogether.

Another difficulty which besets one who would speak of the progress in question is that arising from the technicalities of the subjects to be discussed. Beautiful and important as these subjects are when arrayed in their mathematical dress, and thrilling as they truly are when rehearsed with appropriate terminology in the quiet of one's study, it must be confessed that they are on the whole rather uninviting for the purposes of semi-popular exposition. In order to meet this difficulty it seems best to relegate technicalities which demand expression in symbols to foot notes and, while freely using technical terminology, to translate it into the vernacular whenever essential. Thus it is hoped to avoid the dullness of undue condensation on the one hand and the superficiality of mere literary description on the other.

The end of the last century marks one of the most important epochs in the history of mathematical science. This time, one hundred years ago, the master work of Lagrange (1736–1813), the Mécanique Analytique, had been published about eleven years. The first two volumes of the Mécanique Céleste of Laplace (1749–1827), undoubtedly the greatest systematic treatise ever published, had just been issued. Fourier (1768–1830), whose mathematical theory of heat was destined to play a wonderful rôle in pure and applied mathematics, was a soldier statesman in Egypt, where with Napoleon he stood before the pyramids while the centuries looked down upon them.* Gauss (1777–1855), who with Lagrange and Cauchy (1789–1857) must be ranked among the founders of modern pure mathematics, was a promising but little known student whose Disquisitiones Arithmeticae and other papers were soon to win him the directorship of the observatory at Göttingen. Poisson (1781–1840), to whom we owe in large part the beginnings of mathematical physics, had just started on his brilliant career as a student and professor in the École Polytechnique. Bessel (1784–1846), whose theories of observational astronomy and geodesy were destined soon to assume a prominence which they still hold, was an accountant in a trading house at Bremen. Dynamical astronomy, the favorite science of the day was under the dominating genius of Laplace, with no one to dispute his preëminence, and with only Lagrange and Poisson as friendly competitors in the same field. Rational mechanics as we now know it, was soon to be simplified and systematized by Poinsot (1777–1859), Poisson, Möbius (1790–1868), and Coriolis (1792–1843), who were all at this time under twenty-five years of age. The undulatory theory of light, in which Young (1773–1829), Fresnel (1788–1827), Arago (1786–1853), and Green (1793–1841) were to be the most conspicuous early figures, was just beginning to be considered as an alternative to the emission theory of Newton. The theory of elasticity, or the theory of stress and strain as it is now called, was about to be reduced to the definiteness of formulas at the hands of Navier (1785–1836), Poisson, Cauchy, and Lamé (1795–1870). Planetary and sidereal astronomy, to which so much of talent, time, and treasure have since been devoted, was soon to receive the fruitful impetus imparted to it by the German school of Gauss, Bessel, Encke (1791–1865), and Hansen (1795–1874).

---

*The bombastic words of Bonaparte, "Songez que du haut de ces pyramides quarante siècles vous contemplent," may be excused, perhaps, in view of the fact that Fourier, Monge, and Berthollet were present on the occasion.

The advances that have been made during the past century in analytical mechanics must be measured from the elevated standard attained by Lagrange in his Mécanique Analytique. To work any improvement over this, to simplify its demonstrations, or to elaborate its details, was a task fit only for the keenest intellects. Lagrange had, as he supposed, reduced mechanics to pure mathematics. Geometrical reasonings and diagrammatic illustrations were triumphantly banished from this science and replaced by the systematic and unerring processes of algebra. "Ceux qui aiment l'Analyse," he says, "verront avec plaisir la Mécanique en devinir une nouvelle branche, et me sauront gré d'en avoir étendu ainsi le domaine." The mathematical world has not only accepted Lagrange's estimate of his work, but has gone further, and considers his achievement one of the most brilliant and important in the whole range of mathematical science. "The mechanics of Lagrange," as Mach has well said, "is a stupendous contribution to the economy of thought."*

Nevertheless, improvements were essential, and they came in due time. As we can now see without much difficulty, Lagrange and most of his contemporaries in their eagerness to put mechanics on a sound analytical basis overlooked to a serious extent its more important physical basis. The prevailing mathematical opinion was that a science is finished as soon as it is expressed in equations. One of the first to protest against this view was Poinsot, though the preëminent importance of the physical aspect of mechanics did not come to be adequately appreciated until the latter half of the present century. The animating idea of Poinsot was that in the study of mechanics one should be able to form a clear mental picture of the phenomena considered; and that it does not suffice to put the data and hypotheses into the hopper of our mathematical mill and then to trust blindly to its perfection in grinding the grist. In elaborating this idea he produced two of the most important elementary treatises on mechanics of the century. These are his Éléments de Statique published in 1804, and his Théorie Nouvelle de la Rotation des Corps published in 1834.* In the former work he developed the beautiful and fruitful theory of couples and their composition, and the conditions of equilibrium, as they are now commonly expressed in elementary books. In the latter work he took up the more recondite question of rendering a clear account of the motion of a rigid body. This problem had been treated already by the illustrious Euler, d'Alembert, Lagrange, and Laplace, and it seemed little short of temerity to hope for any improvement. But Poinsot entertained that hope and his efforts proved surprisingly successful. His little volume of about one hundred and fifty pages is still one of the finest models of mathematical and mechanical exposition; and his repeated warning, "gardons-nous de croire qu'une science soit faite quand on l'a réduite à des formules analytiques," has been fully justified. He gave us what may be called the descriptive geometry of the kinetics of a rotating rigid body, the "image sensible de cette rotation"; he clarified the theory of moments of inertia and principal axes; he made plain the meaning of what we now call the conservation of energy and the conservation of moment of momentum of systems which are started off impulsively; and he surpassed Laplace himself in expounding the theory of the invariable plane.

---

* The Science of Mechanics, by Dr. Ernst Mach. Translated from the German by Thomas J. McCormack. Chicago, Open Court Publishing Co. 1893.

Another elementary work of prime importance in the progress of mechanics was Poisson's Traité de Mécanique.

---

* Outlined in a communication to the Paris Academy in 1834. In the introduction to the edition of 1852 he says, "Voici une des questions qui m'ont le plus souvent occupé, et, si l'on me permet de parler ainsi, une des choses que j'ai le plus désiré de savoir en dynamique. "Tout le monde se fait une idée claire du mouvement d'un point, * * * Mais, s'il s'agit du mouvement d'un corps de grandeur sensible et de figure quelconque, il faut convenir qu'on ne s'en fait qu'une idée très-obscure."

Poisson belonged to the Lagrangian school of analysts, but he was so profoundly devoted to mathematical physics that almost all his mathematical work was suggested by and directed towards practical applications. His facility and lucidity in exposition rendered all his works easy and attractive reading, and his treatise on mechanics is still one of the most instructive books on that subject. He was one of the first to call attention to the value of the principle of homogeneity in mechanics,[*] a principle which, as expanded in Fourier's theory of dimensions,[†] has proved of the greatest utility in the latter half of the century. The influence of Poisson's work in mechanics proper, very widely extended, of course, by his memoirs in all departments of mathematical physics, is seen along nearly every line of progress since the beginning of the century.

Of other works which paved the way to the present advanced state of mechanical science, it may suffice to mention the Cours de Mécanique[‡] of Poncelet (1788–1867), the Traité de Mécanique des Corps Solides et de l'Effet des Machines[§] of Coriolis, and the Lehrbuch der Statik[||] of Möbius. To the two former we owe the fixation of ideas and terminology concerning the doctrine of mechanical work, while the suggestive treatise of Möbius foreshadowed a new type of mechanical concepts since cultivated by Hamilton, Grassmann, and others under the general designation of vector analysis.

Following close after the development of the elementary ideas whose history we have sketched came the important mathematical improvements in the Lagrangian analysis due to Hamilton[¶] (Sir W. R., 1805–1865). With these additions of Hamilton, amplified and clarified by the labors of Jacobi, Poisson, and others,[**] analytical mechanics may be said to have reached its present degree of perfection so far as mathematical methods are concerned. By these methods every mechanical question may be stated in either of three characteristic though interconvertible ways, namely: by the equations of d'Alembert, by the equations of Lagrange, and by the equation of Hamilton. Each way has special advantages for particular applications, and together they may be said to condense into the narrow space of a few printed lines the net results of more than twenty centuries of effort in the formulation of the phenomena of matter and motion.

---

[*] See Article 23, Tome I., Traité de Mécanique, 2nd ed., Paris, 1833.

[†] Théorie Analytique de la Chaleur, Paris, 1822.

[‡] Metz, 1826.

[§] Paris, 1829.

[||] Leipzig, 1837.

[¶] "On a general method in dynamics." *Philosophical Transactions*, 1834–35.

[**] For an account of these additions and a complete list of papers bearing on the subject (up to 1857), one should consult the admirable report of Cayley on "Recent progress in dynamics," published in the Report of the British Association for the Advancement of Science for 1857.

Such was the state of mechanical science when the great physical discovery of the century, the law of conservation of energy, was made. To give adequate expression to this law it was only necessary to recur to the Mécanique Analytique, for herein Lagrange had prepared almost all of the needful machinery. So well, indeed, were the ideas and methods of Lagrange adapted to this purpose that they have not only furnished the points of departure for many of the most important discoveries * of the present half century, but they have also supplied the criteria by means of which mechanical phenomena in general are most easily and effectively defined and interpreted.

Of the special branches of analytical mechanics which have undergone noteworthy development during this century, by far the most important is that known as the theory of the potential function. This function first appeared in mathematical analysis in a memoir of Lagrange in 1777 † as the expression of the perturbative function, or force function. It next appeared in 1782 ‡ in a memoir by Laplace. In this memoir Laplace's equation § appears for the first time, being here expressed in polar coördinates. In 1787 ‖ the same equation appears in the more usual form as expressed by rectangular coördinates.

Strange as it now seems when viewed by the light of this end of the century, nearly thirty years elapsed before Laplace's equation was generalized. Laplace had found only half of the truth, namely, that which applies to points external to the attracting masses.¶ Poisson discovered the

---

*Especially those in the theories of electricity, magnetism, and thermodynamics.

† *Nouveaux Mémoires de l'Académie des Sciences et Belles Lettres de Berlin.* See also remarks of Heine, Handbuch der Kugelfunctionen, Band II., p. 342.

‡ *Paris Mémoires* for 1782, published in 1785.

§ $$\Delta^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

$\Delta^2 V$ is called the Laplacian of V.

‖ *Paris Mémoires* for 1787, published in 1789.

¶ That is, Laplace's equation is $\Delta^2 V = 0$, while Poisson's is $\Delta^2 V + 4\pi k \rho = 0$; V being the potential and ρ the density at the point (x, y, z), and k being the gravitation constant.

other half in 1813.* Thus the honors attached to the introduction of this remarkable theorem are divided between them, and we now speak of the equation of Laplace and the equation of Poisson, though the equation of Poisson includes that of Laplace.

Next came the splendid contributions of George Green under the modest title of "An essay on the application of mathematical analysis to the theories of electricity and magnetism."† It is in this essay that the term 'potential function' first occurs. Herein also his remarkable theorem in pure mathematics, since universally known as Green's theorem, and probably the most important instrument of investigation in the whole range of mathematical physics, made its appearance.

We are all now able to understand, in a general way at least, the importance of Green's work, and the progress made since the publication of his essay in 1828. But to fully appreciate his work and subsequent progress one needs to know the outlook for the mathematico-physical sciences as it appeared to Green at this time and to realize his refined sensitiveness in promulgating his discoveries.

"It must certainly be regarded as a pleasing prospect to analysts," he says in his preface, "that at a time when astronomy, from the state of perfection to which it has attained, leaves little room for further applications of their art, the rest of the physical sciences should show themselves daily more and more willing to submit to it." * * * "Should the present essay tend in any way to facilitate the application of analysis to one of the most interesting of the physical sciences, the author will deem himself amply repaid for any labor he may have bestowed upon it; and it is hoped the difficulty of the subject will incline mathematicians to read this work with indulgence, more particularly when they are informed that it was written by a young man who has been obliged to obtain the little knowledge he possesses, at such intervals and by such means as other indispensable avocations which offer but few opportunities of mental improvement, afforded." Where in the history of science have we a finer instance of that sort of modesty which springs from a knowledge of things?

The completion of the potential theory, so far as it depends on the Newtonian law of the inverse square of the distance, must be credited to Gauss, though a host of writers has since contributed many valuable additions in the way of details. Early in the century Gauss had begun the study of the absorbing problems of the day, namely, problems of attractions and repulsions. The prevailing notion of mathematical physicists seems to have been that all mechanical phenomena may be attributed to attractions and repulsions between the ultimate particles of matter and the ultimate particles of 'fluids' associated with matter. The difficulties of action at a distance, without the aid of an intervening medium, happily, did not trouble them at that time; for who shall say that their labors would have been more fruitful if they had stopped to remove these difficulties? Gauss's first memoir in this field relates to the attractions of homogeneous ellipsoidal masses,* and dates from 1813. It was in this memoir that he published a number of the elegant theorems † which are now found in the elementary books on the theory of the potential function. In 1829 he published his theory of fluid figures in equilibrium,‡ and in 1832 there followed one of the most important papers of the century on the intensity of terrestrial magnetic force expressed in what we now call absolute units.§

---

* Poisson's equation was derived in a paper published in *Nouveau Bulletin* * * * *Société Philomatique*, Paris, Dec., 1813.

† Nottingham, 1828.

Six years later he published his wonderful theory of the earth's magnetism || and applied it to all existing observational data. This theory is a splendid application of the potential theory, and his entire investigation is one of the most beautiful and useful contributions to mathematical physics of the century. Well was he qualified, therefore, to complete the theory of the Newtonian potential function in the collection of theorems published in his memoir ¶ of

---

* "Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum," 1813. See Gauss's Werke, Band V., Göttingen, 1877.

† Especially the theorem giving the values of the surface integral

$$\int \frac{\cos(s, n)}{s^2} dS,$$

where dS is an element of any closed surface, s the distance from dS to any fixed point, and n indicates the normal to the surface at dS. This gave the key to the very important theorem of the surface integral of the normal acceleration, or

$$\int \frac{\partial V}{\partial n} dS.$$

‡ "Principia generalia theoriæ figuræ fluidorum in statu æquilibrii," 1829. Werke, Band V.

§ "Intensitas vis magneticae terrestris ad mensuram absolutam revocata." Werke, Band V.

|| Allgemeine Theorie des Erdmagnetismus. Werke, Band V.

¶ "Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungs-Kräfte." Werke, Band V.

1840. This is still the fundamental memoir on the subject of which it treats, and must be regarded as one of the most perfect models of mathematical exposition. In respect to clearness and elegance, indeed, the works of Gauss are unsurpassed. "In his hands," as Todhunter has said,* "Latin and German rival French itself for clearness and precision." "Alles gestaltet sich neu unter seinen Händen," was the tribute † of Bessel; and the lapse of two generations has served only to increase admiration for the genius and industry which made Gauss one of the most conspicuous figures in the science of the nineteenth century.

The importance of the theory of the potential function, when considered in its historical aspects, is found to consist not so much in the rich harvest of results it has afforded in the field of gravitation, as in its direct bearing on the developments of other branches of mathematical physics. For the points of view and the analytical methods of the Newtonian function have been adapted and extended with brilliant success to the interpretation of almost all kinds of mechanical phenomena. Thus it has come about that we have now to deal with many kinds of potential, as logarithmic potential, velocity potential, displacement potential, electric potential. magnetic potential, thermodynamic potential, etc.; each of which bears a more or less close mathematical analogy to the Newtonian function.

In the closing paragraph of his Exposition du Système du Monde, Laplace refers to the immense progress made in astronomy since the geocentric theory was displaced by the heliocentric theory of the solar system. This progress is specially remarkable when we consider that it depended on the discovery, so humiliating to man, of the relatively insignificant dimensions and inconspicuous rôle of our planet. But we agree with Laplace that "Les resultats sublimes auxquels cette découverte l'a conduit sont bien propres à le consoler du rang qu'elle assigne à la Terre, en lui montrant sa propre grandeur dans l'extrême petitesse de la base qui lui a servi pour mesurer les cieux." All astronomy is based on a knowledge of the size, the shape and the mechanical properties of the earth; and it is not surprising, therefore, that a large share of the mathematical investigations of the century should have been directed to the science of geodesy. Founded in the middle of the last century by Clairaut ‡ and his contemporaries; recast by Laplace and Legendre* (1752–1833) in the early part of this century; systematized and extended to a remarkable degree by the German geodesists, led especially by the incomparable Bessel,† this science has now come to occupy the leading position in point of perfection of methods and precision of results. So great, in fact, has been the growth of this science during the century that recent writers have found it desirable to subdivide the subject into two parts, called mathematical geodesy and physical geodesy, respectively, though both parts are nothing if not mathematical.‡

---

* History of the Theories of Attraction and Figure of the Earth, Vol. II., p. 235.

† In a letter to Olbers, 1818.

‡ Clairaut's work, Théorie de la Figure de la Terre, Paris, 1743, was the pioneer work in physical geodesy.

In a former address I have considered somewhat in detail certain of the more salient mathematical problems which have arisen in the study of the earth; § and the present review may hence be restricted to a rapid résumé of the less salient, but perhaps more recondite problems, and to the briefest mention of problems already discussed.

Adopting the convenient nomenclature of geologists, we may consider the earth as made up of four parts, namely: the atmosphere; the hydrosphere, or oceans; the lithosphere, or crust; and the nucleus. Beginning with the first of these we are at once struck by the fact that much greater progress has been made during the century in the investigation of the kinetic phenomena of the atmosphere than in the study of what may be called its static properties. Evidently, of course, the phenomena of meteorology are essentially kinetic, but it would seem that the questions of pressure, temperature, and mass distribution of the atmosphere ought to be determined with a close approximation from purely statical considerations. This appears to have been the view of Laplace, who was the first to bring adequate knowledge to bear on such questions. He investigated the terrestrial atmosphere as one might investigate the gaseous envelope of an unilluminated planet.|| He reached the conclusion that the atmosphere is limited by a lenticular shaped surface of revolution whose polar and equatorial diameters are about 4.4 and 6.6 times the diameter of the earth respectively, and whose volume is about 155 times that of the rest of the earth.* If this conclusion be true our atmosphere should reach out to a distance of about 26,000 miles at the equator and to a distance of about 17,000 miles at the poles. It does not appear, however, that Laplace attempted to assign the distribution of pressure and density, and hence total mass, of the atmosphere within this envelope; and I am not aware that any subsequent investigator has published a satisfactory solution of this apparently simple problem.†

---

* The name of Legendre is famous in geodesy by reason of his beautiful theorem which makes the solution of a geodetic triangle almost as easy as the solution of a plane triangle.

† Bessel's contributions to astronomy and geodesy are collected in Abhandlungen von F. W. Bessel, herausgegeben von Rudolf Engelmann, in drei Bänden, Leipzig, Wilhelm Engelmann, 1875.

‡ See, for example, Die Mathematischen und Physikalischen Theorieen der Höheren Geodäsie von Dr. F. R. Helmert, Leipzig, B. G. Teubner, Teil I., 1880; Teil II., 1884.

§ On the Mathematical Theories of the Earth. Vice-presidential address before the Section of Astronomy and Mathematics of the American Association for the Advancement of Science, 1889. *Proceedings of A. A. A. S.* for 1889.

¶ Mécanique Céleste, Livre III., Chap. VII., and Livre X, Chapts. I.–IV.

On the other hand, the general character of the circulation of the atmosphere and the meteorological consequences

---

* Laplace's equation to a meridian section of this envelope is

$$x^{-1} - x_0^{-1} + \tfrac{1}{2}ax^2\cos^2\varphi = 0,$$

where $x = r/a$, r being the radius vector measured from the center of the earth and a the mean radius of the earth; a is the ratio of centrifugal to gravitational acceleration at the equator of the earth; φ is the geocentric latitude, and x_0 is the value of x for $\phi = \pi/2$.

The problem of the statical properties of the atmosphere may be stated in three equations, namely:

$$\Delta^2 V + 4\pi k\rho - 2\omega^2 = 0, \qquad dp = \rho dV, \qquad p = f(\rho, \tau).$$

In these V is the potential at any point of the atmosphere, p, ρ, τ being the pressure, density and temperature at the same point; k is the gravitation constant; and ω is the angular velocity of the earth. The above equation of Laplace neglects the mass of the atmosphere in comparison with the mass of the rest of the earth. An essential difficulty of the problem lies in the unknown form of the function $f(\rho, \tau)$.

† I have sought a solution with a view especially to determining the mass of the atmosphere. A class of solutions satisfying the mechanical conditions of the following assumptions has been worked out. Thus, assuming $p = c\rho^m$, which includes the adiabatic relation, $p = c\rho^{1.41}$, and the famous Laplacian relation, $\partial p/\partial\rho = 2c\rho$; and the law of Charles and Gay-Lussac, $p = C\rho\tau$; there results

$$\frac{p}{p_0} = \left(\frac{Q}{Q_0}\right)^{\frac{m}{m-1}}, \qquad \frac{\rho}{\rho_0} = \left(\frac{Q}{Q_0}\right)^{\frac{1}{m-1}}, \qquad \frac{\tau}{\tau_0} = \frac{Q}{Q_0},$$

where $$Q = x^{-1} - x_0^{-1} + \tfrac{1}{2}ax^2\cos^2\phi$$

defined above; Q_0 is the value of Q for $x = 1$ and $\phi = \pi/2$; and p_0, ρ_0, τ_0 are the values of p, ρ, τ at the same point $(x = 1,\ \phi = \pi/2)$.

Using the adiabatic law the above formula for ρ leads to a mass for the atmosphere of about 1/1200th of the entire mass of the earth. But since the adiabatic law gives too low a pressure, density, and temperature gradient, this can only be regarded as an upper limit to the mass of the atmosphere. A lower limit of about 1/1000000th of the earth's mass is found by assuming that the mass of the atmosphere is equal to a mass of mercury or water which would give an equivalent pressure at the earth's surface.

thereof, have been brought within the domain of mathematical research if they have not yet been wholly reduced to quantitative precision. The pioneer in this work was a fellow-countryman, William Ferrel,* (1817–1891), who, like Green, came near being lost to science through the obscurity of his early environment. It is a curious though lamentable circumstance, illustrating at once the peculiar shyness of Ferrel and the proverbial popular indifference to discoveries which cannot be patented, that a man who had mastered the Principia and the Mécanique Céleste and who had laid the foundations of our theory of the circulation of the atmosphere, should have found no better medium for the publication of his researches than the semi-popular columns of a journal devoted to medicine and surgery. But such was the medium through which Ferrel's "Essay on the winds and the currents of the ocean"† appeared in 1856. Since that time notable progress has been made at the hands of Ferrel, Helmholtz (1821–1894), Oberbeck, Bezold, and others;‡ so that we may entertain the hope that the apparently erratic phenomena of the weather will presently yield to mathematical expression just as the similar phenomena of oceanic tides and of terrestrial magnetism have already yielded to the powers of harmonic analysis.

When we pass from the atmosphere to the hydrosphere, several questions concerning the nature and properties of their common surface, or what is usually called the sea surface immediately demand attention. The most important of these are what may be distinguished as the static and the kinetic phenomena of the sea surface. Since tidal oscillations belong more properly to hydrokinetics, we may here confine attention to the static phenomena.

Starting from the datum plane fixed by Laplace, the most important contribution to the theory of physical geodesy since his time is found in the remarkable memoir of Sir George Stokes "On the variation of gravity at the surface of the earth."§ Adopting the hypothesis of original fluidity, or the more general hypothesis of a symmetrical arrangement of the strata of the earth, with increasing density towards the center, Laplace had shown that the acceleration of gravity in passing from the equator to the poles should increase as the square of the sine of the latitude.* This conclusion agreed well with the facts of observation; and Laplace rested content in the opinion that his hypothesis was verified. But Stokes showed that the law of variation of the acceleration of gravity at the surface of the sea is wholly determined by that surface, regardless of the mode of distribution of the earth's mass. This, as we now see, of course, is a direct result of the theory of the potential function; for the sea surface is an equipotential surface, and since it is observed to be closely spheroidal, the formula of Laplace follows independently of all hypothesis save that of the law of gravitation. But while Laplace's formula and the arguments by which he reached it throw no light on the distribution of the earth's mass, a slight extension of his methods gives a formula which shows that any considerable differences between the equatorial moments of inertia of the earth would produce a variation in the acceleration of gravity dependent on the longitude of the place of observation.† Thus it is possible by means of pendulum observations alone to reach the conclusion that the mass of the earth is very nearly symmetrically distributed with respect to its equator and with respect to its axis of revolution.

---

* For a biography and autobiographical sketch of Ferrel, and a list of his publications, see Biographical Memoirs of the National Academy of Sciences, Vol. III., pp. 265–309. Washington, 1895.

† In *Nashville Journal of Medicine and Surgery*, Oct. and Nov., 1856.

‡ Some of the most important papers and memoirs on this subject, collected and translated by Professor Cleveland Abbe, have been published by the Smithsonian Institution under the title "The Mechanics of the Atmosphere." Smithsonian Miscellaneous Collections, No. 843, Washington, 1891.

§ Read April, 1849; Mathematical and Physical Papers by G. G. Stokes, Cambridge University Press, 1883, Vol. II.

A question of great interest with which the acceleration of gravity at the sea surface is closely connected is that of the earth's mass as a whole. About two years ago I published a short paper which gives the product of the mean density of the earth and the gravitation constant in terms of the coefficients in Laplace's formula and the dimensions of the earth.‡ It was shown that this product can be

---

*Laplace's formula is $g = \alpha + \beta \sin^2 \phi$, where α is the value of g at the equator, β is a constant, and ϕ is the latitude of the place.

†See Helmert, Geodäsie, Band II., p. 74. The expression for the acceleration is

$$g = \alpha + \beta \sin^2 \phi + \gamma \cos^2 \phi \cos 2\lambda,$$

where α, β, γ are constants, and ϕ, λ are latitude and longitude respectively; and the constant γ involves the difference of the equatorial moments of inertia as a factor.

‡*The Astronomical Journal*, No. 424. This product is expressed thus:

$$k\rho = \frac{2\pi}{T^2} + \frac{3(\alpha s_1 + \beta s_2)}{4\pi a \sqrt{1 - e^2}};$$

wherein k is the gravitation constant, ρ is the mean density of the earth, T is the number of mean solar seconds in a sidereal day, α and β are the first two constants in the formula $g = \alpha + \beta \sin^2 \phi + \gamma \cos^2 \phi \cdot \cos 2\lambda$ for easily computed from existing data to five significant figures with an uncertainty of only one or two units in the last figure, thus making it possible to obtain the mass of the earth to a like degree of precision if the constant of gravitation can be equally well determined. In a subsequent communication to this Society it was explained that the product in question is equal to 3π divided by the square of the periodic time of an infinitesimal satellite which would pass around the earth, just grazing the equator, if there were no atmosphere to impede its progress. The periodic time of such a satellite would be 1 hour, 24 minutes, 20.9 seconds. Attention is called to this subject with the hope that some mathematician may point out another possible relation between the gravitational constant and the mean density of the earth which can be accurately observed, or that some physicist may show how the gravitational constant can be measured directly with a precision extending to five significant figures.

The lithosphere is the special province of the geologist, and we may hence pass on to the nucleus, or chief part of the mass of the earth. Much time and attention have been devoted to the study of the important but intricate problems which the geometers of the early part of the century left to their successors. But while the obscurities and vagaries of our predecessors have been cleared away, it must be confessed that our improved mathematical apparatus has not brought us very far ahead of the positions of Laplace and Fourier as regards the constitution and properties of the nucleus. With respect to the law of distribution of density in the nucleus it may be said that, although Laplace's law * is probably not exact, it is yet quite as nearly correct as our observed information requires.†

---

the acceleration of gravity at the sea surface in latitude φ and longitude λ; a is the half major axis and e is the eccentricity of the earth's spheroid; and

$$s_1 = \tfrac{1}{2}\left\{1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right\}, \quad s_2 = \frac{1-e^2}{4e^2}\left\{\frac{2e}{1-e^2} + \log\frac{1-e}{1+e}\right\}.$$

The resulting numerical value is

$$k\rho = 36797 \times 10^{-11}/(\text{second})^2.$$

* The Laplacian distribution of pressure, density, and potential in the earth are defined essentially (neglecting the effect of rotation) by the following three equations:

$$\Delta^2 V + 4\pi k\rho = 0, \qquad dp = \rho dV, \qquad \frac{\partial p}{\partial \rho} = c\rho;$$

where p, ρ, V are the pressure, density, and potential at any point of the mass, k is the gravitation constant, and c is a constant securing the equality of the members of the last equation.

† With regard to what constitutes an adequate theory in any case, see

Another question of widely general, and of peculiar mathematical interest, is the problem first attacked by Fourier, of the distribution and consequent effects of the earth's internal heat. The most interesting phase of this question is that which relates to the time that has elapsed since the crust of the earth became stable and sufficiently cool to support animal life. It is now nearly forty years since Lord Kelvin* startled geologists especially by telling them that Fourier's theory of heat conduction forbids anything like such long intervals of time as they were in the habit of assigning to the aggregate of paleontological phenomena. On several occasions since then Kelvin has restated his arguments with a cogency that has silenced most geologists if it has not convinced most mathematicians. Quite recently, however, the question has become somewhat less one-sided, since geologists and paleontologists are beginning to defend their position,† while that of Kelvin is being attacked from the mathematical side.‡ My own views on this subject were expressed somewhat at length ten years ago, in the address already referred to, and it seems unnecessary here to go into the matter any further than to reaffirm my conviction that the geologists have adduced the weightier arguments. Beautiful as the Fourier analysis is, and absorbingly interesting as its application to the problem of a cooling sphere § is, it does not seem to me to afford anything like so

---

an instructive paper by Dr. G. Johnstone Stoney on "The kinetic theory of gas, regarded as illustrating nature," *Proceedings Royal Dublin Society*, Vol. VIII. (N. S.), Part IV., No. 45.

* In a memoir "On the secular cooling of the earth," *Trans. Royal Society of Edinburgh*, 1862. Republished in Kelvin and Tait's Treatise on Natural Philosophy, appendix D. Kelvin's latest paper on this subject is entitled "The age of the earth as an abode fitted for life," and is published in *Philosophical Magazine*, January, 1899; also in *Science*, May 12, 1899.

† See Professor T. C. Chamberlain's paper, "Lord Kelvin's address on the age of the earth as an abode fitted for life," *Science*, June 30, 1899; also Sir Archibald Geikie's presidential address to Geological Section of the British Association for the Advancement of Science, Dover meeting, 1899.

‡ Notably by Professor John Perry. See *Nature*, January 3, and April 18, 1895.

§ I have recast this problem of Fourier in two papers published in the *Annals of Mathematics*, Vol. III., pp. 75–88 and pp. 129–144. The solution there given is the only one, so far as I am aware, which lends itself to computation for all values of the time in the history of cooling. A point of much mathematical interest on which this solution depends is the equivalence of the two following series:

$$ru = \frac{2r_0u_0}{\pi}\sum_{n=1}^{n=\infty}\left(\frac{-1}{n}\right)^{n+1} e^{-a^2(n\pi/r_0)^2t}\sin n\pi\frac{r}{r_0},$$

definite a measure of the age of the earth as the visible processes and effects of stratification to which the geologists appeal. In short, the only definite results which Fourier's analysis appears to me to have contributed to knowledge concerning the cooling of our planet are the two following, namely: first, that the process of cooling goes on so slowly that nothing less than a million years is a suitable time unit for measuring the historical succession of thermal events; and secondly, that this process of cooling goes on substantially as if the earth possessed neither oceans nor atmosphere.

It was the well-founded boast of Laplace in the early part of the century that astronomy is the most perfect of the sciences;* and expert contemporary opinion, as we have seen in the case of no less a personage than Green, agreed that the Mécanique Céleste left little room for further advances. Indeed, it would appear that the completeness and the brilliancy of the developments of celestial dynamics during the half century ending with 1825 (the period of Laplace's activity) completely overshadowed all other sciences and retarded to some extent the progress of astronomy itself. The stupendous work of Laplace was chiefly theoretical, however, and he was content in most cases with observational data which accorded with theory to terms of the first order of approximation only. He was probably not so profoundly impressed as men of science at this end of the century are with the necessity of testing a theory by the most searching observational means available. In fact, in elaborating his methods and in applying his theories to the members of the solar system, it has been essential for his disciples to display a degree of ingenuity and a persistence of industry worthy of the master himself. But the prerequisite to progress in celestial mechanics consisted not so much in following up immediately the lines of

---

$$ru = ru_0 - \frac{2r_0u_0}{\sqrt{\pi}} \sum_{n=0}^{n=\infty} \int_{(2n+1)m_0-m}^{(2n+1)m_0+m} e^{-z^2}dz.$$

In these u is the temperature at a distance r from the center of the sphere at any time t; u_0 is the initial uniform temperature of the sphere; r_0 is the radius of the surface of the sphere; a^2 is the diffusivity, supposed constant; and $m = r/(2a\sqrt{t})$, $m_0 = r_0/(2a\sqrt{t})$. It will be observed that when the first series (which is Fourier's) converges very slowly the second converges very rapidly, and vice versa. It will be seen, also, that the series refuse, as they should, to give values of the temperature corresponding to negative values of the time.

* "L'Astronomie, par la dignité de son objet et par la perfection de ses théories, est le plus beau monument de l'esprit humain, le titre le plus noble de son intelligence." Système du Monde, Ed., 1884, p. 486.

investigation laid down by Laplace, as in perfecting the methods and increasing the data of observational astronomy.

The development of this branch of science along with the development of the closely related science of geodesy, is a work essentially of the present century and must be attributed chiefly to the German school of astronomers led by Gauss and Bessel. It is to these eminent minds, as well known in pure as in applied mathematics, that we are indebted for the theories, and for the most advantageous methods of use, of instrumental appliances, and for the refined processes of numerical calculation which secure the best results from observational data. It is a fortunate circumstance, perhaps, considering the irreverence which some modern pure mathematicians show for numerical computations, that Gauss and Bessel began their careers long before the resistless advent of the theory of functions and the theory of groups.

The story of the opportune discovery of the planet Ceres, as related by Gauss himself in the preface to his Theoria Motus Corporum Cœlestium, is well known; but it is less well known that the merit of this magnificent work lies rather in the model groups of formulas presented for the precise numerical solution of intricate problems than in the facility afforded for locating the more obscure members of the solar system. Indeed, the works of Gauss and Bessel are everywhere characterized by a clear recognition of the important distinction between those solutions of problems which are, and those which are not, adapted to numerical calculation. They showed astronomers how to systematize, to expedite, and to verify arithmetical operations in ways which were alone adequate to the accomplishment of the vast undertakings which have since been completed in mathematical geodesy and in sidereal astronomy.

Among the most important contributions of these authors to geodesy and astronomy in particular, and to the precise observational sciences in general, is that branch of the theory of probability called the 'method of least squares.' * No single adjunct has done so much as this to perfect plans of observation, to systematize schemes of reduction, and to give definiteness to computed results. The effect of the general adoption of this method has been somewhat like the

---

* Gauss's fundamental paper in this subject is "Theoria combinationis observationum erroribus minimis obnoxiae," and dates from 1821. Werke, Band IV.

Bessel's numerous contributions to this subject are found in his Abhandlungen cited above, p. 142.

effect of the general adoption by scientific men of the metric system; it has furnished common modes of procedure, common measures of precision, and common terminology, thus increasing to an untold extent the availability of the priceless treasures which have been recorded in the century's annals of astronomy and geodesy.

When we pass from the field of observational astronomy to the more restricted but more intricate field of dynamical astronomy, it is apparent that Laplace and his contemporaries quite underestimated the magnitudes of the mathematical tasks they bequeathed to their successors. Laplace, almost unaided, had performed the unparalleled feat of laying down a complete outline of the 'system of the world'; but the labor of filling in the details of that outline, of bringing every member of the solar system into harmony at once with the simple law of gravitation and with the inexorable facts of observation, is a still greater feat which has taxed the combined efforts of the most acute analysts and the most skillful computers of the preceding and present generation.

It is impossible within the limits of a semi-popular address to do more than mention in the most summary way the extraordinary contributions to dynamical astronomy made especially during the present half-century, contributions formidable alike by reason of their bulk and by reason of the complexity of their mathematical details. An account of the theory of the perturbative function, or of the theory of the moon, for example, would alone require space little short of a volume.* To mention only the most conspicuous names, there is the pioneer and essentially prerequisite work of the illustrious Gauss and the incomparable Bessel. There is the remarkable work of the brilliant Leverrier (1811–1877) and the not less brilliant Adams (1819–1892),† well known to popular fame by reason of what may be called their mathematical discovery of the planet Neptune. Then came the monumental Tables de la Lune‡ from the arithmetical laboratory of the indefatigable Hansen; and this marvellous production was quickly followed (1860) by the equally ponderous, and mathematically more important,

---

* A good account of the progress in dynamical astronomy from 1842 to 1867 is given by Delaunay in his "Rapport sur les progrès de l'Astronomie," Paris, 1867.

† The papers of Adams have been edited by Professor W. G. Adams and supplied with a biographical memoir by Professor J. W. L. Glaisher, under the title: Scientific Papers of John Couch Adams, Cambridge, At the University Press, vol. 1, 1896.

‡ Published by the British Government in 1857.

Théorie du Mouvement de la Lune* from the pen of the admirably fertile and industrious Delaunay. And finally, there is the still more elaborate work, bringing this great problem of the solar system well-nigh to completeness of solution, which, by common consent, is credited to the two preceding presidents of the AMERICAN MATHEMATICAL SOCIETY.†

Probably no mathematico-physical undertakings of the century have yielded so many definite, quantitative, results to the permanent stock of knowledge as the researches in dynamical astronomy. But notwithstanding the astonishing degree of perfection to which this science has been brought, there are still some outstanding discrepancies which indicate that the end of investigation is yet a long way off. The moon, which has given astronomers, as well as other people, more trouble than any other member of the solar system, is still devious to the extent of a few seconds in a century. The earth, also, it is suspected, is irregular as a time keeper by a minute but sensible amount ‡; while it has been proved recently, by the exquisite precision of modern observations, that the earth's axis of rotation wanders in a complex way through small but troublesome angles from its mean position, thus causing variations in the astronomical latitude of a place.§

---

* *Mémoires de l'Académie des Sciences de l'Institut Impérial de France*, Tomes XXVIII., XXIX.

† For an account of the more recent work of Gyldén and Poincaré, reference is made to the presidential address of Dr. G. W. Hill, "Remarks on the progress of celestial mechanics since the middle of the century"; BULLETIN, 2d series, vol. 2, no. 5, p. 125.

‡ The effect of tidal friction on the speed of rotation of the earth appears to have been first explained by Ferrel in a "Note on the influence of the tides in causing an apparent acceleration of the moon's mean position." This paper was read before the American Academy of Arts and Sciences, in December, 1864, only a few weeks before Delaunay read a similar paper before the French Academy. See Ferrel's autobiography cited on p. 144. See also Delaunay's account of his own work in "Rapport sur les progrès de l'Astronomie," Paris, 1867.

§ The cause of such variations is found in the relative mobility of the parts of the earth, especially in the mobility of the oceans and atmosphere. Three types of variation may occur, namely: 1st, that due to sudden changes in the relative positions of parts of the earth's mass; 2d, that due to secular changes in position of those parts; and 3d, that due to periodic shiftings of those parts. Of these the most important appears to be the periodic type. A surprising, and as yet not fully explained, discrepancy brought to light by the discovery of latitude variations is the fact that the instantaneous axis of rotation of the earth makes a complete circuit around the axis of figure in about 428 days instead of in about 305 days as has been supposed from the time of Euler down to the present decade. The discovery of this discrepancy is due to

A question of intense interest to astronomers in the early part of the century is that of the stability of the solar system. Lagrange, Laplace, and Poisson thought they had demonstrated that, whatever may have been the origin of this system, the existing order of events will go on indefinitely. This conclusion seems to have been alike satisfactory to scientific and unscientific men. But with the growth of the doctrine of energy and with the developments of thermodynamics, it has come to appear highly probable that the solar system has not only gone through a long series of changes in the past, but is destined to undergo a similarly long series of vicissitudes in the future. In other words, while our predecessors of a century ago thought the 'system of the world' stable, our contemporaries are forced to consider it unstable.*

But interesting as this question of stability still is, there is no pressing necessity, fortunately, for a determination of the ulterior fate of our planet. A more important question lies close at hand, and merits, it seems to me, immediate and serious investigation. This question is the fundamental one whether the beautifully simple law of Newtonian attraction is exact or only approximate. No one familiar with celestial mechanics, or with the evidence for the law of gravitation as marshalled by Laplace in his Système du Monde can fail to appreciate the reasons for the profound conviction, long held by astronomers, that this law is exact. But on the other hand, no one acquainted with the obstinate properties of matter can now be satisfied with the Newtonian law until it is proved to hold true to a much higher degree of approximation than has been attained hitherto.* For, in spite of the superb experimental investigations made particularly during the past quarter of a century by Cornu

---

Dr. S. C. Chandler, and was announced in the *Astronomical Journal*, no. 248, November, 1891. For the mathematical theory of this subject and for titles of the principal publications bearing on this theory, reference may be made to the author's paper on "Mechanical interpretation of the variations of latitudes," *Astronomical Journal*, no. 345, May, 1895; and to a paper by S. S. Hough on "The rotation of an elastic spheroid," *Philosophical Transactions*, no. 187, 1896.

* See a review of this subject by M. H. Poincaré, "Sur la stabilité du système solaire," in *Annuaire du Bureau des Longitudes* for 1898.

† As to the degree of precision with which the Newtonian law is established by astronomical data see Professor Newcomb's "Elements of the four inner planets and the fundamental constants of astronomy," Supplement to American Ephemeris and Nautical Almanac for 1897, Washington, 1895.

and Baille,[*] Poynting,[†] Boys,[‡] Richarz and Krigar-Menzel,[§] and Braun,[||] it must be said that the gravitation constant is uncertain by some units in the fourth significant figure, and possibly even by one or two units in the third figure;[¶] thus falling along with the sun's parallax, the annual stellar aberration, and the moon's mass amongst the least well determined constants of the solar system. Here then is a fruitful field for research. The direct measurement of the gravitation constant to a much higher degree of precision seems to present insuperable obstacles; but may not the result be reached by indirect means; or, may it not be possible to make the solar system break its Sphinx-like reticence of the centuries and disclose the gravitational mechanism itself?

Just as the theories of astronomy and geodesy originated in the needs of the surveyor and navigator, so has the theory of elasticity grown out of the needs of the architect and engineer. From such prosaic questions, in fact, as those relating to the stiffness and the strength of beams, has been developed one of the most comprehensive and most delightfully intricate of the mathematico-physical sciences. Although founded by Galileo, Hooke, and Mariotte in the seventeenth century, and cultivated by the Bernoullis and Euler in the last century, it is, in its generality, a peculiar product of the present century.[**] It may be said to be the

---

[*] *Comptes rendus*, LXXVI. (1873).

[†] The Mean Density of the Earth, by J. H. Poynting, London, Charles Griffin & Co. (1894).

[‡] *Philosophical Transactions*, no. 186 (1895).

[§] *Sitzungsberichte*, Berlin Academy, Band 2 (1896).

[||] *Denkschriften*, Math. Natur. Classe, Vienna Academy, Bd. LXIV. (1897).

[¶] The results of the investigators mentioned for the gravitation constant are, in C. G. S. units, as follows, the first result having been computed from data given by MM. Cornu and Baille in the publication referred to:

Cornu and Baille (1873)	6668×10^{-11}
Poynting (1894)	6698×10^{-11}
Boys (1894)	6657×10^{-11}
Richarz and Krigar-Menzel (1896)	6685×10^{-11}
Braun (1897)	6658×10^{-11}

Regarding these as of equal weight, their mean is 6673×10^{-11} with a probable error of ± 5 units in the fourth place, or 1/1330th part. This is of about the same order of precision as that deduced by Professor Newcomb from astronomical data.

[**] An admirable history of this science, dealing with its technical aspects, was projected by Professor Isaac Todhunter and completed by Professor Karl Pearson, under the title, A History of the Theory of Elas-

engineers' contribution of the century to the domain of mathematical physics, since many of its most conspicuous devotees, like Navier, Lamé, Rankine, and Saint-Venant, were distinguished members of the profession of engineering; and it is a singular circumstance that the first of the great originators in this field, Navier, should have been the teacher of the greatest of them all, Barré de Saint-Venant.* Other great names are also prominently identified with the growth of this theory and with the recondite problems to which it has given rise. The acute analysts, Poisson, Cauchy, and Boussinesq, of the French school of elasticians; the profound physicists, Green, Kelvin, Stokes, and Maxwell, of the British school; and the distinguished Neumann (Franz Ernst, 1798–1895), Kirchhoff (1824–1887), and Clebsch (1833–1872), of the German school, have all contributed heavily to the aggregate of concepts, terminology, and mathematical machinery which make this at once the most difficult and the most important of the sciences dealing with matter and motion.

The theory of elasticity has for its object the discovery of the laws which govern the elastic and plastic deformation of bodies or media. In the attainment of this object it is essential to pass from the finite and grossly sensible parts of media to the infinitesimal and faintly sensible parts. Thus the theory is sometimes called molecular mechanics, since its range extends to infinitely small particles of matter if not to the ultimate molecules themselves. It is easy, therefore, considering the complexity of matter as we know it in the more elementary sciences, to understand why the theory of elasticity should present difficulties of a formidable character and require a treatment and a nomenclature peculiarly its own.

While it would be quite inappropriate on such an occasion to go into the mathematical details of this subject, I would recall your attention for a moment to the surprisingly simple and beautiful concepts from which the mathematical in-

---

ticity and the Strength of Materials from the time of Galilei to the present time. Cambridge: at the University Press—

Vol. I., Galilei to Saint-Venant, 1886;

Vol. II., Parts I. and II., Saint-Venant to Lord Kelvin, 1893.

A capital though abridged history of the science is given by Saint-Venant in his annotated edition of Navier's Résistance des Corps Solides, Troisième édition, Paris, 1864.

The history of Todhunter and Pearson is dedicated to Saint-Venant, who has been fitly called 'the dean of elasticians.'

* And this illustrious master has left a worthy pupil in M. J. Boussinesq, Professor in the Faculty of Sciences, Paris.

vestigation proceeds rapidly and unerringly to the equations of equilibrium or motion of a particle of any medium. The most important of these are the concept which relates to the stresses on the particle arising from its connection with adjacent parts of the medium, and the concept with regard to the distortions of the particle which result from the stresses. If the particle be a rectangular parallelopiped, for example, these stresses may be represented by three pressures or tensions acting perpendicularly to its faces together with three tensions acting along, or tangentially to, those faces. Thus the adjacent parts of the medium alone contribute six independent force components to the equations of equilibrium or motion; and the stresses, or the amounts of force per unit area, which produce these components are technically known as tractions or shears according as they act perpendicularly to or tangentially along the sides of the particle.* Corresponding to these six components there are six different ways in which the particle may undergo distortion. That is, it may be stretched or squeezed in the three directions parallel to its edges; or, its parallel faces may slide in three ways relatively to one another. These six different distortions, which lead in general to a change in the relative positions of the ends of a diagonal of the parallelopiped, are measured by their rates of change, technically called strains, and distinguished as stretches or slides according as they refer to linear or angular distortion.†

It is from such elementary dynamical and kinematical considerations as these that this theory has grown to be not only an indispensable aid to the engineer and physicist, but one of the most attractive fields for the pure mathematician. As Pearson has remarked, "There is scarcely a branch of physical investigation, from the planning of a gigantic bridge to the most delicate fringes of color exhibited by a crystal, wherein it does not play its part."‡ It is, indeed, fundamental in its relations to the theory of structures, to the theory of hydromechanics, to the elastic solid theory of light, and to the theory of crystalline media.

In closing these very inadequate allusions to this intensely

---

* The terminology here used is that of Todhunter and Pearson, History of the Theory of Elasticity and Strength of Materials, vol. I., note B.

† The terminology and symbology of the theory of elasticity appear to be more highly developed than those of any other mathematical science. A comparison of the terms and symbols of elasticity with those of the older subject of hydromechanics as shown in part on next page is instructive.

‡ History of Elasticity, etc., vol. 1, p. 872.

IN ELASTICITY.

Stresses.

$$\textit{Tractions}\begin{cases} p_{xx} \\ p_{yy} \\ p_{zz} \end{cases}$$

$$\textit{Shears}\begin{cases} p_{yz} \\ p_{zx} \\ p_{xy} \end{cases}$$

Strains.

$$\textit{Stretches}\begin{cases} s_x \\ s_y \\ s_z \end{cases}$$

$$\textit{Slides}\begin{cases} \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{cases}$$

$$\textit{Shifts, or components of displacement}\begin{cases} u \\ v \\ w \end{cases}$$

$$\textit{Shift-fluxions, or space rates of change of shifts}\begin{cases} \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \\ \frac{\partial v}{\partial x}, \text{ etc.} \\ \frac{\partial w}{\partial x}, \text{ etc.} \end{cases}$$

$$\textit{Dilatation, } \theta = s_x + s_y + s_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\textit{Twists}\begin{cases} \tau_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \tau_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \tau_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{cases}$$

Displacement potential in irrotational, or pure, strain.

IN HYDROMECHANICS.

Fluid pressure
p

$$\textit{Component velocities}\begin{cases} u \\ v \\ w \end{cases}$$

$$\textit{Space rates of change of component velocities}\begin{cases} \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \\ \frac{\partial v}{\partial x}, \text{ etc.} \\ \frac{\partial w}{\partial x}, \text{ etc.} \end{cases}$$

$$\textit{Expansion, } \theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\textit{Component spins, or components of molecular rotation}\begin{cases} \xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{cases}$$

Velocity potential in irrotational motion.

practical and abstrusely mathematical science, it is fitting that attention should be called to the magnificent labors of the 'dean of elasticians,' M. Barré de Saint-Venant. It was the object of his life-work to make the theory of elasticity serve the utilitarian purposes of the engineer and at the same time to divest it so far as possible of all empiricism. His epoch-making memoir * of 1853 on the torsion of prisms is not only a classical treatise from the practical point of view but one of equal interest and importance in its theoretical aspects. His investigations are everywhere delightfully interesting and instructive to the physicist; and many parts of them are replete, as observed by Kelvin and Tait,† with "astonishing theorems of pure mathematics, such as rarely fall to the lot of those mathematicians who confine themselves to pure analysis or geometry, instead of allowing themselves to be led into the rich and beautiful fields of mathematical truth which lie in the way of physical research." More important still in a didactic sense, are his annotated edition of Navier's Résistance des Corps Solides of 1864, and his annotated edition of the French translation of the Theorie der Elasticität fester Körper of Clebsch, which appeared in 1883. These monumental works, whose annotations and explanatory notes quite overshadow the text of the original authors, must remain for a long time standard sources of information as to the history, theory, methods, and results of this complex subject. The luminous exposition, the penetrating insight into physical relations, and the scientific candor in his criticism of other authors, render the work of Saint-Venant worthy of the highest admiration.

Closely allied to the theory of elasticity, though historically much older, is the science of hydromechanics. The latter is, indeed, included essentially in the former; and probably the great treatises of the next century will merge them under the title of molecular mechanics. It may seem somewhat singular that the mathematical theory of solids should have originated so many centuries later than the theory of fluids; for at first thought, tangible though flexible solids would appear much more susceptible of investigation than mobile liquids and invisible gases. But a little reflection leads one to the conclusion that it was, in fact, much easier to observe the data essential to found a theory of hydromechanics than it was to discover the principles

---

* "Mémoire sur la torsion des prismes, etc.," published in *Mémoires des Savants étrangers*, Tome XIV., 1855.

† Natural Philosophy, second ed., Part II., p. 249.

which lead to the theory of stress and strain; and the time-interval between Archimedes and Galileo may serve perhaps as a rough measure of the relative complexity of hydrostatics and the theory of flexure and torsion of beams. It must not be inferred, however, that the simplicity of the phenomena of fluids in a state of relative rest extends to the phenomena of fluids in a state of relative motion; for the gap which separates hydrostatics from hydrokinetics is one which has not yet been fully bridged even by the aid of the powerful resources of modern mathematics.

The elements of hydrokinetics, with which branch of hydromechanics this sketch is alone concerned, were laid down by Euler about the middle of the last century.* It is to him that we owe the equations of motion of a particle of a 'perfect fluid.' This is an ideal fluid, moving without friction, or subject, in technical terminology, to no tangential stress. But while no such fluids exist, it is easily seen that under certain circumstances this assumed condition approaches very closely to the actual condition; and, in accordance with the method of mathematico-physical science in disentangling the intricate processes of nature, progress has proceeded by successive steps from the theory of ideal fluids towards a theory of real fluids.

The history of the developments of hydromechanics during this century has been very carefully and completely detailed in the reports to the British Association for the Advancement of Science of Sir George Gabriel Stokes,† in 1846, and of Professor W. M. Hicks,‡ in 1881 and 1882. And the history of the subject has been brought down to the present time by the address of Professor Hicks before Section A of the British Association for the Advancement of Science in 1895, and by the report§ of Professor E. W. Brown to Section A of the American Association for the Advancement of Science in 1898. It may suffice here, therefore, to glance rapidly at the salient points which mark the advances from the state of the science as it was left by Lagrange a hundred years ago.

---

* "Principes généraux du mouvement des fluides." *Histoire de l'Académie de Berlin*, 1755.

"De principiis motus fluidorum." *Novi Commentarii Academiæ Scientiarum Imperialis Petropolitanæ*, Tomus XLV., Pars I., pro anno 1759.

† "Report on recent researches in hydrodynamics," B. A. A. S. Report for 1846.

‡ "Report on recent progress in hydrodynamics," B. A. A. S. Reports for 1881, 1882.

§ "On recent progress towards the solution of problems in hydrodynamics," A. A. A. S. Report for 1898.

Reference should be made also to Professor A. E. H. Love's paper "On recent English researches in vortex-motion" in the *Mathematische Annalen*, Band 30, 1887.

The general problem of the kinetics of a particle of 'perfect fluid' is easily stated. It is this:* given for any time and for any position of the particle its internal pressure, its density, and its three component velocities, along with the forces to which it is subject from external causes; to find the pressure, density, and velocity components corresponding to any other time and any other position. There are thus, in general, five unknown quantities requiring as many equations for their determination. The usual six equations of motion, or the equations of d'Alembert, contribute only three to this required number, namely, the three equations of translation of the particle, since the three which specify rotation vanish by reason of the absence of tangential stress. A fourth equation comes from the principle of the conservation of mass, which is expressed by equating the time rate of change of the mass of the particle to zero. This gives what is technically called the equation of continuity. A fifth equation is usually found in the law of compressibility of the fluid considered.†

Now, the equations of rotation, as just stated, refuse to answer the question whether the particles proceed in their orbits without rotation or whether they undergo rotation along with their motion of translation. This was a critical question, for the failure to press and to answer it seems to have retarded progress for nearly half a century. Lagrange, and after him Cauchy and Poisson, knew that under certain

---

*The statement here given is that of the 'historical method,' which seeks to follow a particle of fluid from some initial position to any subsequent position and to specify its changes of pressure, density, and speed. What is known as the 'statistical method,' on the other hand, directs attention to some fixed volume in the fluid and specifies what takes place in that volume as time goes on.

† The five equations in question are

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{d\,(V\rho)}{dt} = 0,$$

$$\frac{dv}{dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{dw}{dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}, \qquad p = f\,(\rho);$$

in which p is the pressure and ρ is the density at the centroid (x, y, z) of the particles; V is its volume; u, v, w are its component velocities; and X, Y, Z are the force components per unit mass arising from external causes.

conditions[*] the differential equations of motion are integrable, but they do not appear to have understood the physical meaning of these conditions. It remained for Sir George Gabriel Stokes to show that the Lagrangian conditions of integrability correspond to the case of no molecular rotation, thus clearly distinguishing the two characteristic types of what we now call irrotational and rotational motion.[†] Such was the great step made by Stokes in 1845 ; and it furnishes a forcible illustration of the importance, in applied mathematics, of attending to the physical meaning of every symbol and every combination of symbols.

Thirteen years later came the remarkable memoir of Helmholtz (1821–1894) on the integrals of the equations of hydrokinetics for the case of rotational, or vortex, motion.[‡] This memoir is alike wonderful for the directness with which the mathematical argument proceeds to its conclusions and for the clearness of insight it affords of the physical phenomena discussed. In short, it opened up a new realm and supplied the results, concepts, and methods which led the way to the important advances in the science made during the past three decades.

Another powerful impulse was given to hydrokinetics, and to all other branches of mathematical physics as well, by Kelvin and Tait's Natural Philosophy—the Principia of the nineteenth century—the first edition of which appeared in 1867. From this great work have sprung most of the ideas and methods appertaining to the theory of motion of solids in fluids, a theory which has yielded many interesting and surprising results through the researches of Kirchhoff, Clebsch, Bjerknes, Greenhill, Lamb, and others. Of prime importance also are the numerous contributions of Lord Kelvin to other branches of hydrokinetics, and particularly to the theory of rotational mo-

---

[*] That is, when $udx + vdy + wdz$ is a perfect differential, u, v, w being velocity components ; or, when

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

which are the doubles of the components of molecular rotation, vanish, these latter being the conditions for the existence of a velocity potential.

[†] This discovery of Stokes was announced in his fundamental paper "On the theories of internal friction of fluids in motion, and of the equilibrium and motion of elastic solids," *Transactions of the Cambridge Philosophical Society*, vol. 8. Reprinted also in his Mathematical and Physical Papers, vol. I.

[‡] "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen," *Crelle's Journal für die reine und angewandte Mathematik*, 1858.

tion.* In fact, every department of the entire science of hydromechanics, from the preliminary concepts up to his vortex atom theory of matter, has been illuminated and extended by his unrivalled versatility.

When we turn to the more intricate branch of the subject which deals with the motion of viscous fluids, or with the motion of solids in such fluids, it appears that the progress of the century is less marked, but still very noteworthy. This branch is closely related to the theory of elasticity, and goes back naturally to the early researches of Navier, Poisson, and Saint-Venant; but the revival of interest in this, as well as in the less intricate branch of the subject, seems to date from the fruitful memoir † of Stokes of 1845, and from his report to the British Association for the Advancement of Science of 1846. Since then many interesting and useful problems relative to the flow of viscous fluids and to the motion of solids in such media, have been successfully worked out to results which agree fairly well with experiment. But on the whole, nothwithstanding the searching investigations in this field of Stokes, Maxwell, Helmholtz, Boussinesq, Meyer, Oberbeck, and many others, it must be said that difficulties, both in theory and in experiment, of a formidable character remain to be surmounted.‡

Of all branches of hydromechanics there is none of so great practical utility and of such widely popular interest as the theory of tides and waves. These phenomena of the sea are appreciable to the most casual observer; and there has been no lack of impressive descriptions of their effects from the days of Curtius Rufus down to the present time. The mechanical theory of tides and waves is, however, a distinctly modern development whose perfection must be credited to the labors of the mathematicians of the present century. §

---

* "On vortex motion," 1867. *Transactions of the Royal Society of Edinburgh*, vol. 25.

† Cited on p. 160.

‡ An extremely interesting method of experimental investigation has been recently applied with success by Professor Hele-Shaw. See a paper by him on "Stream-line motion of a viscous film," and an accompanying paper by Sir G. G. Stokes on "Mathematical proof of the identity of the stream lines obtained by means of a viscous film with those of a perfect fluid moving in two dimensions." Report of British Association for the Advancement of Science for 1898.

§ An excellent summary of the history and theory of tides, and of methods of observing and predicting them, is given by Dr. Rollin A. Harris in his Manual of Tides, published as Appendices 8 and 9 of the Report of the United States Coast and Geodetic Survey for 1897.

Here, again, progress is measured from the advanced position occupied by Laplace, who was the first to attempt a solution of the tidal problem on hydrokinetic principles. After the fundamental contributions of Laplace, contained in the second and fifth volumes of the Mécanique Céleste, the next decisive advance was that made by Sir George Airy (1801–1892) in his article on Tides and Waves, which appeared in the Encyclopedia Metropolitana in 1842. A quarter of a century later came the renaissance, started undoubtedly by the great memoir of Helmholtz and by the Natural Philosophy of Kelvin and Tait, along with Lord Kelvin's inspiring communications on almost every phase of wave and tidal problems to scientific societies and journals. Then followed the decided theoretical improvements in tidal theory of Professor William Ferrel,* particularly in the development of the tide generating potential and in the determination of the effects of friction. And a little later there appeared the novel investigations of Professor G. H. Darwin, who, in addition to furnishing a complete practical treatment of terrestrial tides,† has extended tidal theory to the solar system and thrown an instructive light on the evolutionary processes whence the planets and their satellites have emerged and through which they are destined to pass in the future.‡

As we reflect on the progress which has been thus summarily and quite inadequately outlined, it will appear that the mathematicians of the nineteenth century have contributed a splendid aggregate of permanent accessions to knowledge in the domain of the more exact of the physical sciences. And as we turn from the certain past to the less certain future one is prone to conjecture whether this brilliant progress is to continue, and, if so, what part the AMERICAN MATHEMATICAL SOCIETY may play in promoting further advances. With respect to these inquiries I should be loath to hazard a prediction or to offer advice. But there appears to be no reason for entertaining anything other than optimistic expectations. The routes along which exploration may proceed are numerous and attractive. We have only to follow the example set by Laplace, Poisson, Green, Gauss, Maxwell, Kirchhoff, Saint-Venant, Helmholtz, and their eminent contemporaries and successors. In commending the works of these great masters, to the younger members especially of the AMERICAN MATHEMATICAL SOCIETY, I would not be understood as urging the cultivation of pure mathematics less, but rather as suggesting the pursuit of applied mathematics more. The same sort of fidelity to research and the same sort of genius for infinite industry which enabled those masters to accomplish the grand results of the nineteenth century may be confidently expected to achieve equally grand results in the twentieth century.

---

* Tidal Researches. Appendix to Report of United States Coast and Geodetic Survey for 1874. Washington, 1874.

† In article on Tides in Encyclopedia Britannica, 9th edition.

‡ Darwin's investigations are published in a series of papers in the *Philosophical Transactions of the Royal Society of London*, Parts I., II., 1879; Part II., 1880; Part II, 1881; Part I., 1882. They are republished in part in Appendix G, Thomson and Tait's Natural Philosophy, second edition. See also the capital semi-popular work, The Tides and Kindred Phenomena in the Solar System, by G. H. Darwin, Boston and New York, Houghton, Mifflin and Co., 1898.

THE STATUS OF IMAGINARIES IN PURE GEOMETRY.

(Read before the American Mathematical Society, October 28, 1899.)

IN teaching the elements of analytical geometry we are practically forced to allow, even to encourage, a slipshod identification of the field of geometry with the field of algebra. We must all have realized the disadvantages attendant on this course. If ever we have the chance of repairing the error—if error indeed it be at that stage—it is in teaching synthetic geometry; but we can repair it then only if we can establish the existence of imaginary elements without the slightest dependence on algebra. Many books refer to the analogy of algebra as affording sufficient basis, others openly rely on algebraic principles; Chasles, for instance, in the Géométrie Supérieure (pp. 54–57) relies essentially on quadratic equations, whose imaginary roots assure him of the existence of imaginary points.

The two chief books that deal with absolutely pure geometry are those of von Staudt and Reye. It is one of the axioms of modern mathematics that von Staudt placed the doctrine of imaginaries on a firm geometrical basis; but logical and convincing as his treatment is, when patiently studied in all its detail, it yet seems to me hardly practicable as a class-room method.

Von Staudt's primary domain is the visible universe; the elements of his geometry, together with the idea of direction, are an intellectual abstraction from the results of observation. He then extends his domain beyond the visible universe by formal definition; to replace the idea of direction he introduces a set of "ideal points," and finds that the nature of an ideal point is the same as that of a common point, and that the relation of the ideal points and lines to one another is precisely that of points and lines in a plane. Thus his second domain is the visible universe increased by one ideal plane.

To attain absolute consistency of language he introduces pairs of feigned, or imaginary, elements. Certain arrangements of pairs of points on a line give rise to a pair of points; certain other arrangements, differing only in their pictured form, apparently do not. Now as we have already extended the domain of geometry beyond the visible universe, the fact that the point pair is not visible cannot be accepted as a proof that it is non-existent. Although the conception of the range of points with its contents is derived from observation, yet we have to take into account the fact that we may not be able to observe the whole. Hence instead of jumping to the conclusion that the points are non-existent, we investigate the alternative hypothesis, namely, that there are still two points, though not pictured. As regards this purely intellectual hypothesis we have to enquire whether it is *tenable*, and whether it is *manageable*.

To prove that the hypothesis is tenable, von Staudt had first to find some device for separating the two elements of a pair, seeing that his geometry accepts a point as element. not a point pair. This difficulty delayed him from 1847–1856; he overcame it finally by a sublimated intellectual artifice, that of the "sense" of a form. He had then to show that every feigned, or imaginary, element is endowed with every formal property of a natural element. His feigned elements are (1) imaginary points, each lying by definition on one real straight line (imaginary elements of a range); (2) imaginary planes, each passing by definition through one real straight line (imaginary elements of an axial pencil); (3) imaginary lines of the first kind, each lying by definition in one real plane and passing through one real point (imaginary elements of a flat pencil); (4) imaginary lines of the second kind, each belonging by definition to a real regulus, but possessing no real point and no real plane. He investigates every possible relation of real and imaginary points, lines, and planes, defining with great care every phrase that has reference to the imaginary elements. Of course all this detail fills up page after page, even unto weariness; but by it he *does* prove that adopting the particular phraseology that assumes the existence of these feigned elements, all verbal consequences are correct. Thus the hypothesis is shown to be tenable.

Von Staudt states explicitly (Beiträge, p. 77) that there are no other imaginary elements; that is to say, that no use of the accepted geometrical processes will lead us outside the domain now defined. He shows also that correlating the real (or observed) elements of two forms does also correlate the imaginary (or inferred) elements. Thus there is no danger of unknown regions of imaginaries opening up before us; we have extended the domain, it is true, but not indefinitely. To complete the proof that the hypothesis is manageable, we must next investigate the nature of this extended domain, in order to determine whether the imaginary elements, thus separately defined, are arranged in accordance with any law of continuity. Von Staudt's results prove, for example, as regards an imaginary straight line of the first or second kind, that the totality of points can be pictured on a real plane, and that there is the same possibility of passing from one point to another as from one point of the plane to another.

Thus the hypothesis is both tenable and manageable; by introducing these formal elements he enlarges the domain of his geometry, without in any way affecting its nature or modifying its laws.

The principal objections to this, as an elementary method, are the following:—

(1) the extreme difficulty and tediousness of the complete proof of the identity of the properties of natural elements and formal elements.

(2) the apparent break in passing from the seen to the unseen, which arouses scepticism as to whether the formal elements can truly be said to "exist." While the reason, if sufficiently trained, is convinced, all natural instincts rebel. The whole thing impresses the natural man as simply a tour de force.

We turn now to geometry as treated by Reye; or rather, as it would naturally have been treated by Reye if he had carried out the plan of his first chapter.* Elements of three kinds are given, absolutely—*a*, *b* and *c*-elements (points, lines, and planes), with a statement of properties and relations to serve instead of definitions. Attaching to these elements the numbers 1, 2, 3, and working with modulus 4, a number of the relations can be obtained arithmetically; for example, $1 + 1 = 2$ expresses that two points determine a line; $3 + 3 = 6 = 2$ [mod. 4], two planes determine a line; similarly $1 + 1 + 1 = 3$, $3 + 3 + 3 = 9 = 1$ [mod. 4], $1 + 2 = 3$, $3 + 2 = 5 = 1$ [mod. 4] give certain of the relations; $2 + 2 = 4 = 0$ [mod. 4] expresses the fact that two lines determine *nothing*. Multiplication by -1 reciprocates; $1 + 2 = 3$ becomes $-1 + (-2) = -3$, that is, $3 + 2 = 1$ [mod. 4]. A trivial suggestion, perhaps; but it brings out clearly the fact that the elements a, b, c are *not* precisely defined as points, lines, and planes.

---

* As I interpret it; it appears however that some others find in it a much closer agreement with the fundamental conceptions of von Staudt.

Reye now makes a very arbitrary assumption, namely, that the elements can be identified with the points, lines, and planes of ordinary space. This identification gives the points at infinity, to account for the disappearing intersection of lines in a plane; it gives rise also to such assumptions as that a line and a conic in a plane may possibly not intersect. Finally, imaginary elements are introduced, with a reference to von Staudt for the justification.

Now as I read Reye's first chapter, in the logical development of the system infinitely distant and imaginary elements must be simply a result of classification. We are *given* all points, all lines, all planes; that is all a, b, and c-elements that can have certain relations to one another, with certain properties that hold without exception. Then we *observe* that points, lines, and planes in the visible universe illustrate, or picture, these properties on the whole; and that we can make the picturing more exact, in detail, if we regard parallel lines as meeting somewhere. In fact, this is a necessary condition of the picturing; for if two parallel lines do not meet, they cannot represent two b-elements that lie in a c-element. Thus the conception of infinitely distant elements arises from the attempt to picture this particular intellectual domain in the visible universe. From this attempt there arises also the distinction between real and imaginary elements. For while the visible points, lines, and planes serve to picture some of the a, b, c-elements, we have no right to assume that they suffice to picture all. Allowing for this, we divide elements into picturable and non-picturable, and examine whether there are any in the second division.

This point cannot be decided until we make some additional statement about the nature of the given elements; perhaps it is enough to affirm the indestructibility of the elements of any given space, whether the elements be given singly and explicitly, or by implication as determined by other elements. Thus, for example, two lines in a plane determine a point, and no rotation of either line can cause this point to pass out of existence. Two point pairs A, A';

B, *B′* on a line are proper for the determination of a third point pair *C*, *C′*, namely, the common harmonic pair; the motion of one element *B′* along the given line causes a change in the positions of *C*, *C′*, but it cannot cause these points to pass out of existence; the common harmonic pair is determinate, however the two pairs *A*, *A′*; *B*, *B′* may be placed. Now picturing the two pairs on a line, it is found that if the segments *A*, *A′*; *B*, *B′* overlap, the third pair is non-picturable. Hence there are elements in the second division; and the only ones that we have detected occur by pairs.

Thus to construct this purely formal geometry we postulate elements of three kinds, *a*, *b*, *c*, with certain relations, in which most of their properties are formulated. In addition to the numerical relations already mentioned, we have to state that any one element can move in any other, adding explicitly that an *a* or *c* moving in a *b* can move only "forward" or "backward," and that elements are indestructible. At any stage we may attempt to picture these formal elements by means of a set of existing elements, subject to the same laws of combination, and may thus obtain a division of our formal elements into those that can be pictured in this way, and those that cannot; the division into real and imaginary results from the attempt to picture these formal elements in ordinary space.

This is merely intended as a suggestion to practical teachers; it is not set forth as a fully worked out scheme. A different formulation of the relations and properties may be preferable, but I believe the principle to be sound. If the elements are given in a purely intellectual manner, and not as abstractions from observation of the visible universe, then the distinction between real elements on the one hand, imaginary elements on the other, is purely a result of classification; the imaginary elements are given together with the real ones, instead of being adjoined to them by definition. We can even decide for ourselves that certain selected elements shall be pictured; but to say how many we can choose arbitrarily, and what others will then fall into the picturable class (real class), is hardly possible without considering von Staudt's theory of chains or strands (Ketten). Any exhaustive investigation of the number and arrangement of the picturable and non-picturable elements of any form must ultimately coincide with von Staudt's; but for the mere introduction of imaginaries into ordinary teaching the plan here suggested for avoiding these long discussions seems feasible. It has the additional advantage of justifying the use of diagrams in proving results that depend on so-called imaginaries.

I may add that in this strictly formal geometry metric relations must of course be introduced independently, by specifying an absolute configuration in the given domain, and classing as metric all relations of any form to this absolute configuration.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR, PENNSYLVANIA,
October, 1899.

NOTES.

THE presidential address of Professor R. S. WOODWARD, which appears in the present number of the BULLETIN, is also published separately in reprint form. Copies may be obtained from the Secretary at twenty-five cents each.

THE advisory board for mathematics at Cambridge University has recommended to the Senate of that institution a number of changes in the regulations relating to the mathematical tripos. It is proposed to omit entirely from Part I. the following subjects:—calculus of variations, elliptic functions, besselian functions, hydrodynamics, and sound, and to limit the extent of the requirements in rigid dynamics, electricity, optics, astronomy, and other subjects. The arrangement of the examination papers is to be changed. No papers are to be devoted exclusively to problems. At least half of the questions set throughout the examination shall be of an elementary character. The classic general restrictions as to methods of solutions for certain papers are to be no longer maintained. The ancient order of merit is to be abandoned. The successful candidates are to be arranged in three classes (wranglers, senior and junior optimes) of three divisions each, the names in each division being placed in alphabetical order. Corresponding changes are proposed for Part II. of the tripos. The Senate returned the proposals to the board for revision.

At one time the largest tripos at Cambridge, the mathematical tripos is now smaller than either the classical or the natural science tripos. For the four years 1869–72, one in eighteen of the resident undergraduates passed the mathematical tripos, Part I.; while for the five years 1895–99, the average falls to one in thirty-five.

ABOUT Easter 1900 the Berlin academy of sciences will mark the 200th anniversary of its foundation by a special celebration to which foreign delegates will be invited. Professor A. HARNACK is preparing an elaborate history of the academy for this occasion.

IN a recent circular the Society for the promotion of quaternions and allied mathematics is announced as now established. The coöperation of Americans in this international organization is solicited, not alone financially but also in the direction of original contributions to the literature of the subject. All communications should be addressed to the secretary for the United States, Professor A. S. HATHAWAY, Rose Polytechnic Institute, Terre Haute, Indiana.

A FAC-SIMILE reprint of Legendre's Théorie des Nombres has just been issued by M. Hermann, of Paris.

THE publishing house of William Ernst and Son, of Berlin, has recently published a historical memoir entitled "Die reine Mathematik in den Jahren 1884–1899," by Professor E. LAMPE. The paper is intended as a memorial monograph of the centennial jubilee of the Technische Hochschule of Berlin celebrated this year. The second part of the memoir presents a short account of the life of SIEGFRIED ARONHOLD, who was professor of mathematics at the Hochschule during the period from 1860 to 1883; the monograph is accompanied by a portrait of Aronhold. The same firm issued the Festschrift of the centennial, a separate volume.

A CATALOGUE of the mathematical library, of the late Professor H. SCHAPIRA, of Heidelberg, has been issued by Gustav Fock, of Leipzig. It is number 164 of the series of catalogues of the latter firm, and contains the titles of about twenty-two hundred different works.

M. A. HERMANN, of Paris, has just published an extensive catalogue, No. 65, of new and second-hand publications in the mathematical sciences. The catalogue contains nearly thirty-seven hundred numbers in mathematics and astronomy.

THE Munich academy of sciences has elected Professors G. DARBOUX, of Paris, and E. BELTRAMI, of Rome, corresponding members.

PROFESSOR P. STÄCKEL has been promoted to an ordinary professorship at the University of Kiel.

Professor A. Sommerfeld, of the Clausthal School of Mines, has accepted a call to the Aachen Polytechnic School as professor of mechanics.

Professor William Harkness, astronomical director of the U. S. Naval Observatory, was retired on December 17, 1899, and is succeeded by Professor S. J. Brown.

Dr. W. H. Maltbie has been promoted to a professorship of mathematics in the Woman's College of Baltimore, Md.

Mr. N. M. Parrish has been made professor of mathematics at the State Normal College, Florence, Ala.

Dr. N. A. Patillo has been elected to a professorship in mathematics in the Randolph-Macon Woman's College, Lynchburg, Va.

The following appointments in mathematics have been recently made: Mr. W. D. Cairns to an instructorship, and Mr. W. H. Sherk, to a tutorship at Oberlin College; Mr. R. T. House, to a tutorship in Miami University; Miss Brooks, to a tutorship at McGill University.

The seventieth anniversary of the birth of Professor B. Christoffel, professor emeritus of mathematics, was recently celebrated at the University of Strassburg.

Dr. E. M. Blake, who has recently returned from study abroad, has been appointed honorary fellow in mathematics at Cornell University.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Abhandlungen zur Geschichte der Mathematik. Heft IX: Festschrift zu M. Cantors siebzigsten Geburtstage, herausgegeben von M. Curtze und S. Günther. (F. Cajori, Notes on the history of logarithms; S. Dickstein, Zur Geschichte der Principien der Infinitesimalrechnung; S. Günther, Nicolaus von Cusa und seine Beziehungen zur mathematischen und physikalischen Geographie; K. Hunrath, Des Rheticus Canon doctrinae triangulorum und Vieta's Canon mathematicus; F. Müller, Zur Terminologie der ältesten mathematischen Schriften in deutscher Sprache; A. Nagi, die Rechenmethoden auf dem griechischen Abacus; P. Stäckel, F. A. Taurinus, ein Beitrag zur Vorgeschichte der nichteuklidischen Geometrie; F. A. Unger, Einige Additionsmaschinen; E. Wohlwill, Entdeckung der Parabelform der Wurflinie; F. Rosenberger, Die Geschichte der exakten Wissenschaften und der Nutzen ihres Studiums; etc.). Leipzig, Teubner, 1899. 8vo. 8 + 657 pp., 2 plates, 1 portrait. M. 20.00

ADAM (C.). See DESCARTES (R.).

BOLYAI (W.). See GAUSS (C. F.).

BRAUNMÜHL (A. VON). Vorlesungen über Geschichte der Trigonometrie. Teil I: Von den ältesten Zeiten bis zur Erfindung der Logarithmen. Leipzig, Teubner, 1899. 8vo. 7 + 260 pp. M. 9.00

CAJORI (F.). See ABHANDLUNGEN.

CURTZE (M.). See ABHANDLUNGEN.

DESCARTES (R.). Oeuvres, publiées par C. Adam et P. Tannery, sous les auspices du ministère de l'instruction publique. Correspondance. Vol. 3: janvier 1640–juin 1643. Paris, Cerf, 1899. 4to. 729 pp. Fr. 25.00

DICKSTEIN (S.). See ABHANDLUNGEN.

DÖLP (H.). Die Determinanten, nebst Anwendung auf die Lösung algebraischer und analytisch-geometrischer Aufgaben, elementar behandelt. 5te Auflage. Darmstadt, Roether, 1899. 8v. 4 + 95 pp. M. 2.00

FESTSCHRIFT zur Feier der Enthüllung des Gauss-Weber Denkmals in Göttingen. Herausgegeben von dem Fest-Comitee. (D. Hilbert: Grundlagen der Geometrie; E. Wiechert: Grundlagen der Elektrodynamik.) Leipzig, Teubner, 1899. 8vo. 92 + 112 pp. M. 6.00

FRENET (F.). Collection of problems in the infinitesimal calculus, translated by A. P. Nenashev. (Russian.) Moscow, 1899. 8vo. 8 + 208 pp. M. 3.00

FRONTERA (G.). See SONNET (H.).

GAUSS (C. F.) and BOLYAI (W.). Briefwechsel. Herausgegeben von F. Schmidt und P. Stäckel. Leipzig, Teubner, 1899. 4to. 13 + 208 pp., 1 plate, 1 facsimile. Cloth. M. 16.00

GRACE (J. H.) and ROSENBERG (F.). Elements of coördinate geometry. Part 2: Conics. London, Clive, 1899. 12mo. 324 pp. (University tutorial series.) 4s. 6d.

GRASSMANN (H.). Schraubenrechnung und Nullsystem. Halle, 1899. 4to. 20 pp. M. 1.50

GÜNTHER (S.). See ABHANDLUNGEN.

HILBERT (D.). See FESTSCHRIFT.

HINN (G. A.). Die Anwendung unendlicher Produkte in der Funktionentheorie. (Progr.) Sächsisch-Regen, 1899. 4to. 33 pp.

HUNRATH (K.). See ABHANDLUNGEN.

KIRSTEIN (O.). Konstruktion eines räumlichen Polarsystems aus einem Polartetraeder und drei Paar konjugierten Punkten. (Progr.). Meseritz, 1899. 4to. 10 pp.

LAMPE (E.). Die reine Mathematik in den Jahren 1884–1899. Nebst Actenstücken zum Leben von Siegfried Aronhold, weil. Professor der Mathematik (1860–1883) an der königl. technischen Hochschule zu Berlin. Mit seinem Bildnisse. Ein Gedenkblatt zur hundertjährigen Jubelfeier der königl. technischen Hochschule zu Berlin. Berlin, Ernst, 1899. 8vo. 48 pp. M. 1.60

Lengauer (L.). Geometrische Wahrscheinlichkeitsprobleme. (Progr.) Würzburg, 1899. 8vo. 62 pp.

Lewenberg (A.). Geometrya rzutowa (tworów pierwiastkowych). (In 18 parts.) Part 1. Warszawa, 1899. M. 1.50

Morale (M.). La rigata razionale d'ordine n dello spazio a quattro dimensioni e sua rigata trasversale, con particolare considerazione al caso di $n = 5$. Palermo, Tipografia matematica, 1899. 8vo. 21 pp.

——. Tre metodi per la costruzione di superficie rigate nello spazio à 4 dimensioni. Palermo, Tipografia matematica, 1899. 8vo. 13 pp.

Moreau (L.). Sur les transformées des radicaux doubles, réels ou imaginaires. Bruxelles, 1899. 8vo. 16 pp. Fr. 1.50

Müller (F.). See Abhandlungen.

Nagi (A.). See Abhandlungen.

Nenashev (A. P.). See Frenet (F.).

Pascal (E.). Repertorio di matematiche superiori : definizioni-formole, teoremi, cenni bibliografici. Vol. II : Geometria. Milano, Hoepli, 1900. 16mo. 18 + 928 pp. Cloth.

Riemann (B.). Elliptische Funktionen. Vorlesungen, mit Zusätzen, herausgegeben von H. Stahl. Leipzig, Teubner, 1899. 8vo. 8 + 144 pp. M. 5.60

Rösch (F.). Ueber die Irreduktibilität der partiellen Differentialgleichung $a(x, y) \frac{\partial z}{\partial x} + b(x, y) \frac{\partial z}{\partial y} + c(x, y)z = 0$.

Rosenberg (F.). See Grace (J. H.).

Rosenberger (F.). See Abhandlungen.

Schmidt (F.). See Gauss (C. F.).

Shaposhnikov (A. L.). Elementary course of mathematical analysis. From lectures given in the Moscow Technical School and in the Constantine Geodetic Institute. Vol. I, part 1 : Introduction to higher algebra and to the analysis of variables. (Russian.) Moscow, 1897. 8vo. 150 pp. R. 1.00

Sonnet (H.) et Frontera (G.). Eléments de géométrie analytique, rédigés conformément au programme d'admission à l'Ecole polytechnique et à l'Ecole normale supérieure. 9e édition. Paris, Hachette, 1899. 8vo. 760 pp. Fr. 8.00

Stäckel (P.). See Abhandlungen, and Gauss (C. F.).

Stahl (H.). See Riemann (B.).

Stolz (O.). Grundzüge der Differential- und Integralrechnung. Theil 3 : Die Lehre von den Doppelintegralen, eine Ergänzung zum ersten Theile des Werkes. Leipzig, Teubner, 1899. 8vo. 8 + 296 pp. M. 8.00

Tannenberg (W. de). Leçons nouvelles sur les applications géométriques du calcul différentiel. Paris, Hermann, 1899. 8vo. 4 + 192 pp. Fr. 6.00

Tannery (P.). See Descartes (R.).

Thibaut (G.). Indo-arische Astronomie, Astrologie und Mathematik. Strassburg, 1899. 8vo. 82 pp. M. 4.00

UNGER (F. A.). See ABHANDLUNGEN.

WIECHERT (E.). See FESTSCHRIFT.

WOHLWILL (E.). See ABHANDLUNGEN.

WOLFF (H.). Ueber die Anzahl der Zerlegungen einer ganzen Zahl in Summanden. (Diss.) Halle, 1899. 4to. 23 pp.

II. ELEMENTARY MATHEMATICS.

BIDDLE (D.). Mathematical questions and solutions from the *Educational Times*, Vol. 71. London, Hodgson, 1899. 12mo. 6s. 6d.

CREFCOEUR (A.). Recueil de formules mathématiques : arithmétique, algèbre, géométrie, trigonométrie, mécanique. Bruxelles, 1899. 12mo. 308 pp. (Autogr.) Fr. 2.00

FIEUX (A.). Eléments d'algèbre (classes de lettres). Paris, Mersch, 1899. 16mo. 84 pp. Fr. 2.00

GANTZER (R.). See SCHUMANN (H.).

GOERING (W.). Die Auffindung der rein geometrischen Quadratur des Kreises und die Teilung jedes beliebigen Winkels und Kreises in eine beliebig eAnzahl gleicher Teile. Dresden, Gewerbe-Buchh andlung, 1899. 8vo. 13 pp., 1 plate. M. 1.00

KÖHLER (A.). See LIEBER (H.).

LIEBER (H.). und KÖHLER (A.). Arithmetische Aufgaben. 2te Auflage. Berlin, Simion, 1899. 8vo. 6 + 222 pp. M. 2.70

MÉMENTO de géométrie, par un groupe d'instructeurs. Paris, Delagrave [1899]. 18mo. 52 pp.

MILNE (W. J.). Solid geometry ; being books 7–10 of "Plane and solid geometry." New York, American Book Co. [1899]. 12mo. 24 + 384 pp. Half leather. $0.75

MODEL answers in mathematics, London matriculation. From June 1893 to June 1899. London, Clive, 1899. 12mo. 176 pp. (University tutorial series.) 2s.

MÜLLER (H.). Die Elemente der ebenen Trigonometrie, mit einer Sammlung von Aufgaben und deren Lösungen. 3te Auflage. Metz, Scriba, 1899. 8vo. 7 + 40 pp., 1 plate. M. 0.80

ORTÙ CARBONI (S.). I complementi dell' algebra elementare per la discussione completa e sistematica dei problemi algebrici di primo e secondo grado, con duemila problemi (risoluti od avviati) d'applicazione dell' algebra e della trigonometria alla geometria. Parte I : Teorie. Livorno, Giusti, 1900. 8vo. 16 + 467 pp. Fr. 5.00

PRESSLAND (A. J.) and TWEEDIE (C.). Elementary trigonometry. Part 2. Edinburgh, Olliver & Boyd, 1899. 12mo. 126 pp. 2s.

SALOMON (A.). Leçons d'arithmétique. Paris, 1899. 8vo. 8 + 273 pp. Fr. 2.00

SCHUBERT (H.). Aufgaben aus der Arithmetik und Algebra für Real- und Bürgerschulen. Heft 1. 2te Auflage. Potsdam, Stein, 1899. 8vo. 4 + 101 pp. Boards. M. 1.20.

——. Dasselbe, mit Resultaten. 4 + 122 pp. M. 1.70

SCHUMANN (H.). Lehrbuch der Elementar-Mathematik für Gymnasien und Realschulen. 5te Auflage, bearbeitet von R. Gantzer. Teil 2: Planimetrie. Berlin, Weidmann, 1899. 8vo. 8 + 244 pp. M. 2.40

SCHUSTER (M.). Geometrische Aufgaben. Ein Lehr- und Uebungsbuch zum Gebrauch beim Unterricht an höheren Schulen. Ausgabe B: Für Progymnasien und Realschulen. Leipzig, Teubner, 1899. 8vo. 7 + 111 pp., 2 plates. Cloth. M. 1.60

SICKENBERGER (A.). Leitfaden der elementaren Mathematik. Teil 2: Planimetrie. 4te Auflage. München, Ackermann, 1899. 8vo. 6 + 123 pp. M. 1.50

——. Teil 3: Stereometrie; Trigonometrie. 3te Auflage. München, Ackermann, 1899. 8vo. 4 + 103 pp. M. 1.20

STEINWELLER (F.). Kurzer Abriss der Geschichte des Rechenunterrichts sowie Beschreibung der wichtigsten Lehrmittel für denselbem. 2te Auflage. Leipzig, Hirt, 1899. 8vo. 48 pp. M. 0.50

TWEEDIE (C.). See PRESSLAND (A. J.).

WALLENTIN (F.). Methodisch geordnete Sammlung von Beispielen und Aufgaben aus der Algebra und allgemeinen Arithmetik. 5te Auflage. Wien, 1899. 8vo. 4 + 329 pp. Cloth. M. 4.20

III. APPLIED MATHEMATICS.

D'ALEMBERT. See VORREDEN.

AMBRONN (L.). Handbuch der astronomischen Instrumentenkunde. Eine Beschreibung der bei astronomischen Beobachtungen benutzten Instrumente, sowie Erläuterung der ihrem Bau, ihrer Anwendung und Aufstellung zu Grunde liegenden Principien. 2 Bände. Berlin, Springer, 1899. 8vo. 9 + 7 + 1276 pp. Cloth. M. 60.00

AMES (J. S.). See CARNOT.

BAKER (W. M.). Elementary dynamics. London, Bell, 1899. 12mo. 252 pp. 3s. 6d.

BAUMHAUER (H.). Darstellung der 32 möglichen Krystallklassen auf Grund der Deck- und Spiegelachsen nebst Beschreibung von Achsenmodellen zur Demonstration der Symmetrieverhältnisse der Krystalle. Leipzig, Engelmann, 1899. 8vo. 5 + 36 pp., 1 plate. M. 2.00

BAYNES (R. E.). See MEYER (O. E.).

CARNOT, CLAUSIUS, and THOMSON. Memoirs on the second law of thermodynamics, translated and edited by W. T. Magie. New York, Harper, 1899. 8vo. 5 + 151 pp. Cloth. (Harper's scientific memoirs, edited by J. S. Ames, No. 6.)

CLAUSIUS. See CARNOT.

CORVISY. See VAN'T HOFF (J. H.).

DIECKMANN (E.). See SCHULTZ (E.).

EDSER (E.). Heat; for advanced students. London and New York, Macmillan, 1899. 12mo. 8 + 470 pp. Cloth. $1.00

EWING (J. A.). The strength of materials. London and New York, Macmillan, 1899. 8vo. 12 + 246 pp. Cloth. (Cambridge University Press.) $3.00

GALILEI. See VORREDEN.

GERLAND (E.). Kurzer Abriss der darstellenden Geometrie zum Gebrauche in Vorlesungen, beim Unterricht und zum Selbststudium. Leipzig, Engelmann, 1899. 8vo. 4 + 50 pp., 26 plates folio. Boards. M. 4.00

GRAETZ (L.). Die Elektrizität und ihre Anwendungen. 8te Auflage (Doppelauflage). Stuttgart, Engelhorn, 1899. 8vo. 14 + 590 pp. M. 7.00

GREAVES (J.). Solutions to the examples in a treatise on elementary hydrostatics. London, Clay, 1899. 12mo. 5s.

GÜNTHER (S.). Handbuch der Geophysik. 2te, gänzlich umgearbeitete Auflage. 2 Bände. Stuttgart, 1899. 8vo. M. 38.00

HENNE (H.). Berechnungsbeispiele über Wasserräder und Turbinen. Ergänzungsheft zu dem Werke Die Wasserräder und Turbinen, deren Berechnung und Konstruktion. Leipzig, Voigt, 1899. 8vo. 72 pp., 8 plates. M. 3.00

HELMHOLTZ, HERTZ, KIRCHHOFF. See VORREDEN.

KOLL (O.). See VELTMANN (W.).

LAGRANGE. See VORREDEN.

LECLAIR (M.). Méthode simplifiée et applications pratiques de la règle à calcul. 14e mille. Paris, Tavernier-Gravet, 1899. 16mo. 60 pp. Fr. 0.75

LEWIS (W. J.). Treatise on crystallography. London, Clay, 1899. 8vo. 692 pp. (Natural science manuals.) 14s.

MAGIE (W. F.). See CARNOT.

MEYER (O. E.). Kinetic theory of gases. Elementary treatise with mathematical appendices. Translated from the revised edition by R. E. Baynes. London, Longmans, 1899. 8vo. 490 pp. 15s.

MINEL (P.). Introduction à l'électricité industrielle : potentiel, flux de force, grandeurs électriques. 2e édition. Paris, 1899. 8vo. 192 pp. F. 2.50

MÜLLER (R.). Leitfaden für die Vorlesungen über darstellende Geometrie an der technischen Hochschule zu Braunschweig. Braunschweig, 1899. 8vo. 7 + 88 pp. M. 2.50

NEUMANN (E.). Ueber die Anwendung eines eigentümlichen Reduktionsverfahrens auf elektrische Probleme. Halle, 1899. 4to. 23 pp. M. 1.80

NEWTON. See VORREDEN.

OERGLER (A.). Ueber die zur disruptiven Entladung in einigen Gasen bei verschiedenen Drucken nothwendige Potentialdifferenz. Berlin, 1899. 8vo. 40 pp. M. 2.00

ROHR (M. VON). Theorie und Geschichte des photographischen Objektivs. Nach Quellen bearbeitet. Berlin, 1899. 8vo. 20 + 435 pp., 4 plates. M. 12.00

SCHMEHL (C.). Die Elemente der darstellenden Geometrie. Zum Gebrauche an höheren Lehranstalten sowie zum Selbststudium. Mit zahlreichen Uebungsbeispielen. 2 Teile. Giessen, Roth, 1899. 8vo. 8 + 95 + 7 + 94 pp. M. 4.00

SCHULTZ (E.). Mathematische und technische Tabellen für Baugewerkschulen. 4te Auflage. Unter gütiger Mitwirkung von E. Dieckmann. Ausgabe ohne Logarithmen. Essen, Baedeker, 1899. 8vo. 8 + 156 pp. Cloth. M. 1.00

STEINMETZ (C. P.). Theorie und Berechnung der Wechselstromerscheinungen. Deutsche, vom Verfasser autorisirte Ausgabe. 1ste Hälfte. Berlin, Reuther & Reichard, 1899. 8vo. 16 + 184 pp. M. 4.00

STEWART (R. W.). Advanced magnetism and electricity. Vol. 4 of Tutorial physics, with questions set in 1894–99 at advanced examinations of Science and art department. London, Clive, 1899. 12mo. 384 pp. (Organized science series.) 3s. 6d.

SUTER (W. N.). Handbook of optics. London and New York, Macmillan, 1899. 12mo. 8 + 209 pp. Cloth. 5s.

TAIT (P. G.). Newton's laws of motion. London, Black, 1899. 12mo. 62 pp. 1s. 6d.

THOMSON. See CARNOT.

VAN'T HOFF (J. H.). Leçons de chimie physique professées à l'université de Berlin. Ouvrage traduit de l'allemand par M. Corvisy. 2e partie : La statique chimique. Paris, Hermann, 1899. 8vo. 165 pp.

VELTMANN (W.) und KOLL (O.). Formeln der niederen und höheren Mathematik sowie für die Teilung der Grundstücke und für Tracirungsarbeiten. Zum Gebrauch beim geodätischen Studium und in der geodätischen Praxis. 3te Auflage. Bonn, Strauss, 1899. 8vo. 3 + 84 pp. Cloth. M. 4.00

VODUŠEK (M.). Neue Theorie der Mondbewegung. (Progr.) Laibach, Fischer, 1899. 8vo. 46 pp. M. 1.50

VORREDEN und Einleitungen zu klassischen Werken der Mechanik: Galilei, Newton, d'Alembert, Lagrange, Kirchhoff, Hertz, Helmholtz. Uebersetzt und herausgegeben von Mitgliedern der Philosophischen Gesellschaft an der Universität zu Wien. Leipzig, Pfeffer, 1899. 8vo. 267 pp. M. 5.00

WALKER (J.). Introduction to physical chemistry. London, Macmillan, 1899. 8vo. 346 pp. 10s.

WATSON (W.). A text-book of physics. London and New York, Longmans, Green & Co., 1899. 8vo. 22 + 896 pp. Cloth. $3.50

WETZSTEIN (G.). Ueber Abweichungen vom Poiseuille'schen Gesetz. (Diss.) München, 1899. 8vo. 59 pp.

L R RY.

THE SIXTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Sixth Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Thursday, December 28, 1899. Beside the usual election of officers and the presentation of annual reports, the occasion was especially marked by the presidential address of President R. S. Woodward, entitled "The century's progress in applied mathematics." An invitation to attend the reading of the address had been extended to the American Physical Society, which was present in a body, the total attendance reaching about eighty persons, of whom the Mathematical Society furnished nearly one-half. President H. A. Rowland of the Physical Society was placed in the chair during the joint session, which was extended to include the reading of the paper by Professor Pupin, noted below. The event afforded a pleasant renewal of the cordial relations of the two scientific bodies and a continued assurance of their hearty coöperation.

The following thirty-eight members of the Society were registered as in attendance at the two sessions :

Dr. E. M. Blake, Professor Maxime Bôcher, Dr. W. G. Bullard, Professor J. E. Clark, Dr. A. Cohen, Professor F. N. Cole, Dr. W. S. Dennett, Dr. F. Durell, Professor A. M. Ely, Professor T. S. Fiske, Mr. A. S. Gale, Miss Carrie Hammerslough, Dr. G. W. Hill, Dr. A. A. Himowich, Professor Harold Jacoby, Mr. S. A. Joffe, Professor Pomeroy Ladue, Professor P. A. Lambert, Professor Gustave Legras, Dr. Emory McClintock, Dr. James Maclay, Dr. Isabel Maddison, Professor Mansfield Merriman, Dr. G. A. Miller, Dr. D. A. Murray, Professor G. D. Olds, Mr. J. C. Pfister, Professor James Pierpont, Professor M. I. Pupin, Professor J. K. Rees, Mr. C. H. Rockwell, Dr. Virgil Snyder, Professor Henry Taber, Miss Mary Underhill, Professor J. H. Van Amringe, Professor L. A. Wait, Professor A. G. Webster, Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair during the separate sessions of the Society. The Council announced the election of the following persons to membership : Professor William Beebe, Yale University, New Haven, Conn.; Dr. J. V. Collins, State Normal School, Stevens Point, Wis.; Professor A. R. Forsyth, Trinity College, Cambridge, England ; Professor M. W. Haskell, University of California, Berkeley, Cal.; Dr. Emilie N. Martin, Philadelphia, Pa.; Mr. C. A. Noble, University of California, Berkeley, Cal.; Mr. E. B. Wilson, Yale University, New Haven, Conn.; Miss R. G. Wood, New Haven, Conn. Four applications for membership were reported.

Reports were received from the Treasurer, the Librarian, and the Auditing Committee. These reports will be printed in the Annual Register, now in preparation. The Secretary reported that the total number of members of the Society was now 337, including twelve life members. Thirty-six new members joined the Society during the year 1899. The total attendance of members at the meetings during the year was 197, and the number of members attending at least one meeting was 110. The number of papers presented was 106, as against 88 in 1898.

On the occasion of the retirement of Professor Harold Jacoby from the position of Treasurer, appropriate resolutions were adopted expressing recognition of his valuable services, extending with a brief interruption from the founding of the Society.

At the annual election the following officers and members of the Council were chosen:

President,	Professor R. S. WOODWARD.
First Vice-President,	Professor E. H. MOORE.
Second Vice-President,	Professor T. S. FISKE.
Secretary,	Professor F. N. COLE.
Treasurer,	Dr. W. S. DENNETT.
Librarian,	Professor POMEROY LADUE.

Committee of Publication,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor FRANK MORLEY.

Members of the Council to serve until December, 1902,

Professor OSKAR BOLZA,
Professor SIMON NEWCOMB,
Professor L. A. WAIT.

The following papers were presented:

(1) Dr. G. A. MILLER: "On the groups which have the same group of isomorphisms."

(2) Professor **Maxime Bôcher**: "On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic."

(3) Dr. **Virgil Snyder**: "On cyclical quartic surfaces in space of n dimensions."

(4) Dr. **Virgil Snyder**: "On the geometry of the circle."

(5) Mr. **W. B. Fite**: "A proof that the commutator subgroup of a group may contain operators which are not commutators."

(6) **J. E. Campbell**, M.A.: "On the types of linear partial differential equations of the second order (in three independent variables) which are unaltered by the transformations of a continuous group."

(7) Professor **L. E. Dickson**: "Proof of the existence of the Galois field of order p^r for every integer r and prime number p."

(8) Dr. **E. M. Blake**: "On plane movements whose point loci are of order not greater than four."

(9) Professor **R. S. Woodward**: Presidential address, "The century's progress in applied mathematics."

(10) Professor **M. I. Pupin**: "The propagation of electrical waves over non-uniform conductors."

(11) Professor **Henry Taber**: "On the singular transformations of groups generated by infinitesimal transformations."

(12) Dr. **J. I. Hutchinson**: "On certain relations among the theta constants."

(13) Professor **E. O. Lovett**: "Singular solutions of Monge equations."

Mr. Fite's paper was presented to the Society through Dr. G. A. Miller, and Mr. Campbell's paper through the Secretary. In the absence of the authors, Mr. Fite's paper was read by Dr. Miller, Dr. Hutchinson's by Dr. Virgil Snyder, and those of Mr. Campbell, Professor Dickson, and Professor Lovett were read by title.

The presidential address appeared in the January number of the Bulletin (pp. 133–163), and has also been published in *Science*, vol. 11 (new series), no. 263, pp. 41–51; no. 264, pp. 81–92. The papers of Professor Dickson, Dr. Snyder, and Professor Taber are contained in the present number of the Bulletin. Mr Campbell's paper will be published in the *Transactions*. Abstracts of the remaining papers are given below.

The main object of Dr. Miller's paper was the determination of all the possible groups whose group of isomorphisms or whose group of cogredient isomorphisms is the symmetric group of order six. It was shown that these groups are the direct products of abelian groups and an infinite system of groups which contains just one group for every power of two and is thus analogous to the system of Hamiltonian groups whose order is a power of two.

The following theorems were also proved: If a group is generated by two characteristic subgroups that have only identity in common, its group of isomorphisms is the direct product of the groups of isomorphisms of these two characteristic subgroups. The group of isomorphisms of an abelian group A is abelian whenever A is cyclical, and it is non-abelian when A is non-cyclical. The necessary and sufficient condition that a cyclical group of order n is the group of isomorphisms of some group is that n is of the form $p^{\alpha}(p-1)$, p being an odd prime number. If the group of cogredient isomorphisms G_1 of a group G transforms one of its operators A_1 of order p^{α}, p being any prime number, into its k^{th} power and if $k \not\equiv 1 \bmod p$, then at least one operator of order p^{α} corresponds to A_1 in the isomorphism between G and G_1; if $k \equiv 1 \bmod p^{\gamma}$, but $\not\equiv 1 \bmod p^{\gamma+1}$, then at least one operator of order $p^{\alpha+\gamma'}$ $(\gamma' \leqq \gamma)$ corresponds to A_1 in the isomorphism between G and G_1.

In Professor Bôcher's paper, which has now appeared in the *Transactions* (Volume 1, number 1, pp. 40–52), the questions which the author considered on pp. 280–281 of the last volume of the BULLETIN, are discussed by another method by means of which it is possible to treat the subject much more exhaustively.

In the geometry of the circle, as discussed by Dr. Snyder, the circle is taken for generating element analogous to the sphere in Lie's higher spherical geometry. A linear equation between the five coördinates defines the ∞^2 circles which cut a fixed circle at a constant angle. Several problems in construction were easily solved, *e. g.*, to find a circle cutting three given ones at given angles. The elements of differential geometry were given, and the differential equation of all envelopes of circles expressed. In this geometry a quadratic complex gives the bicircular quartics when the point circles form one coördinate complex, and a new system of circular curves of order 24 when the coördinates are unrestricted. Some important properties of both systems of curves were obtained.

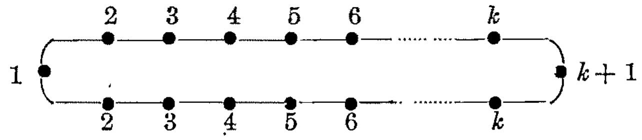
Mr. Fite's paper was in abstract as follows: Let G be a group of order p^m, p being a prime number, which has an abelian group of cogredient isomorphisms, generated by four independent operators P, Q, R, S; and let A, B, C, D be the operators of G that correspond to P, Q, R, S, respectively. The commutator subgroup of G will contain operators which are not commutators of G whenever the following conditions are satisfied:

$$B^{-1}AB = h_1A, \quad C^{-1}AC = h_2A, \quad D^{-1}AD = h_3A,$$
$$C^{-1}BC = h_4B, \quad D^{-1}BD = h_5B, \quad D^{-1}CD = h_6C,$$

where h_1, h_2, $\cdots$, h_6 are independent self-conjugate operators of G different from identity. There is a group of order 1024, obtained by establishing a certain isomorphism for G_8^4 written in six distinct systems of elements, which satisfies all of the given conditions. There is also a group of order 256 whose commutator subgroup contains an operator which is not a commutator.

Dr. Blake considered the movements of a rigid body or system one of whose planes slides upon a fixed plane and is further restricted to those movements which cause all points of the system to generate curves of order not greater than four. A list of all known movements of this character is given in Part I. and the problem of a complete enumeration is proposed. The locus of a straight line oblique to the plane of movement may be a scroll of order either not greater than four or greater than four. The author has discussed these scrolls for all known movements of the former kind. They are the movements of the ellipsograph (article by the author in manuscript); movements whose centrodes are congruent conics (*American Journal of Mathematics*, vol. 21, p. 257); the degenerate case of three-bar movement whose centrodes are limaçons; and Roberts's generalization of the conchoid mechanism of Nicomedes. The author's present paper treats the last two cases in Parts II. and III.

In Professor Pupin's paper the conductor is a long wire having coils 1, 2, $\cdots$, k, $k+1$ interposed at equidistant points, as in the figure.



An electromotive force of type Ee^{ipt} is impressed at point 1. The problem discussed leads to the integration of the differential equation of a plane wave in an absorbing medium, namely,

$$L\frac{dx^2}{dt^2}+R\frac{dx}{dt}=\frac{1}{C}\frac{d^2x}{ds^2},$$

where s is the distance of any point from point 1, x is the current at that point, t is the time, and L, R, C are the inductance, ohmic resistance, and capacity, respectively, of the wire per unit length.

The integral has to satisfy $k+2$ boundary conditions, one at each coil. To formulate these, let x_1, x_2, $\cdots$, x_{ks}, $x_{(k+1)s}$ be the currents at points 1, 2, $\cdots$, k, $k+1$. Also let L_0, R_0, C_0, be the inductance, capacity, and resistance, respectively, of any one of the $k+2$ equal coils. Then at point 1

$$\frac{\delta x_1}{\delta s}=\frac{1}{2}ipCEe^{ipt}-\frac{C}{2}(-p^2L_0+ipR_0)x_1=\frac{1}{2}D_0-\frac{H}{2}x_1;$$

at point $k+1$.

$$\frac{\delta x_{(k+1)s}}{\delta s}=-\frac{H}{2}x_{(k+1)s};$$

and at any other point a

$$\left(\frac{\delta x_{as}}{\delta s}\right)_1-\left(\frac{\delta x_{as}}{\delta s}\right)_2=-Hx_a,$$

where a may be any integer from 1 to k.

The integral does not seem to be obtainable by ordinary methods. The method of successive eliminations which the author developed in his paper on "Electrical oscillations in a loaded conductor," read at the meeting of the Society, February 25, 1899, succeeds in finding an integral.

Thus the current in the section between the points a and $a+1$ at a distance ξ from a is given by

$$x_{as+\xi}=$$

$$\frac{[\frac{1}{2}h_0+h_0'] \cos\left(\frac{al}{k+1}+\xi\right)\mu-d\cos\left\{\frac{(k+1-a)l}{k+1}-\xi\right\}\mu}{\mu\sin\mu l}$$

$$+\frac{h_0}{\mu}\left\{x_s\sin\left[\frac{(a-1)l}{k+1}+\xi\right]\mu+x_{2s}\sin\left[\frac{(a-2)l}{k+1}+\xi\right]\mu\right.$$

$$\left.+\cdots+x_{as}\sin\mu\xi\right\},$$

where

$$h_0 = C(-p^2L_0 + ipR_0), \quad -\mu^2 = C(-p^2L + ipR),$$

$$h_0' = h_0'\left\{ x_s \cos\frac{k\mu l}{k+1} + x_{2s}\cos\frac{(k-1)l\mu}{k+1} + \cdots + x_{ks}\cos\frac{l\mu}{k+1}\right\}$$

The currents $x_1, x_s; \cdots x_{ks}, x_{(k+1)s}$ are determined from the following $k+2$ simultaneous equations:

$$(\sigma_1 + 1)\,x_1 - x_s = \frac{\rho}{2} + \frac{\sigma_1}{2}x_1,$$

$$(\sigma_1 + 2)\,x_s - x_{2s} - x_1 = 0,$$

$$(\sigma_1 + 2)\,x_{2s} - x_{3s} - x_{2s} = 0,$$

....

$$(\sigma_1 + 2)\,x_{ks} - x_{(k+1)s} - x_{(k-1)s} = 0,$$

$$(\sigma_1 + 1)\,x_{(k+1)s} - x_{ks} = +\frac{\sigma_1}{2}x_{(k+1)s},$$

where

$$\sigma_1 = \frac{h_0 \sin\dfrac{\mu l}{k+1}}{\mu} - 4\sin^2\tfrac{1}{2}\frac{\mu l}{k+1}, \qquad \rho = \frac{\tfrac{1}{2}pcEe^{ipt}\sin}{\mu}\,\frac{\mu l}{k+1}$$

By putting $\sigma_1 = -4\sin^2\psi$, we obtain any current

$$x_{ms} = -\frac{\dfrac{\rho}{2}\cos 2(k-m+2)\psi}{\sin 2\psi \sin(2k+2)\psi}.$$

This disposes of the waves of forced period. The waves of free period can be obtained by putting

$$\sin 2\psi\sin(2k+2)\psi = 0, \qquad \therefore\ (2k+2)\psi = \nu\pi,$$

where ν may be any integer from 0 to $2k+2$.

The theta constants considered by Dr. Hutchinson are the values of the theta functions with half integer characteristics for zero arguments. Relations among the thetas of two different Göpel systems are considered. From these can be derived the expression of all the theta constants in terms of those belonging to the same Göpel group (or system). This is the direct generalization of the results already well known for the hyperelliptic theta functions (depending upon two arguments).

Professor Lovett's paper, which is intended for publication in the *Transactions*, employs Lie's theory of infinitesimal transformations to construct a method for determining the singular solutions of Monge and Pfaff equations.

F. N. COLE.

COLUMBIA UNIVERSITY.

THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE Sixth Semi-Annual Meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held on December 28 and 29, 1899, at the University of Chicago. The following members of the Society were in attendance:

Professor Oskar Bolza, Professor E. W. Davis, Professor Thomas F. Holgate, Dr. Kurt Laves, Professor H. Maschke, Professor John A. Miller, Professor E. H. Moore, Professor Alexander Pell, Professor D. A. Rothrock, Professor G. T. Sellew, Professor E. B. Skinner, Dr. H. E. Slaught, Dr. H. F. Stecker, Professor C. A. Waldo, Dr. J. V. Westfall, Professor H. S. White, Professor Mary F. Winston, Professor J. W. A. Young.

Professor E. H. Moore, Vice-President of the Society, occupied the chair during the first of the four sessions, after which Professor E. W. Davis presided. The Christmas meeting being the regular time for the election of officers of the Section, the Secretary was re-elected and Professors H. B. Newson and C. A. Waldo were elected members of the programme committee. The time and place of the next meeting were fixed for Saturday, April 14, 1900, at Northwestern University, Evanston, Ill.

The following papers were presented:

(1) Mr. R. E. MORITZ: "A generalization of the process of differentiation."

(2) Professor E. D. ROE: "On the transcendental form of the resultant."

(3) Dr. E. J. WILCZYNSKI: "An application of Lie's theory to hydrodynamics."

(4) Mr. F. R. MOULTON: "On the question of the stability of certain particular solutions of the problem of four bodies, together with particular solutions of the problem of n bodies of the Lagrangian type.

(5) Professor L. E. DICKSON: "The canonical form of linear homogeneous substitutions in a general Galois field."

(6) Professor L. E. DICKSON: "The cyclic subgroup of the simple group of linear fractional substitutions of determinant unity in two non-homogeneous variables with coefficients in an arbitrary Galois field."

(7) DR. J. V. WESTFALL: "On a category of transformation groups in space of four dimensions."

(8) Professor OSKAR BOLZA: "The elliptic sigma functions considered as a special case of the hyperelliptic sigma functions."

(9) Professor ALEXANDER PELL: "Calculation of the integral

$$\int_0^\infty e^{-\left(px^2+\frac{q}{x^2}\right)} \begin{matrix}\sin\\ \cos\end{matrix} \left\{ rx^2 + \frac{s}{x^2} \right\} dx."$$

(10) Professor JOHN A. MILLER: "Concerning certain modular functions of square rank."

(11) Professor R. J. ALEY: "A new collinear set of three points connected with a triangle."

(12) Professor H. MASCHKE: "Note on the unilateral surface of Moebius."

(13) Professor C. A. WALDO: "On a family of warped surfaces connected by a simple functional relation."

(14) Professor H. S. WHITE: "Plane cubics and irrational covariant cubics."

Mr. Moritz's paper was presented to the Society through Professor Davis, and was read by him in the absence of the author. Mr. Moulton's paper was presented through Dr. Laves. In the absence of the authors, Professor Roe's paper was read by Professor White, Dr. Wilczynski's by Professor Bolza, Professor Dickson's by Professor Moore, and Professor Aley's by the Secretary.

After the regular papers had been presented, a general discussion upon the topic "Limits of functions of one or more variables" was opened by Professor Moore.

Mr. Moritz's paper was a contribution to the doctrine of pure forms. Let $\underset{n}{+}$ denote the symbol of a combinatory process of the nth order, $\underset{n}{-}$ and $\overset{n}{-}$ its two inverses, so that if $a \underset{n}{+} b = c$, then $c \underset{n}{-} a = b$, and $c \overset{n}{-} b = a$; and let $a \underset{n}{+} M_n = a$, for all values of a, define M_n, the modulus of the process. The process of the $(n+1)$th order is built up from the process of the nth order, as multiplication is built up from addition. Three general theorems were proved: I. If M_n is the modulus of a process of the nth order, then $M_n\,{}_n\!\overset{-}{+}\!{}_1 M_n$ is indeterminate. II. If any process possesses a modulus, and is subject to the sub-associative law, viz.:

$$a \overset{+}{_n} b \overset{+}{_n} c = a \overset{+}{_n} (b\,{}_n\!\overset{+}{-}\!{}_1 c),$$

then

$$a \overset{-}{_n} b = (a \overset{-}{_n} c)\,{}_n\!\overset{-}{-}\!{}_1 (b \overset{-}{_n} c).$$

III. If in addition to the hypothesis of theorem II, the process is subject to the sub-distributive law, viz.:

$$a\,{}_n\!\overset{+}{+}\!{}_1(b\,{}_n\!\overset{+}{-}\!{}_1 c) = (a\,{}_n\!\overset{+}{+}\!{}_1 b) \overset{+}{_n} (a\,{}_n\!\overset{+}{+}\!{}_1 c)$$

then

$$M_n\,{}_n\!\overset{-}{+}\!{}_1 a = M_{n-1}.$$

The symbol Y_n was defined by the equation

$$Y_n = \lim_{h=M_n}\{ f(x \overset{+}{_n} h) \overset{-}{_n} f(x\,{}_n\!\overset{-}{+}\!{}_1 h)\};$$

for $n=1$ this becomes $Y_1 = \frac{dy}{dx}$. For any process subject to the conditions of theorem I., Y_n is indeterminate. If, moreover, the conditions of theorem II. and III. exist for the process in question, then by theorem II.

$$Y_n = (M_n\,{}_n\!\overset{-}{+}\!{}_1 a)^{n}(M_n\,{}_n\!\overset{-}{+}\!{}_1 a)$$

and by theorem III. $Y_n = M_{n-1} \overset{-}{_n} M_{n-1}$, that is, it will be possible to evaluate Y_n provided Y_{n-1} can be evaluated. For $n=2$ all the above conditions exist, and we have in fact

$$Y_2 = \lim_{h=1}\left\{ \log_h \frac{f(xh)}{f(x)} \right\}.$$

The symbol $\frac{dy}{dx}$ was used to denote the forms thus defined, and the term quotiential coefficient was applied to them. A table of quotiential coefficients accompanied the paper.

Professor Roe's paper found its origin in two letters of Gordan, one to Hermite (*Comptes Rendus*, 127 (1898), p. 539), the other to the author, and arose from an attempt to remove certain difficulties suggested in the former of these and to work out the formulas contained in the latter. Three functions

$$f = x^m + a_1 x^{m-1} + \cdots + a_m,$$
$$\varphi = x^n + b_1 x^{n-1} + \cdots + b_n,$$
$$\psi = 1 + b_1 x + b_2 x^2 + \cdots + b_n x^n,$$

where the roots of ψ are the reciprocals of those of φ, are assumed, and the transcendental form of the resultant of f and ψ is obtained as

$$R = \sum_{\nu=0}^{\nu=\infty} \sum \frac{(-1)^{\rho_1+\rho_2+\cdots}}{\lambda_1^{\rho_1} \lambda_2^{\rho_2} \cdots \rho_1 ! \, \rho_2 ! \cdots} \{S_{\lambda_1}(a) \cdot S_{\lambda_1}(b)\}^{\rho_1} \{S_{\lambda_2}(a) \cdot S_{\lambda_2}(b)\}^{\rho_2} \cdots$$

where $\rho_1 + \rho_2 + \cdots = \nu$ and $S_\lambda(a)$, $S_\lambda(b)$ are respectively the sums of the nth powers of the roots of f and φ. It is seen, however, that R is also the resultant of f and ψ for any other degrees than m and n and the conclusion is reached: *The transcendental form expresses all binary resultants formally in the same form and is a universal formula comprehending them all. It is the most general formula for binary resultants.*

The finite form of the resultant $R_{m,n}$ is reached by dropping all the terms of R of weight greater than mn which, when expressed in the a's and b's for given functions f and ψ, are identically zero. We have,

$$R_{m,n} = \sum \sum \frac{(-1)^{\rho_1+\rho_2+\cdots}}{\lambda_1^{\rho_1} \lambda_2^{\rho_2} \ldots \rho_1 ! \, \rho_2 ! \ldots} S_\lambda(a) S_\lambda(b),$$

with $\lambda_1\rho_1 + \lambda_2\rho_2 + \cdots \leqq mn$. This $R_{m,n}$ is still very general, formally representing all resultants $R_{\mu,\nu}$, where $\mu\nu = mn$, in the same form, until the a's and b's for given equations are substituted, when it becomes definite. It may also represent expressions which are not resultants.

A one parameter group can, as Lie has noticed, be interpreted as representing the steady motion of a fluid. In Dr. Wilczynski's paper, Lie's suggestion was followed out, a number of general theorems being thus obtained. The theory was then applied to the special cases which from this standpoint are the simplest and most important, viz., that the groups are projective, linear, or linearoid. If the forces which act upon the unit of mass have a potential, the most general ternary projective group which can represent a steady fluid motion is the general linear group. This motion was studied in detail, one of the results being that, if the coördinate planes are appropriately chosen, the projection of the orbit of every point of the fluid upon the plane of xy is a conic section. All of these conic sections are similar and similarly placed.

If the motion is periodic, and also in other cases, the orbits themselves are conics in parallel planes; in the periodic case, of course, ellipses. A special case is the ellipsoid whose particles describe such elliptic orbits, a problem treated by Dirichlet and Dedekind. Interesting results were also found for the more difficult problem when the group is linearoid. The paper will be offered for publication to the *Transactions.*

At the Columbus meeting of the Society, August, 1899, a paper was presented by Mr. Moulton exhibiting twenty-eight particular solutions of the problem of four bodies, under the limitation that one of them should be infinitesimal. The first part of his present paper discussed the question of the stability of the motion of the infinitesimal body. Stability was defined as follows: The particular solutions have been already defined by those constant values of the coördinates and projections of the velocity of the infinitesimal body, satisfying the differential equations of motion. It is now supposed that the initial conditions are slightly different from those of the exact solutions,

$$x_0, \quad y_0, \quad z_0, \quad \frac{dx_0}{dt}, \quad \frac{dy_0}{dt}, \quad \frac{dz_0}{dt}.$$

Suppose $x = x_0 + x'$, etc., at the origin of time, where x', y', etc., are small quantities in terms of which the differential equations are now developed. All terms of degree higher than the first in the new variables are neglected, and the resulting linear system solved. If the roots of the *characteristic equation* are all pure imaginaries the generating solution is said to be *stable;* otherwise it is said to be *unstable.*

In the case where all four of the bodies lie in a line two of the roots of the characteristic equation are found to be real and two pure imaginary for all masses of the finite bodies. In the case where the infinitesimal body does not lie in the same line with the others, the roots of the characteristic equation are all pure imaginaries, or all complex, depending upon the relative masses of the finite bodies. In the case where the finite bodies are equal in mass and are at the vertices of an equilateral triangle, the roots of the characteristic equation are all pure imaginaries.

The second part of the paper took up particular solutions of the problem of n bodies of the Lagrangian type. The Lagrangian solutions of the problem of three bodies may be defined, from the standpoint of their derivation, as those solutions in which the ratios of the mutual distances of the bodies are constants; or, from the standpoint of their other properties, as those solutions in which each of the three bodies describes a conic section in the same period, and in such a manner that the center of gravity is at one of the foci of each conic, and that the law of areas holds for each body considered separately. Adopting the second definition, the necessary and sufficient conditions for solutions of this type of the problem of n bodies were written in algebraic form. It was proved that for any given n bodies there are $\frac{1}{2}n!$ solutions in which the bodies all lie in a line. There are doubtless solutions in which the bodies do not all lie in a line, but they are not proved to exist in the general case. In any particular example where the masses are given there is no difficulty in finding solutions if they exist. The associated problem, given n arbitrary points on a line, to find n masses such that, if they are placed at these points, solutions of the Lagrangian type will exist was also solved.

A more extended paper on this subject by Mr. Moulton, of which the present one will form a part, will be published in the *Transactions*.

The result due to Jordan (Traité des Substitutions, pp. 114–126) on the canonical form of a linear substitution on m variables with integral coefficients taken modulo p, a prime, may be readily generalized to substitutions in the $GF\,[p^n]$. Instead of following Jordan's method of proof, Professor Dickson in his first paper gave a simpler proof by induction. The theorem was supposed to hold for substitutions in the field which have the characteristic determinant (in the parameter K)

$$[F_k(K)]^{\alpha-1}[F_l(K)]^{\beta}\cdots \qquad (F_k,\ F_l,\ \cdots \text{ irreducible})$$

and was then proved to hold for every substitution in the field, which has the characteristic determinant

$$F_k^{\alpha}F_l^{\beta}\cdots \qquad (m \equiv k\alpha + l\beta + \cdots).$$

If we proceed by induction from $m-1$ to m indices, we fail to prove that part of the theorem concerning the conjugacy of the new indices. The author constructs explicitly the general substitution in the $GF[p^n]$ which is commutative with any given substitution in the field. In particular the number of the former substitutions was determined and found to agree with that derived by a different analysis by Jordan (Traité, p. 136) for the case $n = 1$.

Professor Dickson's second paper leads to a revision and a generalization to the $GF[p^n]$ of certain results due to Professor Burnside (*Proceedings of the London Mathematical Society*, vol. 26, pp. 58–106) upon groups of substitutions with integral coefficients taken modulo p, a prime. Numerous variations from Burnside's method of treatment were introduced, partly to avoid the separation of the two cases $p \equiv 1$ and $p \equiv -1 \pmod 3$ and to include the cases $p = 2$ and $p = 3$, and partly to minimize the calculations by the frequent use of known general theorems.

Burnside's results are incorrect in two places. The factor 2 should be deleted from $\frac{N}{2(p-1)^2}$ on pages 103 and 104; in fact, the statements made on page 103, lines 15–23, do not lead to the conclusion stated. Of substitutions having the canonical form

$$x' = x, \quad y' = y + x, \quad z' = z + y,$$

there exist three conjugate sets of N/p^2 substitutions, instead of one conjugate set of $3N/p^2$ substitutions as stated by Burnside, p 102, lines 1–6.

The generalization leads to an interesting result in the particular case $p^n = 2^3$. We have then a simple group of order 20160. The distribution of its cyclic subgroups in sets of conjugates is given by the following table:

960	conj.	cyclic	groups	of	order	7	with	5760	subs.	period	7
2016	"	"	"	"	"	5	"	8064	"	"	5
630	"	"	"	"	"	4	"	1260	"	"	4
630	"	"	"	"	"	4	"	1260	"	"	4
630	"	"	"	"	"	4	"	1260	"	"	4
315	"	"	"	"	"	2	"	315	"	"	2
1120	"	"	"	"	"	3	"	2240	"	"	3
								1	"	"	1

Total number of distinct substitutions 20160.

It follows at once that this group is not isomorphic with the alternating group on 8 letters having the order 20160. In fact, the latter contains substitutions of periods 6 and 15.

The two groups contain the same number of substitutions of period 7, the same number of periods 4 and 2. This non-isomorphism was first proved by Miss Schottenfels under the direction of Professor Moore. The corresponding investigation for three non-homogenous variables is now being made by Mr. T. M. Putnam.

Both of Professor Dickson's papers will appear in the *American Journal of Mathematics.*

Lie, in an article in the *Leipziger Berichte* for October, 1899, proceeding from Helmholtz's assumptions, omitting one, namely, the monodrom axiom, has determined all the groups, satisfying the given conditions under the most general interpretation of the axiom, of "unrestricted motion" (freie Beweglichkeit). He finds, besides the groups of euclidian and non-euclidian motion, five others. Kowalewski, a pupil of Lie, in his inaugural dissertation, has extended Lie's investigations in space of three dimensions to that of four and five. Besides the euclidian and non-euclidian groups, he finds in space of four dimensions three others A, B, and C. From analogy with the groups determined by Lie, he makes the statement, without proof, that group A satisfies the monodrom axiom, while B and C do not. By suitably choosing the points of general position, which are to be held fixed, and by considering the invariants of these points, Dr. Westfall proved that A satisfies the monodrom axiom, without being compelled to resort to the long integration otherwise necessary. In the case of group B, he showed that, contrary to Kowalewski's statement, the group may or may not satisfy this axiom, according to the mutual situation of the points held fixed.

To prove that A always satisfies the monodrom axiom he proceeds as follows: 1. Hold the special triplet of general points

$$P_1 = (0, 0, 0, 0),\ P_2 = (0, 0, z_3, z_4),\ P_3 = (0, 0, z_3', 0)$$

fixed and obtain the subgroup $(x_1 p_2 - x_2 p_1)$ which has closed path curves. 2. The three invariants of this point triplet are positive and never equal to zero. They are, further, independent with respect to z_3, z_4, and z_3'. If we choose any real positive values whatever, and set them equal to these invariants, we can solve for z_3, z_4, and z_3'. 3. From the form of the general invariant of the group we see that all triplets of points of general position have positive invariants which can never be equal to zero. If, therefore, we choose any triplet of points whatever, we can always determine z_3, z_4, and z_3' so that the invariants of the one triplet will be equal to the invariants of the other, and we can, therefore, transform one into the other by means of a transformation of the group. In the special case the path curves were closed, and we therefore know they are closed in general.

By suitable choice of the points to be held fixed and by consideration of the invariants in the case of group B, the following results were proved: 1. All groups obtained by holding fixed a triplet of general points whose invariants are all negative have closed path curves. 2. These are all the subgroups that have this characteristic.

Professor Bolza's paper gave a sketch of the theory of the elliptic sigma functions as they appear in the light of the theory of the hyperelliptic sigma functions. Section 1 contains systems of associated integrals of the first, second, and third kinds; section 2, Weierstrass's theta functions, (a) as functions of u, (b) in the Riemann surfaces; section 3, the Al-functions and the function $p(u)$; section 4, the sigma functions and the functions $\wp u$ and their invariantive properties; section 5, the partial differential equation of the sigma functions and the recursion formulæ for their expansion into power series. The paper has now been published in the *Transactions* (Vol. 1, No. 1, pp. 53–65).

Professor Miller's paper dealt with the properties of certain modular elliptic functions, the rank of which is a square number. The major part of the discussion was given to the particular cases of functions whose ranks are 9 and 4 respectively. Certain theorems used in the treatment of these special cases are true when the rank is any square number, as was proved. In the course of the discussion two groups were found; the one a quaternary group of linear substitutions of order 648, the other a ternary group of order 192. The letters of the substitutions of both these groups, the elements of the groups, and the operational character of the elements, were derived from the consideration of the X-functions, Z-functions, and $\sigma_{\lambda,\mu}$-functions, defined in many places, chiefly by Professor Klein. In case the rank $n = 9$ it is possible to define $t_a (a = 1, \cdots, 4)$ as linear homogeneous functions of Z_a, and thereby to find a group of monomial substitutions on t_a which is holoedrically isomorphic with the group G_{648} referred to above. It was shown that there are an infinite number of forms in t_a, which are invariant under G_{648}, and in accordance with the Hilbert law the basis of this system of forms was found. The function $\sigma_{\lambda/3,\,\mu/3}(w_1, w_2)$ was obtained as a linear homogeneous function of t_α, and by means of this the basis of the infinite system was expressed as rational functions of g_2, g_3, and $\triangle$, where g_2, g_3, and $\triangle$ have the signification usually given them in the treatment of elliptic functions.

The treatment of modular functions the rank of which is four differs in some of its details from that of the preceding, because the definition of $X_\alpha(u)$ in case n is even differs slightly from that of $X_\alpha(u)$ in case n is odd. However, the ends to be reached in the two cases are nearly identical. In both cases an interesting linear homogeneous connection has been found between

$$\sigma_{\frac{\lambda}{n},\,\frac{\mu}{n}}(nu,\ w_1,\ w_2)\ \text{ and }\ X_{\frac{\alpha}{n^2}}(u,\ w_1,\ w_2).$$

In Professor Aley's paper it was pointed out that the isogonal conjugate of the Brianchon point of any triangle, the center of the inscribed circle, and the center of the circumscribed circle are collinear.

Professor Maschke explained briefly the connection between the well known Moebius unilateral paper strip and the ruled surfaces of the third order, illustrating his demonstration by a model. A short account of the paper has been published in the *Transactions* (Volume 1, number 1, p. 39).

Professor Waldo's paper contained a demonstration of the following theorem : The equation

$$f\left(\frac{px}{p-z},\ \frac{qy}{q-z}\right)=0$$

represents a skew surface with two rectilinear directors whose equations are

$$(y=0,\quad z=p)\quad\text{and}\quad(x=0,\quad z=q),$$

and a plane curve director lying in the XY plane, having for its equation $f(x, y)=0$, the common perpendicular to the two rectilinear directors being chosen for Z-axis. By this theorem the equations of the whole family of skew surfaces having two distinct rectilinear directors can be at once written down and their principal deformations shown under cartesian form. The surfaces having for curvilinear directors $x - y = 0$ and $xy - \frac{1}{2} = 0$ were studied in detail and models exhibited showing their principal types.

Professor White's paper was a further development of the topic considered in the paper presented by him at the Columbus meeting of the Society. Each mixed concomitant (2, 2) of the cubic defines (as in the paper referred to) two covariant nets of conics. These are polars of two cubics of the syzygetic sheaf; the totality of such is exactly that entire sheaf of cubics. But these concomitants (2, 2) and all the concomitants (3, 3) serve to define also *four* covariant sheaves of cubics, not in the syzygetic sheaf, intimately connected on the one hand with the four inflexional triangles, and on the other hand with the eighteen collineations of the cubic into itself. This paper will be published in the *Transactions.*

THOMAS F. HOLGATE.
Secretary of the Section.

EVANSTON, ILL.

ON CYCLICAL QUARTIC SURFACES IN SPACE OF *N* DIMENSIONS.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, December 28, 1899.)

THE generation of the cyclide as the envelope of spheres which cut a fixed sphere orthogonally and whose centers lie on a quadric can readily be generalized to space of n dimensions.

In ordinary space it appears that the same surface is the envelope of five different systems; that the quadric loci of centers are all confocal and the associated spheres are all orthogonal; that the possibilities of the system are exactly coextensive with the ∞^{13} possible cyclides.

Let

$$(1)\quad (x_1 - ix_{n+2})\sum_{r=1}^{n} y_r^2 - 2\sum_{r=1}^{n} x_{r+1}y_r + (x_1 + ix_{n+2}) = 0$$

be the equation of a sphere in R_n; it contains $n + 2$ homogeneous constants x_r, and if the radius be denoted by

$$\frac{ix_{n+3}}{x_1 - ix_{n+2}},$$

the quadratic identity

$$(2) \qquad x \equiv \sum_{r=1}^{n+3} x_r^2 \equiv 0$$

will exist among these $n + 3$ numbers x_r, which may be called the homogeneous *coördinates of the sphere.*

Two spheres a, b will intersect orthogonally when

$$\sum_{r=1}^{n+2} a_r b_r = 0,$$

the terms defining the radii of the two spheres not occurring; hence any linear equation of the form

$$(3) \qquad \sum_{r=1}^{n+2} a_r x_r = 0$$

represents the ∞^n spheres which cut a fixed sphere orthogonally.

Now consider a quadratic equation of the form

$$(4) \qquad \varphi_2(x_1, x_2, \cdots, x_{n+2}) = 0$$

which does not contain x_{n+3}, and make it simultaneous with (3); between the two $x_1 + ix_{n+2}$ may be eliminated, leaving a quadratic equation among the point coördinates of the centers of the variable spheres.

Hence, equations (3) and (4) define the ∞^{n-1} spheres which cut a fixed sphere orthogonally, and whose centers lie on a quadric surface M_{n-1}^2. These spheres envelop a new surface whose equation may be found as follows:

Let $x_1 - ix_{n+2}$ be replaced by x_1, as $x_1 + ix_{n+2}$ has been eliminated between (3) and (4). Similarly, let $x_1 + ix_{n+2}$ be eliminated between (3) and (1). Then, with a slight change in the meaning of the coefficients, the problem reduces to that of finding the envelope of the sphere

$$s \equiv x_1 \left[a_{n+2}(y_1^2 + y_2^2 + \cdots + y_n^2) - a_1\right] - 2 \sum_{r=1}^{n} x_{r+1} (a_{n+2} y_r - a_{r+1}) = 0,$$

subject to the condition, say $f(x_1, x_2, \cdots, x_{n+1}) = 0$.

Since the sphere s is to touch the envelope f, it may be regarded as the equation of the point of contact, and f as the equation of the surface itself, in tangential coördinates, hence

$$\sum \frac{\partial f}{\partial x_i} x_i = \sum \frac{\partial s}{\partial x_i} x_i = 0,$$

and corresponding coefficients must be proportional,

$$\frac{\partial f}{\partial x_r} = \lambda \frac{\partial s}{\partial x_r}.$$

Between these $n + 1$ linear equations in x_r and the equation $s = 0$ the numbers x_r and λ may be eliminated, giving

$$(a) \quad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} & a_{n+1} \sum\limits_{r=1}^{n} y_r^2 - a_1 \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} & a_2 - 2y_1 a_{n+2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_n} & a_3 - 2y_2 a_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n+2} \sum\limits_{1}^{n} y_r^2 - a_1 & a_2 - 2y_1 a_{n+2} & \cdots & \cdots & 0 \end{vmatrix} = 0$$

or the Hessian of f bordered by the coefficients of x_r in s.

This proves that the envelope is a cyclical surface, and the number of constants in the most general surface of this kind, viz., $\frac{1}{2}(n^2 + 5n + 2)$, exactly coincides with the number of constants in f, $\frac{1}{2}n(n + 3)$, plus the number in (3), $n + 1$, so that all quartic cyclical surfaces can be generated in this way. This does not show, however, in how many ways the same surface may be generated.

Let the two forms

$$\kappa, \quad \varphi_2 + a_n x_{n+2}^2$$

be subjected to any linear transformation, such that $\kappa = 0$ may go into itself, and φ_2 may become a sum of squares of the form

$$(5) \qquad F \equiv \sum_{r=1}^{n+2} b_r x_r^2 = 0\,;$$

further, suppose the restriction be also imposed that x_{n+3} shall go into itself. Equation (5) represents a complex of spheres; in the vicinity of any of its spheres z it may be replaced by the tangent linear complex

$$\sum_{r=1}^{n+3} \frac{\partial F}{\partial z_r} x_r = 0,$$

so that all of the spheres which belong to F and touch z must touch it in points of its $M^{2,2}_{n-2}$ of intersection with the fundamental sphere of the tangent complex. When

$$\sum_{r=1}^{n+3} \left(\frac{\partial F}{\partial z_r}\right)^2 = 0,$$

these two spheres touch each other. The $M^{2,2}_{n-2}$ reduces to a point, and z is a singular sphere.

Let $\partial F/\partial z = t_r$; now t is also a sphere which touches z, and the whole tangent pencil can be represented in the form

$$m_r = t_r + \lambda z_r = b z_r + \lambda z_r = z_1 (b_r + \lambda).$$

The sphere m touches z, which is also a sphere, hence

$$\sum_{r=1}^{n+3} m_r x_r = 0, \qquad \sum_{r=1}^{n+3} z_r^2 = 0,$$

or, replacing z_r by its values,

$$A \equiv \sum \frac{m_r^2}{b_r + \lambda} = 0, \qquad \sum \frac{m_r^2}{(b_r + \lambda)^2} = 0.$$

Those values of m which satisfy these two equations define the singular sphere of a quadratic complex for every value of λ, hence the whole pencil of complexes have the same surface of singularities. The original complex is contained in the series, corresponding to $\lambda = \infty$.

The λ climinant of these two equations will give the equation in tangential coördinates of the surface of singularities; the surface is seen to be of class $4n$.

Among the quadratic complexes of the pencil are $n + 3$ simple ones counted twice, corresponding to $\lambda = -b_r$. When $\lambda = -b_{n+3}$ this becomes the complex of points in R_n which, combined with $x = 0$ and with the other terms of the complex

$$\sum_{r=1}^{n+2} \frac{x_r^2}{b_r - b_{n+3}} = 0,$$

defines a general cyclical surface. But the envelopes of the other $n+2$ systems corresponding to $\lambda = -b_r$ $(r=1, \cdots, n+2)$ define the same surface. Now $x_r = 0$ represents the totality of spheres which cut a fixed sphere orthogonally, and when associated with $A = 0$ defines all those spheres whose centers also lie on a quadric, hence by (a) they envelop a cyclical surface. The fundamental spheres of these $n+2$ systems are by (3) mutually orthogonal, and the quadrics are confocal since $A = 0$. Hence

An M^4_{n-1} in (euclidean) R_n which contains the absolute as a double $M^{1,2}_{n-2}$ can be generated in $n+2$ ways as the envelope of those hyperspheres which cut a fixed hypersphere orthogonally and whose centers lie on a $M^{2,2}_{n-2}$. The fixed spheres are mutually orthogonal and the quadrics are confocal.

The $M^{2,2}_{n-2}$, intersections of the M^2_{n-1} and the hyperspheres, are all focal spreads of the cyclical surface. Through the center of each sphere passes a bitangent cone M^2_{n-1} whose elements are perpendicular to the elements of the asymptotic, cones of the M^2_{n-1}.

By giving λ different values in the system

$$\sum_{r=1}^{n+2} \frac{x_r^2}{a_r - \lambda} = 0, \qquad x_{n+3} = 0,$$

a series of confocal cyclical surfaces is obtained. By substituting the coördinates of a point sphere in the equation, n different values for λ can be found; hence, *n cyclical surfaces of a confocal system pass through every point in space;* by applying the tangent complex to each and using (3) it appears that *these cyclical surfaces intersect orthogonally.*

For $n=2$ these surfaces (bicircular quartic curves) have been systematically studied, from a different point of view, by Casey, Darboux, Cox, Loria, and others; for $n=3$ (cyclides) by Casey, Maxwell, Cayley, Darboux, Reye, Loria, Bôcher, Domsch, Loewy, Moutard; and many special points have been noticed in numerous other papers.

The method here given is a generalization of that first employed by Darboux, using Lie's more general coördinates. The latter were first systematically employed in an article by the author, read at the Toronto meeting of the AMERICAN MATHEMATICAL SOCIETY, and published in the BULLETIN, volume 4 (new series), pp. 144–154.

For $n=4$, the number of distinct types is 58, and for larger values of n the number of types has not been determined.

CORNELL UNIVERSITY,
November 2, 1899.

ON THE SINGULAR TRANSFORMATIONS OF GROUPS GENERATED BY INFINITESIMAL TRANSFORMATIONS.

BY PROFESSOR HENRY TABER.

(Read before the American Mathematical Society, December 28, 1899.)

By means of r independent infinitesimal transformations

$$X_j = \sum_1^n{}_i\xi_{ji}(x_1, x_2, \cdots, x_n)\frac{\partial}{\partial x_i} \qquad (j = 1, 2, \cdots, r)$$

we may construct a family of transformations

$$(1) \qquad x_i' = x_i + \sum_1^r{}_j a_j X_j x_i + \tfrac{1}{2}\sum_1^r{}_j\sum_1^r{}_k a_j a_k X_j X_k x_i + \cdots$$
$$\equiv f_i(x_1, \cdots, x_n, a_1, \cdots, a_r) \qquad (i = 1, 2, \cdots, n)$$

with r essential parameters $a_1, a_2, \cdots, a_r$. The transformations defined by these equations, for assigned values of the a's, may be denoted by T_a. Each transformation of this family is paired with its inverse.

For finite values of the parameters a, the transformation T_a (provided it is not illusory) belongs to a one parameter group generated by the infinitesimal transformation

$$a_1X_1 + \cdots + a_rX_r.$$

As the a's approach certain limiting values, one or more of which is infinite, T_a may have a definite finite transformation T as a limit. The transformation T may be regarded as a transformation of the family, and, if equivalent to a transformation T_b with finite parameters, can be generated by an infinitesimal transformation of the family (namely, $b_1X_1 + \cdots + b_rX_r$), but not otherwise.*

Let it be assumed that

$$X_jX_k - X_kX_j = \sum_1^r{}_s c_{jks}X_s \qquad (j, k = 1, 2, \cdots, r),$$

---

*Thus, if the transformation T_a, for one or more of the a's infinite, is finite and definite, but is not equivalent to a transformation of the family with finite parameters, the transformation T_a cannot be generated by an infinitesimal transformation of the family. To this extent Lie's theorem on p. 65 of the Transformationsgruppen, vol. 1, requires modification.

the c's being constants. Then, by the chief theorem of Lie's theory, the family of transformations (1) forms a group G with continuous parameters; and each transformation of G is, in general, generated by an infinitesimal transformation of the group. Thus from

$$(1) \qquad x_i' = f_i(x_1, \cdots, x_n, \; a_1, \cdots, a_r) \qquad (i = 1, 2, \cdots, n)$$

and

$$(2) \qquad x_i'' = f_i(x_1', \cdots, x_n', \; b_1, \cdots, b_r) \qquad (i = 1, 2, \cdots, n)$$

we derive

$$(3) \qquad x_i'' = f_i(x_1, \cdots, x_n, \; c_1, \cdots, c_r) \qquad (i = 1, 2, \cdots, n)$$

where

$$(4) \qquad c_j = \varphi_j(a_1, \cdots, a_r, \; b_1, \cdots, b_r) \quad (j = 1, 2, \cdots, r).$$

For finite values of the a's and b's it may happen that every branch of one or more of the functions φ is infinite. In this case, while each of the transformations T_a and T_b is generated by an infinitesimal transformation of the group, the transformation T_bT_a, resulting from their composition, cannot be generated thus, and the group cannot properly be said to be continuous. A transformation of G which cannot be generated by an infinitesimal transformation of G may be termed *essentially singular.*

In what follows I shall signify by T_a, T_b, etc., transformations of the groups with finite parameters generated, respectively, by the infinitesimal transformations

$$a_1X_1 + a_2X_2 + \cdots + a_rX_r, \quad b_1X_1 + b_2X_2 + \cdots + b_rX_r, \quad \text{etc.}$$

Group G may contain a transformation T_a (generated by an infinitesimal transformation) which, in composition with every transformation of some one (or more) subgroups of G with one parameter, in particular with the infinitesimal transformation of such subgroup, results in an essentially singular transformation. Such a transformation T_a I term *non-essentially singular*. The values of the a's for which T_a is non-essentially singular may be termed *critical* values of the parameters. The critical values of the parameters are included among those for which one or more of the roots of the equation in ρ

$$\begin{vmatrix} \sum_1^r {}_j a_j c_{j11} - \rho, & \sum_1^r {}_j a_j c_{j21} \quad , & \cdots \\ \sum_1^r {}_j a_j c_{j12} \quad , & \sum_1^r {}_j a_j c_{j22} - \rho, & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots \end{vmatrix} = 0,$$

is equal to an even multiple, not zero, of $\pi\sqrt{-1}$. This condition is necessary but not sufficient. That is to say, for values of the a's for which one or more of the roots of this equation is an even multiple, not zero, of $\pi\sqrt{-1}$, T_a is not necessarily singular.

I find that every transformation of G, in particular every essentially or non-essentially singular transformation, can be obtained by the composition of two non-singular transformations. Also that, corresponding to every essentially singular transformation T of G, a non-singular transformation $T_{a'}$, whose parameters are functions of a variable λ, can be found which can be made to approach as nearly as we please to T by taking λ sufficiently small, and such that $\lim_{\lambda=0} T_{a'} = T$.*

In every group G containing essentially singular transformations which I have examined, non-essentially singular transformations also exist, and any non-singular transformation T_a whatever combined with some one, or more, non-singular or non-essentially singular transformations T_b results in an essentially singular transformation. These relations undoubtedly hold invariably.

Let $\mathfrak{F}_a$ denote the bilinear form

$$\sum_1^r {}_\mu \sum_1^r {}_\nu \left(\sum_1^r {}_j a_j c_{j\nu\mu}\right) u_\mu v_\nu,$$

and let I denote the bilinear form $\sum_1^r {}_\mu u_\mu v_\mu$. The coefficient of $u_\mu v_\nu$ in the bilinear form

$$\frac{e^{\mathfrak{F}_a} - I}{\mathfrak{F}_a} = I + \tfrac{1}{2}\mathfrak{F}_a + \tfrac{1}{3!}\mathfrak{F}_a^2 + \cdots$$

is a power series in the a's, which may be denoted by $P_{\mu\nu}(a)$, and which is convergent for all finite values of the a's. Let now $\triangle_a$ denote the determinant

---

* These theorems were given for subgroups of the projective group by the author in a paper read at the February Meeting of the Society, 1899.

$$\begin{vmatrix} P_{11}(a), & P_{12}(a), & \cdots \\ P_{21}(a), & P_{22}(a), & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

and let $A_{\mu\nu}$ denote the first minor of $\triangle_a$ relative to $P_{\nu\mu}(a)$. Then, if we put

$$a_j = \varphi_j(\bar{a}_1, \cdots, \bar{a}_r, tb_1, \cdots, tb_r) \quad (j = 1, 2, \cdots, r),$$

regarding $\bar{a}_1, \cdots, \bar{a}_r, b_1, \cdots, b_r$ as fixed and the a's as functions of t, the latter satisfy the differential equations

$$\triangle_a \frac{da_j}{dt} = A_{j1}b_1 + A_{j2}b_2 + \cdots + A_{jr}b_r \quad (j = 1, 2, \cdots, r).$$

The A's are integral functions (transcendental or rational) of the a's, and $\triangle_a$ vanishes only if one or more of the roots of the characteristic equation of $\mathfrak{F}_a$ is an even multiple, not zero, of $\pi\sqrt{-1}$.*

The functions φ are multivalued. Returning to the original notation, let

$$c_j = \varphi(a_1, \cdots, a_r\,;\, b_1, \cdots, b_r) \quad (j = 1, 2, \cdots, r).$$

For assigned values of the a's and b's the difference between any two branches of $\varphi_j(a, b)$ is equal to

$$2\pi\sqrt{-1}\sum_1^r {}_k\, m_k\, \psi_{kj}(a, b),$$

where the m's are integers, and the ψ's are rational functions of the coefficients of the bilinear form $e^{\mathfrak{F}_a}e^{\mathfrak{F}_b}$.

The equations defining the transformations of one group may restrict the functions φ_j to fewer branches than in the case of another group of the same structure (*Zusammensetzung.*) Consequently, of two groups of the same structure, one may be continuous and the other may be discontinuous, that is, may possess essentially singular transformations.†

---

* If we denote by T_{tb} the transformation of G whose parameters are $tb_1, tb_2, \cdots, tb_r$, which is generated by the infinitesimal transformation $b_1X_1 + \cdots + b_rX_r$, the transformation $T_{tb}T_{\bar{a}}$ is essential singular only for those values of t for which the determinant is zero of the bilinear form

$$\frac{e^{\mathfrak{F}_{tb}}e^{\mathfrak{F}_{\bar{a}}} - I}{\mathfrak{F}_{tb}\mathfrak{F}_{\bar{a}}}.$$

† My attention was first drawn to this fact by my pupil, Mr. S. E. Slocum.

If, however, the adjoined of any group G is discontinuous, G itself, and, of course, every group of the same structure is discontinuous. The bilinear form $\mathfrak{F}_a$ is closely related to the adjoined group. In fact, if ϕ_a denotes the matrix of $\mathfrak{F}_a$, the infinitesimal equations of the adjoined are

$$(a_1', a_2', \cdots, a_r) = (1 + \partial t\phi_a)(a_1, a_2, \cdots, a_r);$$

and we have

$$e^{\mathfrak{F}_\beta} e^{\mathfrak{F}_a} = e^{\mathfrak{F}_\gamma}$$

where

$$\gamma_j = \varphi_j(a_1, \cdots, a_r, \beta_1, \cdots, \beta_r) \qquad (j = 1, 2, \cdots, r).$$

Clark University,
December, 1899.

PROOF OF THE EXISTENCE OF THE GALOIS FIELD OF ORDER p^r FOR EVERY INTEGER r AND PRIME NUMBER p.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, December 28, 1899.)

Existence proofs have been given by Serret[*] and by Jordan.[†] The developments used by Serret are lengthy but quite in the spirit of Kronecker's ideas. The short proof by Jordan, however, assumes with Galois the existence of imaginary roots of an irreducible congruence modulo p.

The proof sketched in this note proceeds by induction. Assuming the existence of the $GF[p^n]$, we derive that of the $GF[p^{nq}]$, q being an arbitrary prime number. Since the $GF[p]$ exists, being the field of integers taken modulo p, it will follow that the $GF[p^q]$ exists, and by a simple induction that the $GF[p^r]$ exists for r arbitrary.

We employ the lemma: *A factor of $x^{p^{nm}} - x$, belonging to and irreducible in the $GF[p^n]$, can be of degree m' if and only if m' divides m.* In particular, the irreducible factors of $x^{p^{nq}} - x$ are of degree q or 1. But the product of the distinct[‡] linear factors $x - \nu_i$ belonging to the $GF[p^n]$ is $x^{p^n} - x$.

---

[*] Algèbre supérieure, 2, pp. 122–142.

[†] Traité des substitutions, pp. 16, 17.

[‡] Two functions belonging to the $GF[p^n]$ are called distinct if one is not the product of the other by a constant, a mark of the field.

Every such factor $x - \nu_i$ *is a simple factor of* $x^{p^{nq}} - x$. The statement being evident for the factor x, we assume $\nu_i \neq 0$, and proceed to prove that $x - \nu_i$ is not a factor in the field of

$$\frac{x^{p^{nq}-1} - 1}{x^{p^n-1} - 1} \equiv (x^{p^n-1})^{r-1} + \cdots + (x^{p^n-1}) + 1,$$

where $$r \equiv (p^{nq} - 1)/(p^n - 1).$$

But for $x = \nu_i \neq 0$, this function reduces in the $GF[p^n]$ to the value r, which is evidently not divisible by p.

It follows that the quotient $(x^{p^{nq}} - x)/(x^{p^n} - x)$ breaks up into factors of degree q belonging to and irreducible in the $GF[p^n]$. Any such factor may be used to define the $GF[p^{nq}]$.

MÉRAY'S INFINITESIMAL ANALYSIS.

Leçons nouvelles sur l'analyse infinitésimale et ses applications géométriques. Par M. Ch. Méray, professeur à la Faculté des Sciences de Dijon. Paris, Gauthier-Villars. Première partie: *Principes généraux*, 8vo., 1894, xxxiii + 405 pp.; Deuxième partie: *Étude monographique des fonctions principales d'une seule variable*, 1895, xi + 495 pp.; Troisième partie: *Questions analytiques classiques*, 1897, vi + 206 pp.; Quatrième partie: *Applications géométriques classiques*, 1898, xi + 248 pp.

As a work of art—and mathematics is preëminently a fine art—this monumental work of Méray's is a masterpiece; as a treatise to induct students of mathematics into the mysteries of the infinitesimal analysis under the direction of any other than the author* its success is hardly so unqualified. Constructed so as to require technically as preliminary training nothing but a knowledge of elementary algebra and the theory of systems of linear algebraic equations, the first volume would indicate either or both of two conditions: that the author is a most marvelous teacher, that his pupils are of a race superior to that brilliant body of mathematicians now at Paris. It is one thing to be possessed of sufficient elementary knowledge to undertake the serious study of Méray's treatise and quite another thing for that preliminary training to have developed the power of abstraction necessary to work through it.

---

* The author states in the preface to the first volume (1894) that for twenty-four years he has found his method uniformly satisfactory in teaching.

Méray seems never to have had a particular idea, except at second thought or second hand. He lisped in generalities, for the general ideas came. Near the beginning of his career he wrote an elementary geometry in which the usual distinctions between the geometry of space and that of the plane were abandoned; the perusal of his present work leaves one with the impression that it must have been a lifelong regret to the author that his Euclid did not spring forth in n dimensions.

The first volume, which is an elaboration of the author's Nouveau précis d'analyse infinitésimale published in 1872, comprises in four hundred pages all the general notions of analysis from the integral number to the integrals of partial differential equations, without mention of a particular function, a straightforward, iron-heeled, uninterrupted march of general principles. The power of abstraction demanded is something terrific; the reader experiences all of the symptoms of that peculiar malady encountered by travellers at high altitudes, and if modern theories be true, some of the author's ethereal oxygen ought to be replaced by the carbon dioxide of regions nearer us common mortals. It is upon this ground that the volume cannot be judged as cordially on pedagogic as upon æsthetic principles; this criticism would be more unjust had not the author addressed his work not only to savants but primarily to students. It may be remarked here however, parenthetically, that the first volume is not half so disheartening when accompanied by the second.

But on the other hand when the work is contemplated as the consistent and logical development of an idea, as the persistent adherence to a method until it had mastered the difficulties interposed by its critics, as the epic of a human ideal, Méray's work is a triumphant success. No student who has had sufficient discipline to maintain him in its perusal can read the work without being profoundly grateful; no student who is endeavoring, for his own peace of mind and safety of soul, to construct his own theory of the number system and to frame his own theory of functions, can afford not to have availed himself of Méray's admirable work; many who cannot accept all of his dogmas—and dogmas are as deplorable in science as in religion—will recall with regret that they never sat under the author's tutelage.

A detailed analysis of this extensive treatise would ten times exceed the space that the BULLETIN can place at the disposal of the reviewer; in order to respect this restriction of space and at the same time acquaint the reader with the gist of Méray's method, the writer undertakes a succinct account of the author's memoir* on the theory of functions to which reference has already been made, noting what modifications or additions have been made in the treatise.

In the first chapter of the memoir, and the first three chapters of the treatise, Méray constructs the ensemble of fictitious numbers employed by analysis, without the aid of any other notions than those of the integral number and of the addition of such numbers. He banishes all consideration of arguments from his study of imaginaries. His notion of variant (almost identical with that of infinite rational sequence), employed to give precision to the most difficult concept of analysis, makes his exposition of the irrational both clear and luminous. In the preface to the first volume of the treatise, Méray substantiates the claim made by his friends that the latter solution was first given by him. The invention of this theory of incommensurables has been credited to Heine and the first use made of it to Lipschitz, Du Bois-Reymond, and G. Cantor. Heine's memoir entitled "Die Elemente der Functionenlehre" appeared in the seventy-fourth volume of *Crelle's Journal* with the date of 1872; Méray's Nouveau Précis is of the same date, 1872, but three years previously he had presented the theory at a congress of learned societies in Paris, in a note entitled "Remarques sur la nature des quantités définiies par la condition de servir de limite à des variables données" published in the *Revue des Sociétés savantes* (*Sciences mathématiques*, vol. 4, p. 284, 1869).

---

*Nouveau Précis d'Analyse infinitésimale, Paris, F. Savy, 1872; 8vo, xxiii + 310. This work is now included in the list of publications of Gauthier-Villars. The memoir was dedicated to M. Faurie, General Inspector of secondary instruction, and was seriously designed to bring about a revolution against the traditional methods of teaching the infinitesimal calculus. For convenience of comparison the order of ideas exhibited in the subjects of the chapters is here appended:—1° preliminaries; 2° generalities on series; 3° series arranged in the ascending powers of the independent variables; 4° olotropic functions, classification of their derivatives, general properties; 5° limits of convergence of the developments of olotropic functions; 6° composite functions; 7° principles of the discussion of functions; 8° various propositions on olotropic or quasi-olotropic functions of a single variable; 9° fundamental principle of the integral calculus; 10° theory of implicit functions; study of some implicit functions of a single variable defined by a unique equation; 12° analytic theory of the exponential function and of its related functions; 13° inverse functions; 14° ordinary differential equations, principal case; 15° general case of the preceding; 16° total differential equations; 17° simple integrals.

The number system thus established in a manner that leaves nothing to be desired in the way of cold logic, the author proceeds to present the properties of series, and in particular those of integral power series. He loses no opportunity to impress his conviction that all analytic functions are identical with integral series. To Méray's mind no function exists which cannot be expanded by Taylor's theorem. All other functions he rules out as "phantasies of the mathematical imagination;" but does the reader clearly understand the meaning of the last phrase? To the same exile he consigns the non-euclidean geometry. The demands of analysis required that the idea of the number system be enlarged to include certain fictitious elements that could equally be dubbed fantastic creatures of a mathematical imagination; Méray fell in with the call of the hour and led the procession. For this, among other reasons, it is difficult to understand why the author refuses to respect the discoveries of the last twenty years and accord to the notion function the possibility of a similar enlargement. Méray is a pure mathematician, an analyst of power, and yet he would prescribe a definite boundary to the theory of functions and to all physical phenomena to come within the ken of men because "Tout phénomène naturel est représentable exactement par des séries entières, approximativement par leurs premiers termes, dont des observations de plus en plus précises fournissent empiriquement les coefficients dans l'ordre même où l'Analyse les range."

Lie averted the current objections to his method from being hurled against his volume on differential equations by limiting it to differential equations that admit of infinitesimal transformations; this was thoroughly scientific and did not carry with it the absurd denial of the existence of differential equations not admitting of infinitesimal point transformations, the number of which equations is legion. Similarly no complaint could be lodged against Méray had he contented himself with the study of functions developable by Taylor's theorem, without defining analysis to be the systematic study of the general properties of functions. He got hold of a large idea, and exhausted it ten, perhaps twenty, years ago; since then it seems to have been master of the man. Would that the same vigorous analytical power had been turned in the meantime on those spurned subjects.

The exposition of the general properties of integral series is considerably simpler in the treatise than in the memoir because of the introduction of the author's new demonstration* of a lemma of Cauchy; the demonstration gives the lemma the position of a new theorem. The discussion of the continuity and convergence of series is admirable.

Passing to the notion of function, of which more or less has been remarked already, the author defines a function to be olotropic in the areas S_x, S_y, $\cdots$, with the olometers δ_x, δ_y, $\cdots$, when, for every system of numbers, x_0, y_0, $\cdots$, taken within the areas in question, we can develop the function in a convergent integral power series in $x - x_0$, $y - y_0$, $\cdots$, provided that the moduli of these differences are respectively less than δ_x, δ_y, $\cdots$. The areas S_x, S_y, $\cdots$, are any whatsoever having simple or multiple contours.

The successive derivatives of a function of one or more variables are naturally defined after Lagrange's theory of analytic functions and Arbogast's method of derivatives, the classical objections to these being avoided by considering only olotropic functions. The derivatives are obtained algebraically without reference to infinitesimals by developing the series $f(x + h, y + k, \cdots)$ and throwing the development into the form $f(x, y, \cdots) + hf_x' + kf_y' + \cdots$, the quantities f_x', f_y', $\cdots$ being convergent series.

A capital element in the generation of new functions is the operation which the author calls *cheminement;* this operation is the ensemble of the following: To obtain in X, Y, $\cdots$ the value of a function $f(x, y, \cdots)$ it is necessary to start from an initial system x_0, y_0, $\cdots$, and interpolate, if possible, between it and the proposed system X, Y, $\cdots$ intermediate systems x_1, y_1, $\cdots$; x_2, y_2, $\cdots$; $\cdots$; x_k, y_k $\cdots$, such that the differences between the corresponding values of the variables in two consecutive systems shall have moduli less than the olometers of the function for the values of the smaller index; then to calculate the function $f(x_1, y_1, \cdots)$ by Taylor's formula, by putting

$$f(x_1, y_1, \cdots) = f[x_0 + (x_1 - x_0), \quad y_0 + (y_1 - y_0), \cdots];$$

then $f(x_2, y_2, \cdots)$ in a similar manner by means of $f(x_1, y_1, \cdots)$; and finally $f(X, Y, \cdots)$ from $f(x_k\, y_k, \cdots)$.

The author studies the theory of indefinite integrals in the eighth chapter of the treatise under the title inverse calculus of derivatives. Up to this point inclusive it has been a question only of isolated functions, that is to say functions considered independently, as if no other functions existed. The remainder of the volume is occupied with general operations which imply the composition of functions.

---

*[Bulletin des Sciences mathématiques](), vol. 15 (1891).

In the tenth chapter of the treatise's first volume the author demonstrates the convergence of the elementary developments of the integrals of a system of total differential equations, by a direct examination of these developments themselves and without the intervention of auxiliary differential equations. It is at this juncture that Méray introduces an auxiliary notion, attributed to Weierstrass but employed by Méray since 1868 and given at § 139 of the Nouveau Précis, and § 300 of the first volume. This auxiliary artifice he calls the *majorante* and defines it as follows: Given an olotropic function $f(x, y, \cdots)$ of any variables, we call majorante of this function at $x_0, y_0, \cdots$ every function $\varphi(x, y, \cdots)$ of the same variables, possessed of the property that, for $x = x_0$, $y = y_0$, $\cdots$, its value and that of all its derivatives shall be real and positive, and superior to the moduli of the corresponding values of $f(x, y, \cdots)$ and its derivatives.

The memoir contains practically nothing of the chapter XII. of the first volume on partial differential equations; the material is drawn largely from two papers published by the author with the collaboration of M. Riquier.*

In the last chapter of the volume on systems of total differential equations all reference to unsolved equations and to those of order higher than the first is omitted. In his Nouveau Précis the author demonstrated for the first time the possibility of resolving normally the integral equations with regard to the arbitrary constants; a demonstration neither without use nor without value.

As remarked before, the first volume contains no reference to any special form of function; the chapters of the memoir relating to particular functions were reserved for the second volume.

Among the elements of the first chapter we find the author's more recent demonstration† of the existence of roots of every integral algebraic equation; this demonstration is as simple and elegant as could be desired.

The limits of space prescribed forbid calling attention to the details of this second volume. Certain striking ones

---

**Annales scientifiques de l'École Normale supérieure*, vol. 6 (1889), vol. 7 (1890).

† "Méthode directe fondée sur l'emploi des séries pour prouver l'existence des racines des équations entières à une inconnue par la simple exécution de leur calcul numérique," *Bulletin des Sciences mathématiques*, vol. 15 (1891).

however demand a word. In the third chapter the author develops the theory of radicals as a simple corollary of his theory of implicit functions elaborated in the first volume. He incorporates his memoir* which treats of this question in harmony with the fundamental point of view of Abel, whom Méray places with Lagrange as the first to found the infinitesimal analysis on a solid basis. The author's method is long, but he avoids all trigonometric considerations and thus adds a further illustration that the intrusion of geometric facts is not a necessity in analysis.

In the fourth chapter he reproduces and improves the method given in the Nouveau Précis for extending the capital discovery of Puiseux on algebraic irrationals to the roots of any olotropic equation. A principle is introduced under the name principle of the conservation of the number of roots which later brings into evidence certain fundamental properties of biperiodic functions; the principle is one of those which exist only by virtue of extending the ordinary algebraic operations to imaginaries. The author regrets, as does the reader, that the inflexible rigor which he has made his law would not permit of abridging these earlier chapters.

The Napierian logarithm, being the first transcendent having a derivative, naturally suggests itself as the first transcendent to be studied. The author then presents the exponential as the simple result of the analytic inversion of the logarithm. The inversion of the indefinite integral of a rational differential gives other functions, no longer indefinitely olotropic as the exponential, but indefinitely meromorphic; in this way the circular functions are introduced. Méray returns to the differential equation as the starting point for his elliptic functions, regarding his demonstration† of the non-nullity of the moment of the periods as sufficiently simple, and considering the dominant rôle assumed by the theta functions in modern expositions as artificial. As usual, it is the most general function that he studies.

As to the functions of Euler studied in the last chapter, they also have their origin in integration, not directly or indirectly from indefinite integrals considered as functions of their principal variables, but as functions of parametric variables which accompany the principal variable in certain artificial definite integrals; by the latter Méray means the integral of $f(x)$ from x_0 to X when the path of integration contains any value of x_1 singular for $f(x)$, or when the path is unlimited in any direction.

---

* "Théorie des radicaux fondée exclusivement sur les propriétés générales des séries entières," *Revue bourguignonne de l'Enseignement supérieur*, vol. 1 (1891).

† "Sur l'existence effective des deux périodes des fonctions elliptiques," *Ann. scient. de l'École Normale sup.*, 3d series, vol. 1 (1894).

The course of the development of the whole treatise is admirably exhibited by the subjects and order of the chapters of the several volumes, which are appended below in conclusion.* The elegant applications presented in the last two parts are more than worthy of a direct reference, for they demonstrate again the success of the method in the hands of the author.

I. 1° Preliminary generalities comprising a review of the fictitious quantities upon which the speculations of modern analysis are founded; fractions, positive and negative quantities; 2° continuation of the preceding, variants in general; incommensurable quantities; 3° continuation of the two preceding chapters, imaginary quantities; 4° series in general; 5° integral series; 6° derivatives of olotropic† functions, usual genesis of these functions; 7° fundamental properties of functions which are olotropic within given areas; 8° inverse calculus of derivatives; 9° composite functions; 10° essential principle of the theory of total differential equations; 11° implicit functions in general; 12° essential principles of the theory of partial differential equations; 13° further study of immediate systems of total differential equations; addition, 1° on an essential property of integral polynomials in a single variable.

---

* The first volume contains a list of Méray's principal publications. The works have already been referred to; the memoirs are to be found in the following journals: *Nouv. Ann. de Math.*, 1st ser., vol. 13 (1854), 3d ser., vol. 8 (1889), vol. 9 (1890), vol. 11 (1892); *Comptes rendus*, vol. 40 (1855), vol. 106 (1888); *Ann. di Matem.*, 1st ser., vol. 3 (1860); *Ann. scient. de l'Ec. Norm sup.*, 1st. ser., vol. 4 (1867), 2d ser., vol. 6 (1877), vol. 8 (1879), vol. 12 (1883), 3d ser., vol. 1 (1884), vol. 2 (1885), vol. 6 (1889), vol. 7 (1890); *Revue des Sociétés savantes* (*Sc. math., phys., et nat.*), 2d ser., vol. 3 (1868), vol. 4 (1869); *Liouville's Journal*, 3d ser., vol. 10 (1884); *Bull. des Sc. math.*, 2d ser., vol. 12 (1888), vol. 5 (1891); *Revue bourguignonne de l'Enseign. sup.*, vol. 1 (1891), vol. 2 (1892); *Ann. de la Fac. des Sc de Toulouse*, vol. 4 (1891).

† Mathematics of necessity mutilates language occasionally in the construction of its technical terms, but it is to be regretted that Méray did not give heed to a correction in the form of this term suggested by the editors of the *Bulletin des Sciences mathématiques* in 1874; orthography should respect etymology and the Greek derivative is ὅλος. It is interesting to observe that Méray used this term in 1872, but that not until 1875 did its successful rival *holomorphic* appear; see Briot and Bouquet's Theory of elliptic functions, 2nd edition, p. 14. In the preface to the first volume Méray anticipates a day when both will disappear, namely at that epoch when mathematicians shall have opened their eyes to the accuracy of the conceptions of Lagrange and have ceased to contemplate functions non-integrable, continuous but without derivatives, et cætera-

II. 1° Olotropic functions of a single variable, in general; 2° meromorphic functions of a single variable, in general; 3° simple radical function; 4° study of the principle critical phases of an implicit function of a single variable defined by a unique equation; 5° Napierian logarithm and exponential function; 6° circular functions; 7° development of circular functions in series of simple fractions and in factorial series; 8° summary theory of elliptic functions; 9° continuation of the preceding, biperiodic functions in general; 10° continuation of the two preceding chapters, development of biperiodic functions in series of simple fractions or of circular functions, and in factorial series; 11° continuation of the three preceding, salient points of the theory of biperiodic functions of the second order; 12° continuation of the four preceding chapters, canonical elliptic functions; 13° notions on eulerian functions.

III. 1° Indefinite integration of current differentials; 2° calculation of certain definite integrals not demanding the knowledge of indefinite integrals; 3° elementary differential equations; 4° partial differential equations of the first order; 5° questions of maxima and minima; 6° real multiple integrals, additions: 1° elementary proposition to be substituted for Cauchy's lemma in the general theory of functions; 2° direct demonstration of Cauchy's lemma; 3° on the non-nullity of the differential determinant with respect to arbitrary constants of the general integrals of a system of total differential equations; 4° on the possibility of making the modulus of an indefinitely olotropic function increase without limit; 5° on an extended case in which interpolation permits of representing a function with an indefinite approximation.

IV. 1° Preliminaries, rectifications, quadratures, cubatures; 2° contacts in general; 3° contacts of surfaces and of lines with figures of the first degree; 4° enveloping figures; 5° contacts of first order between the sphere or the circle and given figures; 6° salient properties of ordinary surfaces; 7° contacts of higher orders of a line with the circle and the sphere; 8° questions relating to contacts of second order of a surface with the circle and right line; addition, principal formulæ in polar coördinates.

E. O. LOVETT.

PRINCETON UNIVERSITY.

NOTES.

THE first number has just appeared of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY, published by the Society with the coöperation of Harvard University, Yale University, Princeton University, Columbia University, Haverford College, Northwestern University, Cornell University, The University of California, Bryn Mawr College, The University of Chicago; edited by ELIAKIM HASTINGS MOORE, ERNEST WILLIAM BROWN, THOMAS SCOTT FISKE; New York, The Macmillan Company. The first number, which consists of 96 pages, contains the following articles: —"Conics and cubics connected with a plane cubic by certain covariant relations," by H. S. WHITE; "Formentheoretische Entwickelung der in Herrn White's Abhandlung über Curven dritter Ordnung enthaltenen Sätze," by P. GORDAN; "Sur la définition générale des fonctions analytiques, d'après Cauchy," by E. GOURSAT; "On a class of particular solutions of the problem of four bodies," by F. R. MOULTON; "Definition of the abelian, the two hypoabelian, and related linear groups as quotient groups of the groups of isomorphisms of certain elementary groups," by L. E. DICKSON; "Note on the unilateral surface of Moebius," by H. MASCHKE; "On regular singular points of linear differential equations of the second order whóse coefficients are not necessarily analytic," by M. BÔCHER; "The elliptic sigma functions considered as a special case of the hyperelliptic sigma functions," by O. BOLZA; "On the groups which are the direct products of two subgroups," by G. A. MILLER; "On certain crinkly curves," by E. H. MOORE; "A new definition of the general abelian linear group," by L. E. DICKSON.

A NEW Annual Register of the AMERICAN MATHEMATICAL SOCIETY, including a list of members, directory, list of publications, constitution, by-laws, and the reports of the Treasurer and Librarian has recently been published and distributed to the members of the Society. Copies of the Register may be obtained from the Secretary.

AT the third ordinary meeting of the eighteenth session of the Edinburgh mathematical society, January 12, 1900, an address was delivered by Dr. PEDDIE on "The dissipation of energy in vibrating matter." Professor GIBSON's recent papers on the theory of proportion will be published in full in the society's *Proceedings*.

THE Circolo matematico di Palermo elected the following council for the years 1900–1902 at its recent annual meeting:—Resident members, Professors M. L. Albeggiani, M. Gebbia, F. Gerbaldi, G. B. Guccia, G. Torelli; Non-resident members: Professors E. Beltrami, of Rome, L. Bianchi, of Pisa, A. Capelli, of Naples, V. Cerruti, of Rome, L. Cremona, of Rome, P. Del Pezzo, of Naples, A. Del Re, of Naples, G. Loria, of Genoa, G. Mittag-Leffler, of Stockholm, E. Pascal, of Pavia, G. Peano, of Turin, S. Pincherle, of Bologna, H. Poincaré, of Paris, A. Tonelli, of Rome, V. Volterra, of Turin.

THE following papers were announced for the regular monthly meeting of the London mathematical society held January 11, 1900:—"A problem in resonance, illustrative of the mechanical theory of selective absorption," by Professor H. LAMB, and "Elementary distributions of plane stress," by Mr. J. H. MICHELL.

At the preceding meeting of December 14, 1899, the list of papers presented was as follows:—"A method for extending the accuracy of mathematical formulæ," and "Central difference formulæ," by Mr. W. F. SHEPPARD; "Circular cubics," by Mr. A. B. BASSETT; "The theorem of residuation, being a general treatment of the intersections of plane curves at multiple points," by Dr. F. S. MACAULAY; "The genesis of the double gamma functions, by Mr. E. W. BARNES; "On the expression of spherical harmonics as fractional differential coefficients," by Mr. J. ROSE-INNES; "Sums of greatest integers," by Dr. G. B. MATHEWS.

AT the meeting of the Royal Society of Edinburgh on December 19, 1899, Professor TAIT presented a "Note on the claim recently made for Gauss to the invention of quaternions," whose object was to show that what Professors Klein and Sommerfeld ascribed to Gauss in their treatise Ueber die Theorie des Kreisels was not the Hamiltonian quaternion, but a particular and limited kind of strain which consisted of a simple rotation combined with an isotropic expansion, and thus involving four constants only. Dr. C. G. KNOTT contributed a detailed criticism of the section devoted to the discussion of the theory of quaternions in the treatise cited.

THE annual meeting of the British mathematical association was held January 20, 1900, and the following papers were presented:—"Dynamical applications of the theory of correspondences," by Professor R. S. BALL; "Triangles triply in perspective,'' by Mr. J. A. THIRD; ''The teaching of indices and surds,'' by Professor R. W. GENESE; ''Illustrations of prismatic equations,'' by Mr. T. J. BROMWICH; '' A note on the focoids,'' by Mr. R. F. DAVIS.

ON the twenty-fourth of December, 1899, the Physico-mathematical society of Kazan celebrated a jubilee in honor of the twenty-fifth year of the scientific career of Professor VASILIEV, who has been president of the society for fifteen years.

AMONG the various congresses in process of organization in connection with the approaching Paris Exposition there is to be one devoted to the history of the sciences. This congress proposes to consider the development of all branches of scientific knowledge. Professor PAUL TANNERY is president, and Dr. SICARD DE PLAUZOLES is secretary of the organizing committee. The official address of this committee is 10 boulevard Raspail, Paris.

THE last volume of the *Revue de Métaphysique et de Morale* contains an extended discussion on the axioms of geometry between Professor H. POINCARÉ and Mr. B. RUSSELL, occasioned by the essay of the latter on the foundations of geometry. The discussion is continued in the January number of the current volume. In this connection it may be added that Mr. FREDERICK PURSER contributes an elaborate defense of the Kantian theory of the axioms of geometry to the twenty-fifth number of *Hermathena*, edited by members of Trinity College, Dublin.

AS ONE of the by-products of the Encyclopædia of the mathematical sciences now in process of construction, Teubner, of Leipzig, announces the preparation of an elaborate series of text-books in the field of pure and applied mathematics, and seeks the coöperation of mathematicians in the project, whether or not they be members of the contributing staff of the Encyclopædia. The preliminary announcement is very meagre, but more precise details are promised at an early date. The following members of the editorial staff of the Encyclopædia, among others, have signified their approval of the undertaking :—Messrs. Bôcher, Brunel, Castelnuovo, Dingeldey, Enriques, Harkness, Kohn, Krazer, von Lilienthal, Mehmke, Netto, Pincherle, Segre, Seliwanoff, Simon, Stäckel, Staude, Vahlen, Voss, Wiman, Wirtinger, and Zeuthen.

PROFESSOR V. BJERKNES, of the University of Stockholm, has just issued the first volume of his '' Vorlesungen über hydrodynamische Fernkräfte nach C. A. Bjerknes' Theorie." This volume, illustrated with forty figures, gives an exposition, together with the author's original contributions, of the theories which have constituted the life work of his father Professor C. A. Bjerknes of the University of Christiania, and experiments which have brought to light the analogies between electrostatic and electrodynamic phenomena and the apparent attractions experienced by solid bodies plunged into moving liquids.

THE approaching volume of the *Jahresberichte* of the German mathematical association will contain, among others, the following memoirs:—"Numerical solution of equations," by R. HAUSSNER; "The kinetic problems of scientific technology," by K. HEUN; "Calculus of variations," by A. KNESER; "Graphic methods," by R. MEHMKE; "The theory of manifoldnesses," by A. SCHOENFLIES; "General dynamics," by PAUL STÄCKEL; "The theory of finite groups," by E. STEINITZ.

THE first volume of a French edition of TCHEBYCHEV's collected works has just appeared. The volume, which is edited by Professors MARKOF and SONINE, contains thirty-four memoirs and a portrait of their author. A German translation of Vaseliev's memoir on the mathematical work of Tchebychef is to be published immediately by Teubner, of Leipzig.

M. HERMANN, of Paris, announces the appearance of a memoir of Professor G. HUMBERT on singular abelian functions.

PORTRAITS of the following mathematicians are to be had of B. G. Teubner, of Leipzig:—Moritz Cantor, Alfred Clebsch, Hermann Grassmann, Hermann von Helmholtz, Leopold Kronecker, Sophus Lie, Nikolaj Lobatschefskij, and P. L. Tchebychef. With the exception of that of Grassmann, which is a woodcut, all are heliogravures.

THE Brussels academy of sciences has awarded its prize of six hundred francs to Professor LÉON AUTONNE, of the University of Lyons, for his work in geometry.

PARIS ACADEMY OF SCIENCES.—Professor CHARLES MÉRAY, of Dijon, has been elected a correspondent of the section of geometry.—The nominees of the academy for the post of astronomer at the Bureau des Longitudes held by the late Professor Tisserand are MM. RADAU and BIGOURDAN.—The

Bordin prize in geometry was not awarded but M. JULES DRACH received an honorable mention. This was to have been awarded in 1898 for a successful memoir on a study of questions relative to the determination as to properties and applications of systems of orthogonal curvilinear coördinates in n variables, indicating in particular and in a manner as precise as possible the degree of generality of these systems. But one memoir, of insufficient merit, was received in 1898; accordingly the competition was held open for 1899. Three new memoirs were received by the academy, only one of which was retained. This bore the unique device "L'esprit ne peut se soumettre qu' à ce qui est esprit." The commission consisted of Professors Poincaré, Picard, Darboux, M. Lévy, and Appell.—The Francoeur prize in geometry was awarded to M. LE CORDIER, with a very honorable mention to M. LE ROY.—The Poncelet prize was given to M. COSSERAT for the whole of his contributions to geometry and mechanics.—The extraordinary prize in mechanics, six thousand francs, was assigned to M. BAILLES for his treatise on the geometry of indicator diagrams, supplementary prizes being given to MM. CHARBONNIER, GALY-ACHÉ and PERRIN.—The Montyon prize was awarded to M. PARTIOT, the PLUMEY prize to M. BONJOUR, and the Fourneyron prize to M. A. RATEAU.—The Lalande prize in astronomy was awarded to Mr. W. R. BROOKS for his discoveries in connection with comets, and the Valz prize to M. NYRÉN, of Pulkowa, for his work in sidereal astronomy.—Of the general prizes, the Arago medal was conferred on Professor G. G. STOKES on the occasion of his jubilee at Cambridge, and the Petit D'Ormoy prize in mathematics was given to M. MOUTARD.—In 1900 the subject for the Grand Prize in the mathematical sciences is "to perfect in some important point the investigation of the number of classes of quadratic forms having integral coefficients in two indeterminates." The subject of the Bordin prize for the same year is "to develop and perfect the theory of surfaces applicable on a surface of revolution." The Francoeur, Poncelet, Montyon, Lalande, and Valz prizes will be awarded the same year. The *Comptes rendus* for December 18, 1899, contains full details relative to all the prizes of the academy.

UNIVERSITY OF CALIFORNIA.—The following courses in mathematics are announced for the second term of the present session:—By Professor M. W. HASKELL: Selected topics in higher mathematics, two hours; Theory of algebraic forms, three hours; Spherical harmonics, two hours; Seminar, three hours.—By Professor G. C. EDWARDS: Differential equations, three hours.—By Mr. A. B. PIERCE: Analytical projective geometry, three hours.—By Dr. E. J. WILCZYNSKI: The hypergeometric function, two hours; Calculus of variations, one hour.—By Mr. A. W. WHITNEY: Theory of functions of a complex variable, three hours.—By Mr. N. L. PERRY: Determinants and theory of equations, two hours.

THE department of mathematics of the University of Pennsylvania announces a course of twelve lectures on space analysis to be given by Dr. ALEXANDER MACFARLANE during the month of February, 1900.

PHYSICS was the most favored subject, and mixed mathematics rather more popular than pure mathematics, with the successful candidates for the B.Sc. degree at the recent examinations of London University.

THE public discussion of the proposal to remodel the mathematical tripos at Cambridge University (see BULLETIN, 2d series, vol. 6, No. 4, January, 1900, p. 168) was, on the whole, favorable to the scheme; though the senior wrangler naturally found more than one advocate for his continued existence. Generally speaking, the younger mathematicians, many of whom had themselves headed the tripos, were urgent that the present system is prejudicial to sound mathematical learning, and, in particular, to mathematical research. The older members of the university and certain of the private tutors, while more or less willingly admitting that the system produced imperfect results, denied that the abolition of the order of merit was the proper remedy. They would prefer to see older and more experienced examiners appointed, together with some rearrangement of the marks assigned to the harder questions, without indicating precisely how these suggestions would meet the difficulties of the case. The question is exciting keen discussion in private, inasmuch as the title of senior wrangler is interwoven with the texture of Cambridge tradition. As regards titles it has been suggested that the first Smith's prizeman might bear the title of first wrangler; this would have the advantage of finality, since there is no further examination to displace the candidate. The question comes up for decision in the next Cambridge term.

THE Council of the Royal astronomical society has awarded the society's gold medal of the year 1899 to Professor H. POINCARÉ for his researches in celestial mechanics.

PROFESSORS G. DARBOUX and H. MOISSAN are the delegates of the Paris academy of sciences to the approaching celebration of the second centenary of the Berlin academy of sciences.

Professor G. G. Stokes, of Cambridge University, has been elected a foreign associate of the section of mathematics and physics of the Brussels academy of sciences.

Professor W. Burnside has been made an honorary fellow of Pembroke College, of Cambridge University.

Mr. B. H. Crenshaw has been promoted to an assistant professorship of mathematics at the Alabama Polytechnic Institute.

Mr. N. L. Perry has been appointed instructor in mathematics at the University of California. Dr. J. V. Westfall and Mr. August von Ende have been made instructors in mathematics at the University of Iowa. Mr. F. E. Ross, graduate student of mathematics at the University of California, has been appointed instructor in mathematics at the University of Nebraska.

Dr. John Eugene Davies, professor of mathematical physics in the University of Wisconsin, died at Chicago, January 23, 1900.

The deaths are announced of Mr. Francis Guthrie, professor of mathematics in South African College, aged sixty-eight years, and of Judge J. B. Stallo, of Cincinnati, Ohio, a well known patron of philosophy and mathematics, aged seventy-six years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALASIA (G.). Geometria e trigonometria della sfera. Milano, Hoepli, 1900. 16mo. 6 + 207 pp. (Manuali Hoepli.)

BOLYAI. Wolfgangi Bolyai de Bolya Tentamen iuventutem studiosam in elementa matheseos purae elementaris ac sublimioris methodo intuitiva evidentiaque huic propria introducendi, cum appendice triplici. Editio secunda. Tomus I: Conspectus arithmeticae generalis. Mandato academiae scientiarum hungaricae suis adnotationibus adiectis ediderunt Iulius König et Mauritius Réthy, academiae scientiarum hungaricae sodales. Leipzig, Teubner, 1899. 4to. 12 + 679 pp., 11 plates, 1 portrait. Half leather. M. 40.00

BONNEL (J. F.). Les atomes et hypothèses dans la géométrie. 3e édition, revue et augmentée. Paris, Hermann, 1899. 8vo. 200 pp. Fr. 5.00

BOYER (J.). Histoire des mathématiques. Paris, Carré et Naud, 1899. 8vo. 226 pp., boards. (Bibliothèque de la *Revue générale des sciences.*) Fr. 5.00

FATTOR (L.). Sulle serie del Fibonacci; nota. Venezia, Visentini, 1899. 8vo. 10 pp.

FORMULAIRE de mathématiques publié par la *Revue de mathématiques* (*Rivista di matematica*). Directeur: G. Peano. Tome II, No. 3: Logique mathématique; arithmétique; limites; nombres complexes; vecteurs; dérivées; intégrales. Torino, Bocca, 1899. 8vo. 200 pp.

JUNG (H.). Ueber die kleinste Kugel, die eine räumliche Figur einschliesst. (Diss.) Marburg, 1899. 8vo. 26 pp.

KIEPERT (L.). Grundriss der Differential- und Integral-Rechnung. Teil II: Integral-Rechnung. 7te Auflage des gleichnamigen Leitfadens von weiland M. Stegemann. Hannover, Helwing, 1899. 8vo. 20 + 617 pp. M. 11.50

KLUSSMANN (R.). Systematisches Verzeichnis der Abhandlungen, welche in den Schulschriften sämtlicher an dem Programmtausche teilnehmenden Lehranstalten erschienen sind. Nebst zwei Registern. Vol. III: 1891–1895. Leipzig, Teubner, 1899. 8vo. 8 + 342 pp. M. 5.00

KÖNIG (J.). See BOLYAI.

LEGENDRE (A. M.). Théorie des nombres. Réimpression fac-simile de la troisième édition (1830). 2 volumes. Paris, Hermann, 1899. 4to. 800 pp. Fr. 40.00

MILLER (G. A.). On several classes of simple groups. (*Proceedings of the London Mathematical Society*, Vol. 31, No. 688.) 8vo. Pp. 148–150.

PEANO (G.). See FORMULAIRE.

PRZEWALSKI (I.). Analytic geometry, with a collection of problems. 4th edition. Moscow, 1899. 8vo. 310 pp. (Russian.) M. 7.00

RÉTHY (M.). See BOLYAI.

RICHARD (J.). Leçons sur les méthodes de la géométrie moderne. Paris, Hermann, 1899. 8vo. 210 pp. Fr. 6.00

STEGEMANN (M.). See KIEPERT (L.).

VECCHI (M.). Sulla funzione ζ di Riemann; gli zeri della funzione ζ. Parigi, Hermann, 1899. Fr. 1.50

II. ELEMENTARY MATHEMATICS.

ANTOLINI (G.). Nozioni elementari di algebra per le scuole tecniche per la prima classe delle scuole normali conforme agli ultimi programmi ministeriali. Empoli, Traversari, 1899. 16mo. 57 pp. Fr. 0.90

ARBUZOV (V.), MININ (A.), MININ (V.), and NAZAROV (D.). Systematic collection of problems in arithmetic, for use in gymnasia, seminaries, etc. 8th edition. Moscow, 1899. 8vo. 244 pp. (Russian.) R. 0.75

BAUR (L.). Sammlung arithmetischer Aufgaben. Durch Erläuterungen, Andeutungen und vollständig ausgeführte Auflösungen eingerichtet für Lehrer und Lehramts-Kandidaten (Reallehrer, Kollaboratoren, Volksschullehrer etc.), sowie zum Selbststudium. Horb, Christian, 1899. 8vo. 8 + 211 pp. Cloth. M. 3.00

BENOIT (P.). Essai d'une géométrie nouvelle. Saint-Dié, Weick [1899]. 8vo. 153 pp.

BERTRAND (J.) et GARCET (H.). Traité d'algèbre. Première partie, à l'usage des classes de mathématiques élémentaires. 17e édition. Paris, Hachette, 1899. 8vo. 4 + 330 pp. Fr. 5.00

BLUMBERG (I.). Complementary algebra, preceded by a paper on the application of algebra to geometry. 6th edition, revised. St. Petersburg, 1899. 8vo. 148 pp. (Russian.) R. 1.00

CHEVREL. SEE FITZ-PATRICK.

EICHHORN (W.). Arithmetisches Regelheft nebst Wiederholungstafeln. Mit einer Begleitschrift als Vorwort und Gebrauchsanweisung. Als Ergänzung zu einem jeden Lehrbuche der Arithmetik zusammengestellt. In 4 Heften. (1. Quarta (Quinta): Rechnen als Vorstufe der Arithmetik. 40 pp. M. 0.40.—2. Untertertia: Grundrechnungsarten mit allgemeinen Zahlen; Gleichungen. 32 pp. M. 0.40—3. Obertertia: Proportionen, Potenzen, Wurzeln, Gleichungen. 42 pp. M. 0.40.—4. Untersekunda: Logarithmen, Reihen, Zinseszins- und Rentenrechnung. 23 pp. M. 0.30.) Leipzig, Teubner, 1899. 8vo.

ESTEL (V.). See SÄRCHINGER (E.).

F. (F.). Solutions des exercices du Cours élémentaire de géométrie. Paris, Poussielgue [1899.] 16mo. 81 pp. (Collection d'ouvrages classiques rédigés en cours gradués, conformément aux programmes officiels.)

FITZ-PATRICK ET CHEVREL. Exercices d'arithmétique, énoncés et solutions, avec une préface de J. Tannery. Deuxième édition, augmentée des énoncés de 800 problèmes d'arithmétique théorique et commerciale, d'un résumé et exercices d'arithmétique commerciale. Paris, Hermann, 1899. 8vo. 690 pp. Fr. 10.00

FURST (S. W.). Mensuration, with special application of the prismoidal formula. Harrisburg, Pa., Myers, 1899. 16mo. 72 pp. Cloth. $0.50

GAROET (H.). See BERTRAND (J.).

GLÖSER (M.). Grundzüge der allgemeinen Arithmetik für die dritte Classe der österreichischen Realschulen. 4te Auflage. Wien, Pichler, 1899. 8vo. 3 + 116 pp. Cloth. M. 1.30

——. Lehrbuch der Arithmetik für die erste und zweite Classe der österreichischen Realschulen. 4te Auflage. Wien, Pichler, 1899. 8vo. 5 + 209 pp. Cloth. M. 1.80

KLAS (A.). Die Dreiteilung und Fünfteilung des Winkels auf dem Wege der elementaren Geometrie, allein mit Lineal und Zirkel gelöst und dargelegt. Wiesbaden, Ferger, 1899. 4to. 14 pp., 8 plates. M. 1.20

LAZZARI (G.). Manuale di trigonometria piana. Livorno, Giusti, 1899. 16mo. 4 + 124 pp. (Biblioteca degli studenti; riassunti per tutte le materie d'esame nei licei, ginnasi, istituti tecnici, ecc.) Fr. 1.00

LOEWENSTERN. Answers to the questions in theoretical arithmetic set at the entrance examinations of the Polytechnic School. Moscow, 1899. 8vo. 46 pp. (Russian.) R. 0.75

MILNE (W. J.). Key to Milne's Plane and solid geometry. New York, American Book Co. [1899.] 12mo. 313 pp. Half leather. $1.25

MININ. See ARBUZOV (V.).

MOLA (P.). Appunti di trigonometria. Torino, Prete, 1899. 16mo. 33 pp., 2 plates. (Collana scientifica.)

MURRAY (D. A.). Plane trigonometry; for colleges and secondary schools. New York, Longmans, Green & Co., 1899. 12mo. 13 + 206 pp. Cloth. $0.90

NAZAROV (D.). See ARBUZOV (V.).

NIKULTSEV (P.). Course in arithmetic for secondary schools. 5th edition. Moscow, 1900. 8vo. 277 pp. (Russian.) R. 1.00

PINCHERLE (S.). Geometria metrica e trigonometria. 5a edizione. Milano, Hoepli, 1899. 16mo. 4 + 158 pp. (Manuali Hoepli.)

REINBECK. Die planimetrische Aufgabe für Quarta und Untertertia des Realgymnasiums. Uelzen, 1899. 8vo. 12 + 80 pp. M. 2.00

ROBINSON. New higher arithmetic, for high-schools, academies, and mercantile colleges. New York, American Book Company [1899]. 8vo. 527 pp. Half leather. $1.00

SÄRCHINGER (E.) und ESTEL (V.). Resultate zur zweiten Auflage der Aufgabensammlung für den Rechenunterricht in den Unterklassen der Gymnasien. 2tes Heft: Quinta. 58 pp. M. 1.50.—3tes Heft: Quarta. 26 pp. M. 0.70. Leipzig, Teubner, 1899. 8vo. Boards.

SASSENFELD (I.). Die Hauptsätze der Elementar-Mathematik für das Gymnasium. Nach den neuen Lehrplänen bearbeitet. In 6 Lehrgängen. (1. Planimetrie. (Quarta.) 20 pp. 2. Planimetrie. (Untertertia.) 20 pp. 3. Planimetrie. (Obertertia.) 20 pp. 4.

Planimetrie; Einleitung in die Trigonometrie und Stereometrie; Anhang: Die Logarithmen. (Untersekunda.) 24 pp. 5 and 6. Abschluss der Planimetrie, Trigonometrie und Stereometrie; Einleitung in die analytische Geometrie der Ebene. (Obersekunda und Prima.) 64 pp.) Trier, Löwenberg, 1899. 8vo. M. 3.50

——. Lehr- und Uebungsbuch der Arithmetik und Algebra für das Gymnasium. Trier, Löwenberg, 1899. 8vo. 7 + 111 pp. M. 1.80

SHMULEVICH (P.). Collection of 1220 problems in algebra and arithmetic, with complete solutions. 2d edition. St. Petersburg, 1900. 8vo. 466 pp. (Russian.)

TANNERY (J.). See FITZ-PATRICK.

TESTI (G. M.). Corso di matematiche ad uso delle scuole secondarie e più specialmente degli istituti tecnici. Vol. II: Algebra elementare. 2a edizione, intieramente rifatta. Livorno, Giusti, 1900. 8vo. 399 pp. Fr. 2.80

——. Elementi di matematica ad uso degli alunni delle scuole normali. Fasc. 1: Nozioni di algebra, radici quadrate e cubiche. Per la prima classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 8 + 88 pp. Fr. 0.80

——. Fasc. 2: Geometria piana; generalità, eguaglianza, equivalenza e misura. Per la prima classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 7 + 78 pp. Fr. 0.80

——. Fasc. 3: Aritmetica razionale; norme didattiche per l'insegnamento dell' aritmetica nelle scuole elementari inferiori. Per la seconda classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 7 + 105 pp. Fr. 0.90

——. Fasc. 4: Proporzionalità o similitudine; norme per l'insegnamento della geometria piana nelle scuole elementari. Per la seconda classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 60 pp. Fr. 0.60

——. Fasc. 5: Proporzionalità e problemi relativi; norme didattiche per l'insegnamento dell' aritmetica nelle scuole elementari superiori. Per la terza classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 7 + 76 pp. Fr. 0.70

——. Fasc. 6: Geometria solida; norme didattiche per l'insegnamento della geometria solida nelle scuole elementari. Per la terza classe normale. 3a edizione. Livorno, Giusti, 1899. 16mo. 99 pp. Fr. 0 90

WARREN (I.). Elements of plane trigonometry; for schools and universities. 7th edition. London, Simpkin, 1899. 12mo. 194 pp. 3s. 6d.

WROBLEWSKI (W.). Solutions of the collection of exercises in F. Bychkov's Course of elementary algebra. Part IV: Quadratic and cubic equations. St. Petersburg, 1900. 8vo. 286 pp. (Russian.) R. 2.00

III. APPLIED MATHEMATICS.

AIMÉ (G.). See DUHEM (P.).

BARR (J. H.). Kinematics of machinery; a brief treatise on constrained motions of machine elements. New York, Wiley, 1899. 8vo. 5+247 pp. Cloth. $2.50

BILFINGER (G.). Untersuchungen über die Zeitrechnung der alten Germanen. Teil I: Das altnordische Jahr. Stuttgart, 1899. 8vo. M. 2.50

DUHEM (P.). La tension de dissociation avant H. Saint-Claire Deville.—De l'influence de la pression sur les actions chimiques par G. Aimé (1837), avec une introduction par P. Duhem. Paris, Hermann, 1899. 8vo. Fr. 2.00

EWERS (P.). Zur Mechanik der Kanal- und Kathodenstrahlen. (Diss.) München, 1899. 8vo. 52 pp.

FALLER (O.). Eine neue Anschauung über die Reibung. Vorläufige Mitteilung. Vortrag. München, Ackermann, 1899. 8vo. 16 pp. M. 0.40

GERHARDT (F.). Ueber den Gang des Dampfdrucks in den wärmsten Tagesstunden. (Diss.) Berlin, 1899. 8vo. 35 pp.

GRAWINKEL (C.). See THOMPSON (S. P.).

HUMMEL (L.). See REBBER (W.).

LEHFELD (R. A.). Text-book of physical chemistry. London, Arnold, 1899. 12mo. 320 pp. 7s. 6d.

MAFFIOTTI (G. B.). Il planimetro a scure di H. Prytz; teoria e pratica. Torino, Paravia, 1899. 8vo. 31 pp. Fr. 1.60

NOWAK (A.). Beispiele aus der Festigkeitslehre. 2te Auflage. Mittweida, 1899. 8vo. 38 pp., 1 plate. Boards. M. 1.50

PIGNATARI (G.). Piani ed elissi centrali nei sistemi di forma invariabile. Napoli, Priore, 1899. 8vo. 11 pp.

REBBER (W.). Die Festigkeitslehre und ihre Anwendung auf den Maschinenbau. Elementar behandelt zum Gebrauche für Studierende und in der Praxis. 4te Auflage, herausgegeben von L. Hummel. Mittweida, Polytechnische Buchhandlung, 1899. 8vo. 8 + 584 pp. Cloth. M. 12.00

RITTER (A.). Lehrbuch der technischen Mechanik. 8te Auflage. Leipzig, Baumgärtner, 1899. 8vo. 15 + 801 pp. M. 20.0

SAILER (E.). Die Aufgaben aus der darstellenden Geometrie, welche bei der Prüfung für das Lehramt der Mathematik und Physik an den königl. bayerischen humanistischen und technischen Unterrichts-Anstalten in den Jahren 1873 bis 1893 gestellt wurden. München, Ackermann, 1899. 8vo. 75 pp. M. 2.00

SCHLOTKE (J.). Lehrbuch der darstellenden Geometrie. Teil 2: Schatten- und Beleuchtungslehre. 2te Auflage. Dresden, Kühtmann, 1899. 8vo. 4 + 60 pp. M. 2.00

SHULGIN (G.). Nautical astronomy. St. Petersburg, 1899. 8vo 520 pp., 2 maps. (Russian.)

STRECKER (K.). See THOMPSON (S. P.).

THOMPSON (S. P.). Die dynamoelektrischen Maschinen. 6te Auflage. Nach. C. Grawinkel's Uebersetzung neu bearbeitet von K. Strecker und F. Vesper. Teil I. Halle, Knapp, 1899. 8vo. 11 + 374 pp., 12 plates. M. 12.00

VESPER (F.). See THOMPSON (S. P.).

Cook No . ...
L RARY.

MATHEMATICAL INSTRUCTION IN FRANCE.

BY PROFESSOR JAMES PIERPONT.

Introduction.

The following paper falls into two parts. In the first part, I propose to give an account of the way France is educating those of its youth who wish to become mathematicians, to indicate rapidly what positions a young man of talent may hope to reach, how he attains them, and what his duties are in the various stages of his career. In the second part, I wish to call attention to the advantages which Americans can enjoy in studying mathematics in France, particularly in Paris.

An article of this nature seemed to me to be not only of general interest, but also of considerable importance, and for this reason. We American professors discharge a double office: we are not only professors, in that we investigate and teach; but we are also administrative officers. The direction and form that our secondary and higher instruction takes depends in a large measure upon us. It is therefore advisable not only to know what England and Germany have to teach us on this subject, but also to inform ourselves of the system of mathematical instruction current in France. At the present moment our graduate schools are in a period of formation. What is plastic and easy to mould now will fifty years hence be difficult or quite impossible to change. It is patent to every one that our new university life is being moulded largely after German ideals. Ninety-nine per cent. of our younger men have received their inspiration in Germany; it is only natural that we should have German methods before our eyes in shaping the course our graduate instruction shall take. It is at first sight indeed remarkable how ignorant we are of the scheme of mathematical instruction that prevails in France. The land that has produced Cauchy, Poncelet, Laplace, Monge, Hermite is quite ignored by us. How many of us know anything of the famous Classe de mathématiques spéciales, or of the rôle the École Normale plays in forming the elite of French mathematicians. It is a question in my mind whether it is wise for us to imitate so freely German methods, and be so largely dominated by the German way of looking at things. America is not a New Germany. To counteract this excessive German influence, it seems desirable that we should have among us a respectable minority who have spent considerable time under French influences. It is well, so it seems to me, that a part of our gifted youth should have received their inspiration from men like Poincaré and Picard, Darboux and Koenigs, Painlevé, Borel and Hadamard.

The present moment is very auspicious for Americans who wish to study mathematics in France. The French are beginning to be very much awake to the desirability of not remaining longer isolated in this particular. They are beginning to realize that the thousands of Americans who study in Germany, who are there hospitably received, and after one or more happy and prosperous years return to America, become centers of propaganda for German ideas and help to establish between the two countries relations of no mean value, for which France has no equivalent. Certain it is that the American student of mathematics will be well received and given most precious opportunities of study. But of this more in detail in the second part of this paper.

I.

General Remarks on French Instruction.

I begin by observing that practically all instruction given in France is given by the State; for one and the same grade it is therefore sensibly uniform. Remark also that France has made gigantic efforts in the last thirty years to improve instruction in every way. The State and the larger cities have spent prodigious sums to this end. New buildings have been erected in multitude, chairs endowed, laboratories equipped, libraries provided, and salaries improved. The number of scholarships offered to encourage effort, particularly to enable young men of exceptional talent but slender means to go on with their studies, indicates strikingly how eager the French Republic is to avail itself of talent wherever found. So far as we are concerned, the instruction falls under two heads; secondary, given in the lycées; superior, given in the Sorbonne, the Collège de France, the École Normale, and the École Polytechnique. Consider first the lycées. As the sexes are divided from the start we can consider here only those for boys.

ELEMENTARY MATHEMATICAL INSTRUCTION. THE LYCÉES.

The lycées are large schools scattered throughout all the cities of France. They take the children of parents in comfortable circumstances as early as five years of age, and lead them to the *baccalauréat* when they are generally 17 or 18 years old. The *élèves* are of three kinds: 1° *externes;* who like the children of our grammar and high schools live at home and go to school only to recite; 2° *internes* or *pensionnaires,* who live entirely at the lycée, as in our great preparatory schools, Exeter and Andover; 3° *demi-pensionnaires,* who pass the whole day at the lycée and the night at home. As internes children are not received before 6 years of age. Many families do not send their boys to a lycée till they are 10 or 12 years old, preferring to educate them more privately till then. The expenses are fixed by the State and vary from one city to another. For example, in Lyons, Bordeaux, Marseilles I find:

for externes,	from 70 to 300 francs per year;
for demi-pensionnaires,	from 375 to 625 francs per year;
for pensionnaires,	from 650 to 950 francs per year.

Here the first sum in each case corresponds to the lowest class and the last sum to the highest class. In Paris the prices are a little higher, in small towns a little less. The boys wear a peculiar dress or uniform, whose cut is partly regulated by the State. Until about the age of 11 the instruction is the same for all élèves. At this point a bifurcation takes place, into *enseignement classique* and *enseignement moderne.* The latter is without Latin or Greek. As the number of careers open to youths who have had Latin and Greek is much greater, about 80 per cent. of the parents in their uncertainty what talents their boys are destined to develop choose the classical course. It is well, however, to remark that today in France neither Latin nor Greek is necessary to obtain a doctor's degree in mathematics or the natural sciences. For our educators in America this is surely a significant fact. Harvard is, I believe, the only prominent American university which takes a stand approaching this. It is to be regretted that the exigencies of the French scheme of instruction force a choice when the children are at so tender an age. A delay of a few years would permit many an uncertain parent to decide for the modern course.

I wish to give now some details on the instruction given at the lycées. Perhaps the most flagrant source of weakness in our American system of higher education lies in our preparatory and high schools. We mathematicians have only to think of the entrance papers in mathematics to see the truth of this. Certainly at Yale we lose a good part of the Freshmen year training our men into form. It is therefore important for us to compare the training French boys get in mathematics with that which our boys receive.

In the first place I remark that after the age of 7 or 8 French boys are taught by men. These men have all studied at the university and have passed the *examen de licence.* For the higher classes the instructors have passed the extremely severe *examen d'agrégation* in the subjects they propose to teach. It is too humiliating for us to inquire here what our teachers of like grades have done to prepare themselves for their duties. That precious years are often lost to our youth by their inferior instruction is apparent to all. Add at once, however, that a brighter future is in store for us. As the country grows older our educators must see that men teachers take many of the places now occupied by women. On the other hand, the number of students at our universities who are following graduate courses is so large that the greater part of these must be content with positions in high schools and the better preparatory institutions. It is thus to be expected that the quality of the candidates for admission to our freshman classes will improve steadily.

In the limits of this article it is impossible to follow every turn of the instructions at the lycées. For this reason, I can speak only of the classical course. Aside from lacking Latin and Greek, the modern course differs from it but little. The bifurcation begins at the sixth class; the first class is called the *classe de rhétorique.* Here is a program for the week:

Class.	Age.	French, Latin.	French, Latin, Greek.	History, Geography.	History.	Geography.	Modern Language.	Zoölogy.	Geology, Botany.	Arithmetic.	Geometry.	Geometry, Algebra.	Arith., Alg., Geom.	Total.
VI	11	13	...	2	...	....	3	1	...	1	...	...	....	20
V	12	...	13	2	...	....	3	...	1	1	...	...	....	20
IV	13	...	13	...	2	1	2	...	...	...	2	...	....	20
III	14	...	12	...	2	1	2	...	...	...	...	...	3	20
II	15	...	12	...	2	1	2	...	...	...	...	3	....	20
Rhet.	16	...	12	...	2	1–2	3	...	...	...	...	...	2–1	20

In addition to these subjects, the élève has from 1½ to 2 hours per week in drawing, starting with Class VI. The first feature that strikes us in this scheme is the number of hours the French boy has in class work. Next we observe that he has 15 to 16 hours of class work a week in languages and only 1, 2, or 3 in mathematics. Also we note that his studies in arithmetic, algebra, and geometry are constantly intermingled, and not kept apart as with us. To us it is rather surprising that so little time is given to mathematics. French parents complain that their boys forget one week what they learned the week before. We remark also that their native tongue is from the start placed on an equality with the classical languages. The modern language is usually German, since this is required at the entrance examinations of many of the government schools, in particular the École Polytechnique.

Let us see what the French boy learns in the classe de rhétorique, at which time he is 16 years old.

Latin. Selections from Cicero, Livy, Seneca, Tacitus, Horace.

Greek. Selections from Thucydides, Xenophon, Plato, Demosthenes, Homer, Aristotle, Aristophanes.

French. Selections from authors ranging from the time of Montaigne to Hugo, a study of the principal features of the literature at each period, the theatre, the moralists, the eloquence of the church, letters and memoirs, the philosophers and savants, etc.

German. Selections from Lessing, Goethe, and Schiller.

History. History of Europe, 1610–1789.

Geography. Detailed study of France from the geological, economic, and sociological standpoints.

Arithmetic. Properties of integers, fractions, decimals, square roots, etc.

Algebra. Simultaneous linear equations, quadratic equations.

Geometry. Solid geometry.

Astronomy. Elements.

I wish to note here that arithmetic is taught as a science, not as an art. I do not believe our boys ever have the elementary processes of arithmetic: long division, extraction of roots, etc., demonstrated to them till they reach the graduate school of the university. A peculiarity of great interest is this: mathematics is taught by lectures from Class III., *i. e.*, when the élève is only 14 years old, onward. Each lycée has a library which the élèves are encouraged to consult. But it is from his note books rather than from a textbook that he learns his mathematics. To us this seems remarkable indeed. Even in most of our colleges, to say nothing of high schools, the student learns his lessons from his book. To my mind, our custom of teaching mathematics from textbooks to boys of 18 years or more is pernicious.

It is ridiculous to say that our students are too immature to take proper notes at this age. I have inspected a number of note books of boys much younger than this in Paris lycées, and was pleased to find them neatly and fully taken. In this respect Harvard is probably in advance of other universities.

We have now reached a new period in the school life of a French boy. The classe de rhétorique finished, he passes the first part of his baccalauréat. A peculiar feature of these examinations is that they are not held at the lycée, nor under the charge of its professors, but at the university in whose district the particular lycée is situated. Whether the élève has the intention of attending a university is quite immaterial. The degree of baccalauréat is a State diploma. A considerable number of government positions can be occupied by no one who has not this diploma. The State sees fit to entrust the charge of so important examinations to a class of men wholly independent of the lycées, namely, university professors. This is, no doubt, a relic of the days of the first Napoleon, when the lycées formed an integral part of that monstrous institution, *l'Université de France.* The examinations are written and oral; only those who pass the written examination are admitted to the oral. In 1897, 16,625 of these examinations were held in the classical course, and 5,691 in the modern. Of the first only 6,269 were successful, of the second 2,328. We observe that less than one-half were successful in each category, a fact which shows they are no playful affairs. On returning to the lycée, to prepare for the second part of the baccalauréat, the lycéen has a choice of two courses, *classe de philosophie* and *classe de mathématiques élémentaires.* Those who have little taste for mathematics will choose the first, the rest will choose the second. We are interested only in these last. The boys who enter this class have already a pronounced taste for mathematics and the natural sciences. A considerable part of them will later try to enter the great government schools as the École Polytechnique, the École Normale, the École Centrale, École St. Cyr, École Navale, etc. For all these the requirements in mathematics are severe. We shall not be surprised then to see, in the scheme of studies, mathematics spring from two or three hours a week to ten, while Latin and Greek disappear altogether and modern languages have one hour. Here is a scheme of studies:

Mathematics, 10 hours per week; physics and chemistry, 6 hours; anatomy and physiology of animals and plants, 1 hour; philosophy, 2 hours; modern history, 3 hours; French or German, 1 hour.

The total is 23 hours of class-room work for a boy of 17. When we recall that our young men in college from 18 to 22 years of age have but 15 hours, we are compelled to reflect which system errs; is one in excess, or the other in default? After visiting leading lycées in Paris, I can testify that the élèves seemed as buoyant in spirit and healthy in body as our own. The excess, if it be excess, seems to make them only more boisterous in their play hours.

I mention for lack of space only what is taught in mathematics: arithmetic, algebra, plane and solid geometry, conic sections by elementary geometry, plane trigonometry, descriptive geometry, mechanics, theory of simple mechanisms, cosmography. We observe that arithmetic is still taught. The reason for this is that, like geometry, it is taught as a science and not as a compendium of rules, to be learned by heart and practised mechanically. The presence of descriptive geometry is explained by the fact that the government schools just alluded to, being technical, require it for entrance.

At the end of this year, the second part of the baccalauréat is taken. If successful, the élève receives a diploma from the minister of public instruction called the *baccalauréat de l'enseignement secondaire classique.* Those who have followed the modern course receive a similar diploma with the word *classique* replaced by *moderne.* Élèves of the enseignement moderne after passing the first part of the baccalauréat enter this class of mathématiques élémentaires which I have just described, if they propose to enter the government schools mentioned above.

We have reached now a point where another peculiarity of French instruction meets us. Our young laureat who has a taste for mathematical studies does not now enter the university, as he has a right to and as his German or American comrade does. Instead, he returns to his old lycée or enters a large Paris lycée, like St. Louis or Louis-le-Grand, which pays particular attention to fitting its pupils to succeed in the highly competitive examination of the great and celebrated government schools. For us only the École Normale and the École Polytechnique are of interest. The difficulty of entering these is shown by the following figures. In 1897, 1,023 candidates presented themselves at the École Polytechnique; 224, or 22 per cent., were admittd. At the École Normale the number of candidates in science was 260; of these 13, or 5 per cent., were received. The number that will be received is fixed in advance and is limited by the capacity of the school. A student usually tries twice or thrice, before he succeeds or concludes that he had better give it up. For the most persevering, however, there exists a limit. At the École Polytechnique no élève who is 21 years or more of age will be received; at the École Normale, the age limit is 24 years. What these schools are and what makes them so tempting I shall state later. Return now to our candidate. He re-enters, as stated, the lycée, passing into the classe *supérieure* de mathématiques élémentaires. This class prepares him for the arduous work of the classe de mathématiques spéciales. His studies in this preparatory class are largely the same as in the classe de mathématiques élémentaires, only more advanced. They are: arithmetic, plane and solid geometry, descriptive geometry, algebra, plane analytic geometry, physics and chemistry. In passing, observe that from this point on to the agrégation or doctorate the studies of the French boy pursuing mathematics are in the highest degree technical and special. History, literature, philosophy, logic, psychology, in short the humanities have now disappeared from his program of studies never to reappear except 4 hours of literature in the following class and enough German to pass the entrance examinations at the École Normale or Polytechnique.

Pass now to the famous *classe de mathématiques spéciales.* The amount of instruction in mathematics given in this class is I believe without parallel in any country in the world. When I looked over for the first time the field covered by these boys of 18 or 19, I was amazed. It did not seem possible to me that they could survive the treatment. And yet they show no signs of fatigue, as far as I could see. Here is the scheme of their studies: mathematics, 12 hours per week; physics, 4 hours; chemistry, 2 hours; literature, 4 hours; German, 2 hours; design, 4 hours; épure, 4 hours. The course is fitted exactly for the requirements of admission at the Écoles Normale and Polytechnique. The topics examined on at both these schools are the same, but the examination at the École Normale is the more searching. The following works will indicate roughly the scope of the mathematical lectures.

Niewenglowski: Cours d'algèbre, Vol. I., pp. 382; Vol. II., pp. 508. By the same author, Cours de géométrie analytique, I., p. 483; II., p. 292; III., p. 569. For trigonometry I mention the work by Vaquant (1 vol., pp. 401), and for descriptive geometry that of Jarvary (Vol. I., pp. 265; Vol. II., pp. 702). These works are a little fuller than the actual courses; but even when we have made the necessary allowance, the amount that remains seems overwhelming to us. In passing let me add that a very good idea of the courses given in the classe supérieure de mathématiques élémentaires can be formed by perusing the series of textbooks edited by Darboux, namely, the Cours complet de mathématiques. An exception is the arithmétique by Tannery. For this the Traité d'arithmétique by É. Humbert is nearer what is taught.

I have been present at exercises in these two classes at St. Louis and Louis-le-Grand, and although I came at an unexpected moment, the exhibition of dexterity in attacking difficult problems and of accurately and rapidly performing long analytical calculations is nowhere even approached, I believe, in America. I express the sincere wish that our professors may attend these exercises. They will then realize how much we have to learn from the French lycées.

After a year spent in the classe de mathématiques spéciales, those fitting themselves to follow the career of mathematics present themselves at both the École Normale and the École Polytechnique. The élite, *i. e.*, 2 to 5 per cent. of them, succeed at one or the other of these places at the first trial; the rest return to the lycée and repeat the classe de mathématiques spéciales. They go over the same ground, but much that they understood before imperfectly now becomes clearer. They present themselves at the end of this year again. In the second trial about 25 per cent. succeed. It is now a fact that less than half of those who enter the classe de mathématiques spéciales succeed even after a third trial. The rest either go to the Sorbonne as *étudiants libres* or give up their aspirations to become mathematicians and try to enter a government school where the examinations in mathematics are less severe.

This question arises: if only from 2 to 5 per cent. can succeed in the first trial and the rest have to return and go over the same ground again, why is it not more rational to divide this prodigious amount of matter crowded into one year and spread it over two? The answer seems to be this: The government is eager to pick its servants from the best of the nation's youth. This is done by these highly competitive examinations. At the same time if an élève is sufficiently gifted to succeed in one year of mathématiques spéciales, it is to his own and the government's interest to push him forward in his high career. Almost all the young mathematicians of today who are adding new luster to France's fair renown in mathematics succeeded after a single year in this remarkable class.

HIGHER MATHEMATICAL INSTRUCTION.

With this brief sketch of secondary instruction in mathematics, let us turn to the *enseignement supérieure.* This takes us to the Sorbonne, the Collège de France, the École Normale, and the École Polytechnique.

The Sorbonne.

We describe in a few words the relation between the University of Paris and the Sorbonne, a point not clear to all. Before the great upheaval of 1789 there were scattered about in France various universities whose foundation and government were similar to those of other European universities. These universities were all suppressed in 1793. Under Napoleon they were again revived; not, however, as separate corporate bodies. All the universities, the little collèges, and the lycées were united into one great body, *l' Université de France.* This reducéd the universities to an association of faculties with no local self-government worthy of the name. Within recent years the desire became ever stronger to return to the old régime; to give the universities each à separate existence, a budget, a power of local administration; they were however to remain, as elsewhere, more or less dependent upon the *ministère de l'instruction publique.* The change eventually took place in 1896. The Université de Paris to-day consists of 5 faculties; *Faculté des sciences, des lettres, de droit, de médecine,* and *de théologie.* The faculties of science and letters have their offices, lecture rooms, laboratories, etc., in the building called the Sorbonne. Here also are the university library and the central offices of university administration. In passing we remark that these two faculties constitute what is called in Germany the *philosophische Facultät.*

The Sorbonne, as it now stands, is a gigantic and magnificent structure built to replace the old Sorbonne built by Richelieu. It was begun in 1885 and is now all but complete. Very picturesquely it encloses the old church of the Sorbonne also built by Richelieu and containing his magnificent tomb.

Turn now to the mathematical department. Instruction is given by 1° *professeurs* who occupy chairs each covering a definite part of what is called in France the *sciences mathématiques;* 2° *professeurs adjoints;* 3° *chargés du cours;* 4° *maîtres de conférences.* There are no *Privat-docenten* in France. All instructors are appointed for life except in cases which are obviously exceptional or temporary, and receive salaries according to a fixed scheme of advancement. There is no competition among the universities, as in Germany which allows some professors to receive twice the salary they otherwise would or to enjoy exceptional privileges. There are nine chairs which interest us; I add their present occupants.

1° Géométrie supérieure; Darboux.
2° Analyse supérieure; Picard.
3° Calcul différentiel et intégral; Goursat.
4° Mécanique rationelle; Appell.
5° Mécanique céleste; Poincaré.
6° Astronomie; Wolf.
7° Mécanique physique et expérimentale; Koenigs.
8° Physique mathématique; Boussinesq.
9° Physique générale (2 chairs); Lippmann, Bouty.

In mathematics the only professeur adjoint is M. Puiseux, while the chargés du cours and maîtres de conférences are Raffy, Andoyer, Blutel, and Hadamard. Unlike the German universities, where instruction in these subjects is free from the taint of examinations, the universities of France are required to spend a large part of their efforts to fit their students to pass two State examinations, the licence, and the agrégation. These examinations, particularly the latter, require the student to have an elaborate knowledge of certain subjects and it is the duty of the university to see that students of a certain calibre of mind and degree of assiduity are prepared and drilled to pass them. For this reason the courses 3°, 4°, 6°, 7°, 9° are the same each year. On the contrary the courses by Darboux, Picard, Poincaré, and Boussinesq are variable. The wide range and the elegance of the courses given by Darboux, Picard, and Poincaré are known the world over through their published reproductions.

The first peculiarity that meets us is that half the courses are repeated each year almost without change. The next feature that attracts our attention is that no professor gives more than one course of lectures, *i. e.*, two lectures a week; while Darboux, Picard, Lippmann, and Bouty lecture but one semester, or less than four months a year. From this it results that the number of courses the university offers the student during an abode of two or three years is small in comparison with a large German university where each professor gives usually two or three courses a semester which often run through a cyclus of two or three years. We next remark that the maîtres de conférences do not lecture on subjects which interest them, as does the privat docent in Germany, thus swelling the number of courses a German university offers, but that they employ their efforts to drill the students and elucidate more fully the lectures of the professor. Thus during the present semester Hadamard did this for Goursat; Puiseux for Appell, and to some extent for Koenigs; and Raffy for Darboux. For the examination of l'agrégation Andoyer, Hadamard, and Blutel are engaged in this kind of work.

The Examinations.

Perhaps this is the fittest place to state what the examinations for the *licence*, *agrégation*, and *doctorat* are. A young man who wishes to follow the career of mathematics naturally aims at a professorship in a university. To attain such a position he must pass the doctorate; to do this he must previously have passed the *licence*. Now positions at the universities are relatively few, and only the most talented can hope to reach them. On the other hand, the advance in the lycées to a larger salary is more rapid at the start than at the universities, and some who could rightfully aspire to a university career, are forced from pecuniary considerations to choose the lycée. The greater part of the students thus look to the lycées as their resting place for life. But to become professor in a lycée one must have passed the *agrégation*, which presupposes the *licence*. Let us then consider these three examinations as far as they concern us.

The Licence. Up to 1896 there were three sorts of *licence*, viz.: *licence ès sciences mathématiques*, *licence ès sciences physiques*, *licence ès sciences naturelles.* Since then the program has been changed. Besides the 9 mathematical subjects we mentioned above, the *Faculté des sciences* gives instruction in 9 more, chemistry, zoölogy, etc. A certificate of *licence ès sciences* without other qualification in the title, but with mention of the subjects passed on in the text of the diploma, is now given to any one who has passed examinations on 3 of these 18 subjects. The examinations on the different subjects are taken at will; each examination successfully passed entitles the student to a *certificat* for that subject. The examinations are written and oral. The written examinations consist of two parts, an *épreuve écrite* and an *épreuve prâtique;* each lasts about four hours. The first of these two written examinations is more theoretical and abstract, the second is generally a numerical example with extensive calculations. The oral examination is held before a jury of three professors and lasts 15–20 minutes.

Doctorat ès Sciences Mathématiques. There is no title of doctor of philosophy in France. As already mentioned, the

German philosophical faculty is represented in France by the faculty of letters and the faculty of sciences. The first gives the title *docteur ès lettres;* the second faculty gives three titles, *docteur ès sciences mathématiques, docteur ès sciences physiques, docteur ès sciences naturelles.* We are, of course, interested only in the doctorate of mathematical sciences. This requires the *licence ès sciences* with mention of calcul différentiel et integral, mécanique rationelle, and a third certificate at the choice of the candidate. In addition a *thèse* is required, which must contain original discoveries or present the subject in an original manner. The size of these thèses averages 50–60 quarto pages, and their quality—so far as the University of Paris goes, and most of the mathematical doctorates are taken here—is considerably superior to that of the German dissertation.

Remark that the French desire to maintain the exclusiveness of the doctor title. For ten to fifteen years the quality of the thèses has been slowly rising, so that now only a small part of the German dissertations would be accepted at Paris. At the same time the thèses are written much more independently than is usual in Germany. It is not considered good form in Paris to consult the Professor for assistance. A student who is not independent enough to carry on his work without more than the slightest aid has very little chance of a university career.

Besides this thèse, there is a *seconde thèse* on a subject assigned by the faculty, which is generally one that the candidate has an interest in but which is not in the line of the principal thèse. There is no *rigorosum* in the German sense and no promotion at all. Everything turns on the principal thèse. If this contains sufficient original work it is admitted and permission given to publish it. Of late years no thèse has been accepted at Paris which has not been published. Although there is no rigorosum, there is still an oral examination. This is the *soutenance publique* of the two thèses. It is a very solemn affair, the examiners appearing in great state. As far as the principal thesis is concerned, it is a mere form. In regard to the second thesis it is really a mild kind of oral examination. The candidate is expected to have prepared the second subject sufficiently well to be able to answer benevolent questions.

The Agrégation. Unlike the *baccalauréat*, the *licence*, and the *doctorate*, the agrégation is a competitive examination. Each year the number of candidates who will be received is determined in advance by the minister of public instruction, in accordance with the needs of the lycées. For the sciences mathématiques it is at present about 7 or 8 each year; the number of candidates is about 100. For the doctorate in mathematics we said the licence must mention 3 *certificats;* for the agrégation it must mention 4—the extra certificate being the *physique générale.* Also the certificate at the option of the candidate must be in mathematics. Usually at Paris it is Darboux's course on higher geometry. The courses of Picard and Poincaré, being of very elevated nature, are chosen less frequently. To pass the agrégation the student usually spends at least three years, and as follows: Each of the first two years he employs in passing two of the four certificats. Having these off his hands, he turns his whole attention to preparing for the agrégation proper. This examination is the most severe that I have any knowledge of, and is worthy of some detail. Remark first that it is an examination whose object is to choose among the candidates those best fitted to teach in the lycées. The examinations thus turn largely on the subjects there taught. They consist, as almost all examinations in France do, of two parts; a written and an oral part. The first two written examinations take questions from subjects taught in the classe de mathématiques élémentaires and the classe spéciale. The two following examinations which are based on the courses heard at the university are: 1° *Composition sur l'analyse et ses applications géométriques.* This usually involves the theory of partial differential equations and differential geometry of space, 2° *Composition de mécanique rationelle.* Each of these four examinations lasts 7 hours. The papers are the same for all France. Those that reach a sufficiently high standard are called *admissible,* and these, if living in the provinces, must go to Paris, where the final competition is held. Here the examinations consist of two parts. First, two new compositions and, secondly two *leçons.* The compositions consist of the solution of a problem in descriptive geometry, and the solution of a problem in analysis requiring numerical calculation. These two last each four hours. The two leçons treat of subjects taught in the classe élémentaire and classe spéciale and take place as follows: The candidate draws by lot one question among 25. For the leçon on mathématiques élémentaires he has three hours, for the leçon on mathématiques spéciales four hours to think over how he will present the subject of the leçon. They have no books to assist them. They appear individually before the jury and give the leçons as if before a class in the lycée. The examinations last at most an hour each. Some unfortunates are cut down before they have talked ten minutes.

We note the differences between the examination of the doctorate and agrégation. The first tests exclusively the originality of the candidate, the second his ability as a teacher in a lycée. The second is competitive, because it gives the successful candidate the right to demand a place as professor in a lycée; the first is not, because it gives him no right to teach anywhere. Among the candidates who were *admissible* but not successful are chosen the chargés du cours and occupants of other inferior positions in the lycées.

The Collège de France.

This institution is independent of the university and has some 40 chairs, whose occupants are among the most distinguished savants of France. For us, only the four following chairs are of interest.

Mécanique analytique and céleste; Lévy (Hadamard, *suppléant*).

Mathématiques; Jordan.

Physique générale and mathématique; Bertrand (Deprez, *suppléant*).

Physique générale and expérimentale; Mascart.

√ The courses at the Collège de France are free to all; any one can enter a lecture and leave it when he chooses. The audiences in some courses are very mixed. Retired petty merchants, and amiable old ladies who seek a small diversion at no expense are not infrequent visitors. Many of these promptly begin to nod. They cannot well fall quite asleep, as the authorities have taken care to provide benches without backs. Even for those who have no desire to slumber, this is a serious discomfort. For the first time in my life I experienced how painful it is to sit an hour with nothing in front or behind to rest against. Fortunately for us mathematicians, our subjects are too esoteric to attract this kind of audience. The duties of the professors I have mentioned are simply to lecture. They do not come in contact with the student by directing his work or subjecting it to examination. All attendance is voluntary and no official recognition is given for it. Where these chairs are held by strong and enthusiastic men, the lectures given at the Collège de France can become sources of the highest inspiration.

The École Normale Supérieure.

The object of this school is to form the future professors of the lycées and universities, by supplementing the instruction they receive at the Sorbonne and the Collège de

France. As at the École Polytechnique, its élèves live at the school as internes. The École falls into two distinct departments, science and letters. Only the first interests us. The present year 166 candidates presented themselves for admission to the department of science; only 13 can be received yearly. Of these generally from 7 to 9 intend to study the mathematical sciences. The élèves in mathematics spend three years at the school preparing for the agrégation. The plan of study is as follows: During the first year they usually attend Goursat's course at the Sorbonne on the calculus and Lippmann's and Bouty's course on physics, taking at the end of the year the examinations for these two certificates. At the École, during this year, they attend each week a conference by Tannery and one by either Painlevé or Borel. These last each an hour and a half. They are like the conferences given at the Sorbonne, elucidating the topics the élèves have heard there and exercising them in the solution of problems. Problems are given also to the élèves to work out of class and these are then criticised by the professor before the whole class. In the second year the two remaining certificates are taken, viz. the mécanique rationelle by Appell and usually the géométrie supérieure by Darboux. For the first of these Painlevé gives conferences at the École; for the second, Raffy. During the third year the élèves attend what courses they choose at the Sorbonne or the Collège de France. At the École they are drilled in giving leçons preparatory to the agrégation: on algebra by Tannery, on analytic geometry by Borel, on analysis by Painlevé. Élèves of the second and third year also hear courses of an advanced nature given by Painlevé and Borel. Last year they were on the theory of differential equations and ensembles, subjects to which these young savants have made important contributions.

The training the élèves receive at this school is thus a happy mixture of theory and practice. To judge of its efficacy I have only to remark that of the 100 candidates in mathematics at the agrégation 7 or 8 are Normaliens. Of the 7 or 8 who succeed in this terrific competition, all but one or two are again Normaliens.

Let me add that I should do these élèves a grave injustice if I left the impression on the reader that their efforts are bent exclusively to preparing for their examinations. Many of them find time to make extensive studies and researches, often the commencement of a doctor's thèse, in regions remote from their examinations.

A few words on the life at the École. The building itself makes the impression of a fine old palace in renaissance style, with beautiful court yard, fountain, trees, flower beds. Around its walls are busts of distinguished savants and anciens élèves. Instruction and pension are gratuitous for those who afterwards teach. The life at the school must be very pleasant and profitable. Living and studying here together for three years, surrounded by a quiet and studious atmosphere rendered more effective by the distinguished past of the school, subject to the stimulating influence of a generous rivalry among themselves, and enjoying daily and intimate intercourse with their professors, these élèves, the pick of France, certainly have ideal opportunities to unfold their latent talents. The genius of the place as far as mathematics is concerned is the directeur des études scientifiques, M. Tannery. The affectionate respect and gratitude which the élèves cherish for him long after leaving the school is very touching. If we seek the two men whose influence has been most inspiringly and profoundly felt by the present younger generation of mathematicians in France we certainly would mention without hesitation Hermite and Jules Tannery. It is a pleasure to note that the privileges of this celebrated place of learning are open to American students, either in part or even as internes. In the latter case the pension would be 1200 francs a year. For a student of limited means, but of distinguished talents, this is an opportunity of great value.

The École Polytechnique.

Until recent years, this school was famous not only for the great engineers it produced but also for its brilliant mathematicians. Cauchy, Poncelet, Lamé, Liouville, Chasles, Serret, Halphen, Laguerre, Jordan, Hermite, and Poincaré, not to mention a host of other celebrated mathematicians, were all trained here. But the demands made on an engineer to-day are so great that it is impossible to give the élèves more than a glimpse of modern mathematics. The spirit of the place has changed in accordance with this fact. An élève who has mathematical tastes on entering finds them *dis*couraged rather than *en*couraged by his stay at this school. Instead then, of being a great breeding place for mathematicians, as it was during the first half of this century, it has now sunk into insignificance in this respect. I should not devote much space to this school, therefore, if it were not that it still exercises on the élite of the French youth a truly remarkable fascination, which produces this curious and as it seems to me unfortunate change in the destinies of so many of the young men we have been following. We have seen namely that the élèves of the classe de mathématiques spéciales who propose to follow a mathematical career present themselves not only at the École Normale but also at the École Polytechnique. Those who succeed at the École Normale become mathematicians, the remainder have their plans of life changed completely, for statistics show that only an insignificant per cent. of the élèves of the École Polytechnique follow the mathematical career. Observe too that since only 7 or 8 can possibly enter the École Normale, by far the larger part of the successful candidates of each year must needs enter the École Polytechnique. The result of this system, so it seems to me, is that France loses many a good mathematician and receives in lieu a relatively poor engineer or officer.

The élèves remain at the school only two years. Instruction in mathematics is given during both these years, but not continuously. During 3–3½ months each year, the élèves have 3 lectures a week each lasting 1½ hours. The professors are Jordan and Humbert. The lectures are supplemented by *répétiteurs* who are often mathematicians of high distinction. At the present moment we find here Appell, Goursat, Koenigs, and Painlevé. At the same time I must add however that the practical drill the élèves receive is inferior to that given at the Sorbonne, not to mention the École Normale where it reaches its highest excellence.

We have seen how strong the attraction is which these two Écoles exercise on the French youth; that instead of entering the university on passing the baccalauréat as is done elsewhere, they seek to enter these schools. The reason for this is not difficult to find. In France the struggle for existence is severe. The École Polytechnique offers all its graduates a career which is at once rapid, brilliant, and certain. The École Normale assures its graduates at least a professorship in a lycée and prepares its élèves for this or for a university career better and more rapidly than the university can do it.

Closing Remarks on the French System of Mathematical Instruction.

Before leaving the subject of mathematical instruction let me note the difference between the attitude of mind of the French and the German student while preparing for his career as a mathematician. The German university requires 8 semesters to prepare for the doctorate. As there are no examinations till the close, the German student is free to roam about not only in mathematics but in physics, philosophy, history, etc. In fact a part of the rigorosum requires a knowledge of philosophic studies. At the close, the examinations, which are oral, aim not so much at specific encyclopedic knowledge as a largeness of view and maturity of judgment. As we Americans have so often heard at Commers, *Frei ist der Bursch!* In France, on the other hand, from the moment the élève leaves the classe de rhétorique, *i. e.*, from 16 years onward, he has an examination before him at every turn. The plans of his studies are very largely marked out for him; care is constantly taken to see he makes no lost steps and that he is properly prepared to pass his examinations. The university and the École Normale make the impression of a great institution wonderfully arranged to turn out a certain product, in a certain amount, in a certain degree of excellence, with the least loss of time and energy. The Collège de France is the only seat of learning which reminds us of the German university, both in regard to its indifference whether the seeds scattered abroad with lavish hand bring forth fruit, and in respect to the liberty of its professors to teach what they choose and of the student to learn what he will. While making objectively these observations I wish to refrain from passing any judgment as to which scheme is the better. Indeed, can we rightfully say more than that each seems best adapted to the needs and temperament of its people? Remark finally that there is to-day a tendency in Germany in official circles to restrict the *Lern- und Lehr-Freiheit*, however one may be unwilling to confess it. On the other hand strenuous efforts are being made in France to increase the number of *cours libres* which shall be quite independent of the demands of any examination.

CAREER.

We have seen how France trains its mathematicians. The question we now ask is, what are the inducements it offers young men to follow this career? This question is one of no mean importance if we are looking for the causes which make a country eminent for the steady growth in volume and quality of its output in a science. Because, if the career is slow and arduous, if it offers slight chances of reward in comparison with other professions or with commerce, it is evident that many of the best minds will be dis-couraged from entering it. This question has also its lesson for us. If we wish American mathematics to take a place alongside that of France, Germany, and Italy, we must look to it that the conditions under which the American mathematician has to work compare favorably with those which surround his colleague abroad.

We observe in the first place that the position of a young doctor in mathematics is much better in France than in Germany. In Germany the young *Privat-docent* may be 28 or 30 before receiving an *ausserordentliche Professur*. Until this time his salary is practically nothing. In France the young doctor who is Agrégé may enter a lycée as professor in mathematics, if no place as maître de conférences in a university is available. Here the lowest salary for Paris is 5,500 francs, the highest 9,000. Promotion takes place, 1° by choice, 2° according to time in the service. From the sixth, *i. e.*, the lowest class in rank, to the third the number of those that can be advanced each year by choice is equal to that of those that are advanced by *ancienneté*. For the second and first classes, two advancements can be made by choice to one by ancienneté. Advance of those *hors classe* (a vanishing number) is exclusively by choice. The advancements are made December 31 each year, and take place so that there are always 20 per cent. in the sixth class and respectively 18, 18, 16, 14, 14 per cent. in the others. The advantage of this scheme is obvious. It leaves plenty of room to encourage unusual talent to exert itself; at the same time it does not dishearten the others from doing their duty with enthusiasm. The chargés du cours, *i. e.*, those who did not succeed in the agrégation receive from 2,800 to 4,800 francs. The number of hours a professor has to teach in a lycée varies from 12 to 14 a week. The remainder of the time is *strictly* his own.

The majority of young doctors of promise prefer not to avail themselves of the lycées, and generally it is not long that they have to wait for a position as maître de conférences at a university. At Paris the salary for such a position ranges from 5000 to 7000 francs. For this the incumbent gives two conferences a week, each lasting 1½ hours. This may well seem munificent; for he not only has abundant time to carry on his own researches but has also *ample* means to live *well*. It often happens that a young man occupies other positions, as an *examinateur*, or *répétiteur*, or *suppléant* at one of the numerous other institutions of learning at Paris.

At the age of 30 a young maître de conférences may be made professeur. This however rarely happens now. The number of chairs is not large and promotion is not rapid by any means. This is offset by the relatively large salaries and by the fact that the occupants are often active in other capacities, as just mentioned. In Paris the salary of a professor is from 12000 to 15000 francs. In the provinces it varies from 6000 to 11000. The professor is expected to give but one course of lectures, viz., the course his chair calls for. These take place twice a week and last from one to one and one-half hours. Those courses, which are repeated each year, require but little preparation and hence but little labor. On the other hand the stimulating effect which is exercised on the mind of the professor when lecturing on subjects which are interesting him is altogether wanting. The number of hours a professor has to lecture seems to us magnificently small. Even in Germany it is not strange to see an ordentlicher Professor lecture 6 hours a week and even have in addition 2 hours Seminarübungen. Let me give here for comparison the salaries German professors receive. In Berlin these are: *Professor ordinarius*, 5700 Marks and 100 Marks increase each four years, till after 24 years the maximum salary of 8100 Marks is reached; *Professor extraordinarius* 3300 Marks with additions as before till a maximum of 5700 Marks is attained.

In other Prussian universities the salaries are respectively for full professor 4000 to 6000 Marks; for assistant professor 2000 to 4000 Marks. In addition there is here a *Wohnungsgeldzuschuss*, which varies from case to case, but which may roughly be put at 500 Marks.

One feature of the life of a French university professor is certainly very disagreeable and wearisome; we mean the great number of examinations he is obliged to hold. In the last report of the *Doyen* of the Faculté des sciences at Paris, we read that its professors attended 9,946 examinations. Of these, 8,485 were those for the baccalauréat. From July 1 to August 15, and November 1–10, the time of the professor is more or less broken into. To indicate roughly the extent, I would say that on an average a professor may expect to be called on to assist at 12 examinations of baccalauréat, which consume, taking into account the time spent in marking the compositions, five hours each. Besides this, one may calculate on 20 examinations for certificates, of one hour each. In all 80 hours.

In spite of these losses, I think the French professor has much more leisure for research and publication than his German colleague. At any rate, we Americans have a lesson to learn. How many of our professors have from 10 to 12 hours a week to teach, besides faculty meetings, committee meetings, and examinations in harassing multitude! At the present moment our universities for the first two years are gymnasia or lycées, and our professors are tied down to a régime which is proper to secondary instruction. At the same time most of us are expected to be conversant with the latest phases of our science and to advance it by our own researches. This anomalous state of affairs is certainly transitional. Some way, the problem must be solved how to retain all the excellencies of our college life, which in my mind is one of the pearls of our American civilization, and at the same time create an environment for certain men of talent which is conducive to the highest scientific activity.

II.

THE AMERICAN STUDENT OF MATHEMATICS AT PARIS.

In sketching the instruction given in mathematics at the Sorbonne, the Collège de France, and the École Normale, I have indicated at the same time what the American student may get here. I wish now to give more specific information on certain points. What are exactly the advantages and attractions which might induce an American to study in Paris?

To begin with, there is Paris and the French people themselves. To live in Paris as a serious student is an education in itself. There is, in the first place, the wealth of historic associations. Paris is saturated with the spirit of bygone days which to the attentive and loving eye is manifest on every side. Its galleries, churches, palaces, public squares, the Seine, St. Cloud, Versailles, Fontainebleau cause many a page of history to flash out before the mind's eye with startling vividness. Then there is the spirit of the language to learn, and the customs and points of view of those that use it. Some insight gained in this respect, French literature takes on new meanings and presents fresh beauties. But it is impossible to go into detail here; the Théâtre-Français, the Opéra, the Salon, its fêtes and exhibitions, a hundred influences are constantly at work to refine and educate if one is so disposed.

Turn now to the mathematical side. An American student who intends to spend a year here, and since the courses run with a few exceptions through the whole year it is advisable to calculate on at least a year, will do well to come to Paris as early as possible in June. As the lectures close here from the 10th to the 15th of June, he cannot count on hearing anything at this stage; the examinations however keep the university open until well into August and many of the professors are still in town. He can thus look around and prepare for autumn. If he has no friends here, he will do well to visit the bureau of the *Comité de patronage des étudiants étrangers*, directed by the university. Here he will find some one who talks English and who will give him valuable information regarding the university, lodgings, etc. There are several clubs which he can join, and at this bureau he can get letters of introduction to them. I mention the *Club Americain*, the *Cercle des étudiants protestants*, the *Association générale des étudiants*. In these places he will be well received, will soon get acquainted and begin to feel at home.

Many students will prefer not to stay in Paris after this first orientation. The weather is very hot and life is not pleasant or healthful here in the summer months. It will be well, then, to go to a university town like Grenoble or Geneva. Here the Alps are at one's very door and the excursions they offer are without end. Generally speaking the American student, while able to read French fluently, cannot speak it or understand it well when spoken. The summer should be devoted to acquiring proficiency in this respect. At Grenoble and Geneva as well as at Paris, summer courses are given under high auspices and by professors of the university and lycées on French language, literature, art and history, supplemented by practical exercises of a most valuable kind. These should be frequented. The fees are insignificant.

The university opens November 3d; it is well to return some time before to engage lodgings. Here two courses are open to the American; either he can engage rooms in a hôtel meublé or take pension. The latter seems preferable. The habit in Germany of obtaining rooms or pension in private families who have one or two chambers to let is not usual in the quarter about the Sorbonne. Rooms can be had from 30 francs a month up. It is well, however, to count on paying from 40 to 50. Good pension, room and board, can be had for 200 francs a month. The prices, we observe, are a little higher than in Berlin. Still, on 600 or 700 dollars a year a student can live nicely in Paris, spending the summers elsewhere.

The formality of matriculation is quite as simple as in Germany. It is well to bring a passport and the diploma received at college. If one wishes to take his doctorate in mathematics here, he has an advantage not procurable at the German universities; for, as already observed, no Latin or Greek is necessary for this. In regard to the courses he will wish to take; this to be sure depends on what he has already had. It is safe to say he will want to take, at the Sorbonne the courses of Darboux, Poincaré, and Picard; at the Collège de France, those of Jordan and Hadamard; at the École Normale those of Tannery, Painlevé, and Borel. The courses at the Collège de France being altogether public, there is no formality of inscription. To attend courses at the École Normale it suffices to present oneself to M. Tannery. Besides these *cours magistraux*, the American student will derive exceptional benefit from taking part in the conferences (Seminarübungen) directed by Puiseux, Andoyer, Raffy, and Hadamard at the Sorbonne, and by Tannery, Painlevé, and Borel at the École Normale. At the start he will find himself very inferior to his French comrades, as is natural when we consider how long they have been trained in this rude exercise. But in my estimation, this training is of utmost value. It is, perhaps, the most characteristic feature of the French system and should not under any circumstances be neglected.

Besides these courses in mathematics there are numerous courses in mathematical physics and allied subjects which I have not room to mention. Finally, I must call attention to the courses on history, philosophy, literature, etc., given in the Faculté des lettres by some of France's most distinguished savants. To attend these lectures it is necessary to be inscribed in this faculté; but this is a mere form and involves no additional expense.

I come now to the last point, but also a most essential one. How is an American student to obtain the doctorate. At present there are two: the *doctorat de l'État* and the *doctorat de l'Université de Paris*. The first has always been available. the second was created in 1898 by virtue of a law passed the year before, authorizing all universities to create a university doctorate. I believe only Paris has availed itself of this privilege up to the present. Consider the requirements for the doctorat de l'État. In the first place, the American student must have a B.A. or a B.S. to offer as an equivalent of the baccalauréat. To have this accepted, special permission, which will always be granted to a worthy candidate, must be obtained from the minister of public instruction. Next he must pass the *licence* with mention of calculus, mechanics, and a third certificate at will. To pass these three examinations the American student will find it necessary to frequent with assiduity the conferences, in order to acquire the dexterity necessary to solve the searching questions of these written examinations. Finally, there are the two theses already mentioned. As observed above, the principal thesis is ordinarily worked out quite independently by French students. For American students this is not necessary. They will find the professors under whom their work falls quite as ready to talk it over with them and to offer suggestions as in Germany.

Turn now to the doctorate of the university. Here the conditions are the same, except that two instead of three certificates are required. There is no question that the ordinary American student who takes his doctorate in two years in Germany could satisfy the requirements for the doctorate of the university of Paris in the same time; and as to their value as testimonials of work accomplished, I see no difference.

In closing this article I wish to express my thanks to all the gentlemen, too numerous to mention here, who by their courteous and kind attentions assisted me in procuring the materials for it. My gratitude is due in a particular measure to M. Paul Painlevé.

PARIS, *July*, 1899.

APPLIED MATHEMATICS.

Cinématique et Mécanismes, Potentiel et Mécanique des Fluides. Cours professé à la Sorbonne. Par H. POINCARÉ. Paris, Carré et Naud, 1899. 385 pp.

THE certificates granted at the Sorbonne for proficiency in mechanics require the study of a certain programme of subjects—amongst others, those on the title page of this volume. The somewhat curious mixture is probably accounted for by the fact that the students who attend these lectures desire to obtain only a certain degree of information and that in as short a time as possible. To treat four subjects in applied mathematics, within the compass of a single volume is of course impossible, if an adequate account of each is to be given. Nevertheless, M. Poincaré has, on the whole, succeeded in setting forth the main ideas of all of them, and his treatise thoroughly studied will probably fulfil the purpose which it is intended to serve.

The first chapter on kinematics contains the usual investigations of expressions for the motion of a point. As examples, the radii of curvature of a helix and of an ellipse are found directly from the dynamical formulæ. In the next chapter we are given the general kinematics of an invariable figure sliding on a plane. The lecturer goes to the root of the matter by considering the loci of the instantaneous center on the fixed and moving planes; the motion is obtained by rolling one of these curves on the other. The former locus is called the base and the latter the roulette, and M. Poincaré justifies the latter term by the property just mentioned. The name is somewhat unfortunate, as it is usually applied to the locus of a carried point and not to the rolling curve. Several examples are given, including the well known ones of the epi- and hypocycloids. The chapter is really one on geometry and the subject is treated as it should be treated, not as a part of the science of motion with the time element present, but as a purely geometrical science of motion.

The next stage is the motion of a body about a fixed point. As with the plane motion just treated, this can also be reduced to the rolling of one curve on another; but here the fixed and moving curves lie on a sphere with the fixed point as center. Hence, many of the propositions for plane curves can be applied to this extended case. In particular, spherical epicycloidal motion is of interest.

When the solid rotates about an axis and has a translation along the axis, we obtain helicoidal or screw motion. In the general case of the motion of a solid body, it is always possible and generally necessary to divide up the motion into a translation and a rotation. But the instantaneous motion can always be considered to be a uniform "screw" and the investigation thus becomes much simplified. The motion may also be considered as made up of two simultaneous rotations about two axes which do not meet, and the varying positions of these two straight lines as the solid moves constitute the main factor in determining its motion. A short chapter on relative motion and the theorem of Coriolis concludes this portion of the subject.

Under the heading "Mécanismes" M. Poincaré treats the geometrical aspects of machines for transmitting motion. The principal problems to be solved are the transformations of continuous or alternating circular motion into continuous or alternating circular or rectilinear motion. The machines for this purpose can be generally divided into two classes—transmission by hinged bars and that by continuous contact. The geometrical problem to be solved in the latter case is this :—Knowing the motion of both parts and the form of one part, to find that of the other. Since the parts are in contact, the problem reduces to the discovery of the contour of the second part. This is obtained by finding the envelope of the successive positions of one part as seen by an observer rigidly attached to the other. It is thus merely a problem in geometry or rather in kinematics. In most cases where toothed wheels are not present the problem is quite simple and is easily solved as soon as the two motions are given. But when wheels with teeth are used we have to determine the form to be given to the teeth, and further, the form most effective for the particular machine to which they belong. M. Poincaré goes into those points with some detail. He discusses the simplest case of two wheels of different diameters and in the same plane, by means of the work in the earlier chapters, especially that on epicycloidal motion. Then follow more complicated cases, including those where the wheels are not in the same plane. Finally several pages are devoted to eccentrics and other forms of transmission.

The chapters on potential are excellent. The earlier part is perhaps a little condensed, but it contains all that is necessary. Then follow Green's theorem and many applications; and the final chapter is one on the attraction of an ellipsoid, where full use is made of the theorems tending to shorten what may be a long and troublesome piece of analysis.

In hydrostatics, after the usual preliminaries, most of the investigations are concerned with the action of liquids under gravity only. A complete solution of the barometric formula connecting the difference of height of two places with the difference of the barometric readings is given, with numerical results. M. Poincaré then deals with the metacenter of a floating body, which again gives him an opportunity to proceed mainly by geometrical methods, and of this he takes full advantage.

The chapter on hydrodynamics is rather out of proportion in its various parts. One would hardly think that the purpose of the teacher is achieved when wave motion is confined to one short article and the theory of vortices is gone into with some detail. In fact, there are 14 pages in which the general equations are briefly discussed, 7 pages on the efflux of liquids from a containing vessel, 4 pages on small movements of liquids (waves), 25 pages on vortices. Then follow 9 pages on the motion of a solid in a liquid, most of which are taken up with the problem of the mutual effect of two pulsating spheres. But if doubt is expressed as to the reasonableness of M. Poincaré's exclusion or insertion of matter, none can be entertained of the excellence of his manner of exposition. Without long circumlocutions and yet with comparatively few symbols, he gets right into the heart of the problems he is discussing: it may be that he has chosen them from this point of view. No better example of his method can be furnished than the way he reaches the action of two rectilinear vortices on one another. The reading of this will appeal strongly to the student and whet his appetite for more information. In all the hydrodynamical part—and, indeed, in the rest of the book—practically the only results of an at all advanced nature that a student needs are those deduced from Green's theorem. But these have already been dealt with in the chapters on potential. An instructor who wishes to give a short course on hydrodynamics can hardly have a better model for the details than this chapter, and he will not find it difficult to add to it the portions which are necessary in order that his hearers may obtain a general idea of the problems which arise in the subject.

Ernest W. Brown.

SHORTER NOTICES.

Annuaire pour l'An 1900, publié par le Bureau des Longitudes. Paris, Gauthier-Villars.

This handy little volume, brought out for popular and professional use, is as usual improved by the addition of new matter, the omission of portions of no special value, and the alteration of details here and there. Among the additions may be noticed tables of the right ascension of the sun at mean noon and of the right ascension, declination, and parallax of the moon, together with some auxiliary astronomical tables. The magnetic elements for the principal towns in France have been brought down to January 1, 1900. In the "Notices," M. Janssen gives his annual report of the work done at the observatory on the summit of Mont Blanc; he also contributes a note on the use of balloons for astronomical purposes. M. Lippmann describes briefly but clearly the discovery and main properties of the newly found atmospheric gases. The longest article is on the theory and construction of dynamos, and this deserves special mention. As is usual with French writers, M. Cornu begins at the beginning, by explaining what is meant by a field of force, and then gradually leads the reader up to a clear comprehension of the principles used in the actual construction of a dynamo. He promises to resume this subject in a future notice. Finally, the speeches of MM. Bassot, Poincaré, and Loewy at the unveiling of the monument to Tisserand are printed in full, M. Poincaré giving a short account of the services rendered by Tisserand in the domain of celestial mechanics.

One important change, which it is hoped may become universal, has been made—that of reckoning the day from midnight to midnight, and numbering the hours from 0 to 24. This is already adopted in some European railway time tables. The great convenience of this plan will probably induce the editors of official publications, such as the Nautical Almanacs, to adopt it at a not very distant date.

Ernest W. Brown.

Elements of Precise Surveying and Geodesy. By Mansfield Merriman. New York, John Wiley & Sons, 1899. 261 pp.

Professor Merriman's latest work will be found to be of considerable value to one who wishes to become a practising surveyor and may be said to be almost indispensable to a candidate for a post on the Coast and Geodetic Survey. In the government service operations are frequently on a much larger scale than in private practice, and the work culminates in the measurement of a meridian arc—a process requiring extreme care to obtain the accuracy demanded today.

There are eleven chapters, the titles of which will sufficiently indicate the scope of the work. They are: least squares, precise plane triangulation, base lines, leveling, astronomical work, spherical geodesy, spheroidal geodesy, geodetic coördinates and projections, geodetic triangulation, figure of the earth, tables. In these not only are the mathematical parts of the subject fully and clearly set forth, but the practical details, so necessary for successful results are so described that the reader may recognize their relative importance. On page 8, however, a remark is made which is somewhat dangerous for a beginner, namely, that observations affected with mistakes must be rejected. A "mistake" should be very clearly defined. It is true that this remark is qualified by a fuller statement on pp. 44, 45, but a warning should be inserted then and there that this rejection must never take place without very good cause. Much astronomical work has been looked on with suspicion where its authors have rejected doubtful observations. Professor Merriman recommends that negative characteristics for logarithms should be used in computations. This is largely a matter of custom; most computers prefer the use of positive characteristics, inserting —10 when there is likely to be any doubt.

One of the most interesting chapters is that on spherical geodesy, containing a short historical account of the measurement of meridian arcs and the gradual approach to our present knowledge of the form and dimensions of the earth's surface. The whole work is fully illustrated by solved problems, and the references to government reports and other works on allied subjects will enable the reader to obtain everything necessary for understanding every part of the subject.

ERNEST W. BROWN.

Premiers Principes de Géométrie Moderne. Par ERNEST DUPORCQ. Paris, Gauthier-Villars, 1899. vii + 160 pp.

THIS useful book is intended to give to students who have some acquaintance with analytic geometry, a liking for the purely geometric point of view. It is not a work on pure geometry as a self-contained science freed from arithmetic; for instance, the notion of imaginary points is sketched in the preliminary chapter from the historic algebraic standpoint. A purist (say von Staudt) would energetically oppose such statements as: "Le point * * * ne représente rien de géométrique lorsqu'il est imaginaire" (p. 9); and would proceed, at any cost, to give a geometric meaning to a point however imaginary.

The book takes, then, a middle path. The ground won by the application of algebra to geometry shall be handled geometrically whenever convenient. That it is very frequently convenient is to be shown, in a way that shall strengthen the geometric sense. For the cultivation of the geometric sense is a very important* aim of a mathematical course, toward which algebraic methods contribute little.

The ground to be covered is that of the simpler transformations, viz., homography, correlation, inversion, quadratic transformation of a plane, Lie's transformation of lines into spheres.

The body of the book (Chapters II.–V.) is devoted to homography and correlation, and covers the principal properties of the conic, of the range or pencil of conics, of conics "harmonically circumscribed," or apolar, to a given conic; of the quadric, of the range or pencil of quadrics, and of quadrics and cubic curves "harmonically circumscribed" to a quadric.

---

* According to the author's introduction, it is the principal aim; which seems too strong a statement.

The remaining transformations are handled in the last chapter (Chapter VI.) in briefer fashion; and the book ends with seventy exercises whose sources are not stated. The author has carried out his intention artistically and, given that he is addressing *intelligences d'élite*, excellently.

The actual titles of foreign journals should be given. "Ann. de Math." might be an American, a German, or an Italian periodical.

I see no reason for calling a circle discovered by La Hire "cercle de Monge" (p. 64). On the other hand the orthoptic sphere of p. 107 might fairly be called "Sphère de Monge." On this latter page the reference of the footnote should be to No. 86, not No. 80.

The "transformation par rayons vecteurs réciproques" of p. 11 is finally called inversion. It is attributed to Bellavitis (1836). Some assign it to Plücker (1831).

A smaller point of history may also be raised. An interesting theorem occurring three times in the book under different forms (pp. 116, 132, 138) is in effect as follows: that the tangents at the points where a tangent of an asteroid (hypocycloid of class 4) meets the curve again meet on the cusp circle. This theorem is attributed to Laguerre. R. A. Roberts once told me that the theorem stated in the correlative form for the lemniscate, was well known in Dublin and was there attributed to Casey. It would be worth while to ascertain whether Casey anticipated Laguerre.

F. MORLEY.

Opinions et Curiosités Touchant la Mathématique d'après les Ouvrages Français des XVI^e, XVII^e et XVIII^e Siècles. Par GEORGES MAUPIN. Paris, Carré et Naud, 1898. 199 pp.

THE book seems to be addressed, not to the mathematician, nor to the historian of mathematics, but to the general reader, with the view of entertaining him and creating in him a love for the history of the science. In twenty-seven chapters there has been gathered together from old French writers, mostly unknown to our time, a miscellaneous mass of material relating to the squaring of the circle, the value of mathematical studies, pleas for the study of this science in the French rather than in the Latin language, mathematical pursuits in the universities, the extent to which women should master this subject, and the relation of mathematics to religion. Some topics are given which have nothing to do with mathematics—such, for instance, as the history of a conjurer, the expediency of higher education for women, the danger of long sermons, discussions of copyright and piracy and of gambling.

Here and there are points of antiquarian interest to the mathematician. The pathological effusions and conceits of circle squarers seldom lose their charm. M. Maupin quotes from De Vausenville who addressed his solution "to the monarchs of France, of Europe, and of the World." The French academy of sciences disdainfully rejected the solution and referred the memoir to a special commissioner, called, it seems, "le Commissaire des Enfans perdus." The author was furious at this treatment and wrote a letter to D'Alembert in which he said, among other things, that Charles V. erected a statue to the inventor of the art of salting and packing herring; why not do as much for the discoverer of the quadrature of the circle?

As an illustration of how mathematics was dragged into theological discussion, we may refer to Pardies's demonstration of the existence of God, drawn from the consideration of asymptotic spaces. "The knowledge of asymptotic spaces * * * brings out most clearly the greatness and the spirituality of our soul, for by the light of its spirit alone, penetrating, as it does, beyond the infinite, it unveils most clearly things which no sensory experience can reveal. * * * These spaces, though infinite in length, are equal to a circle or other determinate figure. * * * Of all knowledge which man can acquire through his own reasoning, this comprehension of the infinite is without doubt the most admirable." This power of man, which is independent of the sensory organs, was held to demonstrate the existence of the human soul. "Dare I go still further and say that in this demonstration we find also the proof of the existence of God?"

In another chapter of the book is given Varignon's quasi-mathematical argument of the possibility of the real presence of Christ's body in the eucharist. No doubt this seems very strange discourse to modern mathematicians who are fond of speaking of their science as "die absolut klare Wissenschaft," who habitually resent attempts to drag metaphysical speculation into mathematics and are little giving to applying mathematics to theology.

Readers familiar with the name of François Barréme, the arithmetical writer of the seventeenth century, who enjoyed in France much the same popularity as did Adam Riese in Germany, Edward Cocker in England, and Nicholas Pike in the United States, will be interested in the facsimile reproduction of the richly ornamented title page of his arithmetic of 1761, and in his dedicatory verses and other poetical outbursts.

FLORIAN CAJORI.

La Mathématique: Philosophie, Enseignement. Par C. A. LAISANT. Paris, Carré et Naud, 1898. 292 pp.

THIS book is intended for teachers and students of mathematics who are not specialists in this science. It deals with the philosophy and teaching of mathematics, and without much pretension to originality, presents the subject in an attractive and instructive manner. Separate chapters are given to arithmetic, algebra, calculus, theory of functions, geometry, mechanics, and to the practical application and the teaching of these subjects. The author does not discuss methods of teaching in detail, but wisely confines himself to general principles. The book gives a good general idea of mathematical instruction in France.

The author refers repeatedly to Comte's philosophy of mathematics and it is interesting to observe that he is compelled to abandon Comte's definition of mathematics, as the science which deals with "the indirect measurement of magnitudes." M. Laisant points out that Comte's definition does not include "the notion of order, which is inherent in mathematics to the same degree as measurement," and warns the reader against such a definition as carrying with it "a certain confusion which is not without danger." This point is just now of especial importance to us in the United States; for in the West certain theories of teaching arithmetic are being promulgated which assume that all mathematics deals solely with ratio and measurement and that the number concept is primarily and purely metrical. M. Laisant, in his discussion of number, does not find its origin primarily in measurement, but bases it on the cognition of a group of objects which, by mental abstraction, are considered alike. The primary number idea is non-metrical. On this point modern mathematicians are unanimous, and it is a sign of danger when the elementary teachers go in a direction diametrically opposite to the advanced workers and, misled by wrong conceptions, write textbooks which give an unnatural and one-sided development of arithmetic. The best pedagogical results can be reached only when the elementary and the advanced teachers work together. During the seventeenth and eighteenth centuries the writers of arithmetical texts in England were not mathematicians and were out of touch with the advanced workers. That period, with its wretched arithmetical books, serves as a warning of the danger which confronts us in the West.

The author insists on greater attention, in the class-room, to the theory of approximation in computation. He tells us the story of the illustrious Dulong, who was asked, "What is the use of seven or eight decimals in the ratio of refractive indices in double refraction when the experimental determinations disagree in the third place?" Dulong replied with grave irony: "I do not see why one should suppress the last decimals; for, if the first ones are false, possibly the last ones are correct." M. Laisant might well have recommended the revival of a sixteenth century method of multiplication which begins, as in algebra, on the left, with the figure of highest denomination in the multiplier. In decimal multiplication, where only approximate results are usually needed, it is then easy to avoid the computation of the last decimals in the product. This method was favored by Lagrange and is used by E. M. Langley in his Treatise on computation.

The author has a high and, we fear, somewhat exaggerated appreciation of the mathematical research which has been carried on in the United States. That his estimate may not rest on a precise knowledge of affairs in this country is surmised from his statement that Sylvester was called to *Boston* and founded a mathematical journal there.

FLORIAN CAJORI.

NOTES.

The Seventh Summer Meeting of the American Mathematical Society will be held, in affiliation with the American association for the advancement of science, at Columbia University, New York City, beginning Wednesday, June 27th, 1900, and extending through three or more days. Papers intended for presentation at this meeting should be in the hands of the Secretary by June 7th.

The preliminary programme of the International congress of philosophy, to be held at Paris, August 2d–7th, has appeared. The following subjects from the section of logic and history of the sciences are of interest to mathematicians:—1° Algebra of logic and theory of probabilities; theories of ensembles, sequences, and groups; the transfinite. 2° Principles of analysis: number, continuity, theory of functions. 3° The postulates of geometry, their origin and value; intuition in mathematics; non-euclidean geometries. 4° Methods of geometry; analytical geometry; projective geometry; geometrical calculus. 5° Principles of mechanics, their nature and value. 6° Methods of mathematical physics; theory of errors and approximation. 7° The origins of the infinitesimal calculus. 8° The genesis of the notion of imaginary and the progressive elucidation of the theory of functions. 9° History of the discovery of the Newtonian gravitation, and of its influence upon the development of mechanics and physics.

The following papers were presented at the regular meeting of the Edinburgh mathematical society on February 9th: "Remark on Dr. Peddie's proof of a theorem in potential," by Mr. R. F. Muirhead; "A general mechanical description of the conic sections," by Mr. Alexander Morrison; "On Bessel functions and spherical harmonics," by Mr. John Dougall.

At a meeting of the Cambridge philosophical society on February 5th, Dr. A. C. Dixon presented a paper "On differential equations with two independent variables."

The programme of the meeting of the Texas academy of science, held February 16th, included a paper by Professor L. E. Dickson, of the University of Texas, entitled "An elementary account of the greater problems solved by the modern group theory."

THE eighth volume (1900) of the *Jahresbericht* of the German mathematical association will consist of three parts. The first will contain an account of the annual meeting held at Munich, September 17–22, 1899, the constitution and bylaws of the association, a list of its four hundred and forty members, and appreciative biographical notices accompanied by portraits of six of its members deceased during the year 1899: C. I. GERHARDT, by M. Cantor; SOPHUS LIE, by Friedrich Engel; EUGEN VON LOMMEL, by Ludwig Boltzmann; FRIEDRICH MEYER, by G. Riehm; HERMANN SCHAPIRA, by C. Koehler; KARL SCHOBER, by W. Wirtinger. The second part of the volume will contain the papers presented at the Munich meeting (see BULLETIN, current volume, pp. 33–34). The third will contain three extensive reports: "The analytico-arithmetical theory of algebraic functions of two variables," by K. HENSEL; "The kinetic problems of scientific technology," by K. HUEN; "Curve and point manifoldnesses," by A. SCHOENFLIES.

BEGINNING with the January number of this year the *Nouvelles Annales de Mathématiques* inaugurates a monthly supplement containing features quite analogous to the "Notes" and the "New Publications" of the BULLETIN. This supplement consists of a chronicle of general mathematical intelligence, a bibliographical bulletin of recent periodicals and recent works, and a petite correspondance. The *Bolletino di Bibliografia e Storia delle Scienze Matematiche*, *L'Enseignement Mathématique*, and the *Bibliotheca Mathematica* have also recently promised to give their readers mathematical news. The *Nouvelles Annales* proposes occasionally to include portraits accompanied by brief biographical sketches; Lagrange is the first of the series.

IN the future the *Mathematical Gazette* of the British mathematical association will be issued six instead of three times per year. The aim of the association in publishing the *Gazette* is to supply a journal of direct and special interest to teachers of mathematics. Among the special features announced are articles suggestive of improvements in methods of teaching, others on subjects not satisfactorily treated in the textbooks, reviews of books of the first importance and of groups of textbooks on kindred subjects giving elementary presentations of the history and treatment of the subject. The second volume which begins with the present year will contain a series of articles by Professor CHARLOTTE ANGAS SCOTT, on von Staudt's Geometrie der Lage.

THE lectures on differential equations delivered by Professor É. PICARD at the decennial of Clark University in July, 1899, are appearing in the *Revue Générale des Sciences.*

MESSRS. Deighton Bell and Company, of Cambridge, England, now offer the second edition of Cayley's Elliptic functions at seven and a half shillings.

MESSRS. D. Appleton and Company, of New York, have recently published Elements of the differential and integral calculus based on the Lehrbuch der Differential- und Integral-Rechnung of Professors W. Nernst and A. Schoenflies, by Professor J. W. A. YOUNG, of the University of Chicago, and Mr. C. E. LINEBARGER, of the Lake View high school, Chicago.

THE proposed changes in the examinations for the mathematical tripos at Cambridge University, abolishing the order of merit and the senior wrangler, were defeated in the senate of the university, February 15th, by a vote of one hundred and fifty-two to one hundred and thirty-nine.

CORNELL UNIVERSITY. Among the courses announced for the summer session, July 5th to August 16th, are courses in algebra, geometry, trigonometry, analytical geometry, and infinitesimal calculus, given by Professor WAIT and Drs. MURRAY and MILLER, and the following advanced courses :—By Professor L. A. WAIT : Analytic geometry, based on Salmon's conic sections, three hours ; Differential calculus, based on the treatises of Todhunter and Williamson, three hours ; Integral calculus, three hours.—By Dr. D. A. MURRAY : Differential equations covering parts of the author's textbook, three hours.—By Dr. G. A. MILLER : Projective geometry, an elementary course in Cremona's projective geometry, three hours ; Theory of functions of a complex variable, the elements of the theories of Cauchy, Riemann, and Weierstrass, three hours ; Introduction to the theory of groups and theory of numbers, three hours ; seminar in groups, two hours.

DURING the summer semester of 1900 the several universities mentioned below offer the following courses in mathematics, the lectures beginning about April 20th and closing about July 31st :

UNIVERSITY OF ERLANGEN. By Professor P. GORDAN : Differential equations, four hours ; Invariants, four hours ; Exercises in seminar, three hours.—By Professor M. NOETHER : Synthetic geometry with exercises, four hours ; Analytical mechanics, four hours ; Mathematical exercises.

UNIVERSITY OF GÖTTINGEN. By Professor G. BOHLMANN: Analytical geometry, four hours.—By Professor F. SCHILLING: Differential calculus, four hours; Axonometry and perspective, one hour; Exercises in seminar, two hours.—By Professor D. HILBERT: Theory of differential equations, four hours; Selected chapters from the theory of surfaces, two hours; Exercises in seminar.—By Professor F. KLEIN: Theory of elasticity, four hours; Exercises in seminar, two hours.—By Dr. E. ZERMELO: Elliptic functions, two hours.—By Dr. J. SOMMER: Theory of algebraic curves, four hours.

UNIVERSITY OF HEIDELBERG. By Professor L. KOENIGSBERGER: Differential and integral calculus, four hours; Theory of lines and surfaces, four hours; Mathematical seminars, two hours.—By Professor M. CANTOR: Analytical geometry of the plane, four hours; Arithmetic and algebra, three hours.—By Professor F. EISENLOHR: Theory of probabilities, three hours; Mechanics, four hours.—By Professor K. KOEHLER: Synthetic geometry of the plane, three hours. —By Professor G. LANDSBERG: Theory of functions, three hours; Theory of determinants, two hours.

UNIVERSITY OF KIEL. By Professor L. POCHHAMMER: Analytical geometry of space, four hours; On differential equations in one independent variable, four hours; Exercises in mathematical seminar, one hour.—By Professor P. STÄCKEL: Higher analysis, first part, four hours; Projective geometry, two hours; Partial differential equations, three hours; Exercises in mathematical seminar, one hour.

UNIVERSITY OF MUNICH. By Professor G. BAUER: Analytical geometry of space; Exercises to the preceding course; Mathematical seminar.—By Professor F. LINDEMANN: Integral calculus; Theory of conformal representation and linear differential equations; The fundamental concepts of geometry; Mathematical seminar in the resolution of higher equations.—By Professor A. PRINGSHEIM: Selected chapters from the theory of functions; Definite integrals.—By Dr. H. BRUNN: Elements of higher mathematics for general students.—By Dr. K. DÖHLEMANN: Descriptive geometry, with exercises; Geometric transformations.—By Dr. E. RITTER VON WEBER: Ordinary differential equations; Elements of geometry.—By Dr. A. KORN: Analytical mechanics; Selected chapters in potential theory.—By Dr. GÖTTLER: General theory of algebraic curves; Methodology and pedagogy of mathematical instruction in secondary schools.

UNIVERSITY OF TÜBINGEN. By Professor H. STAHL: Elementary analysis, three hours; Theory of functions, three hours; Seminar, two hours.—By Professor A. VON BRILL: History of the theory of algebraic functions, two hours; Analytical mechanics, five hours; Seminar, two hours.—By Professor L. MAURER: Higher analysis, two hours; Exercises to the preceding course, one hour; Potential theory, two hours.

AT the meeting of the Deutsche Mathematiker-Vereinigung, last September, Professor HILBERT sketched briefly a new method for establishing existence theorems in the case of boundary value problems, and gave two applications of the method—one, to the proof that, on a given surface, a shortest line exists connecting two points, the other, to supplying the final step in 'Dirichlet's Principle,' as employed by Riemann in his dissertation. In the latter case, Professor Hilbert gives a direct proof that a function exists satisfying the boundary conditions and rendering the double integral

$$\int\int\left\{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right\}d\sigma$$

a minimum. (Cf. the forthcoming *Jahresbericht der Deutschen Mathematiker-Vereinigung.*) The method promises to admit of wide application to the boundary value problems of mathematical physics and geometry. In the coming summer semester at Göttingen, Professor Hilbert intends to give a course in which the essential features of the method, together with some of its applications, will be treated in detail.

PROFESSOR SIMON NEWCOMB has been elected correspondent of the Bureau des Longitudes of Paris.

PROFESSORS H. G. ZEUTHEN, of Copenhagen, and G. MITTAG LEFFLER, of Stockholm, have been elected correspondents to the section of geometry of the Paris academy of sciences, the former in place of the late Sophus Lie.

PROFESSOR A. GUTZMER, of the University of Jena, has been made professor ordinarius at the same institution.

DR. L. RAFFY, maître de conférences of the faculty of sciences of the University of Paris has been promoted to an adjunct professorship of mathematics.

MR. ALBERT M. SAWIN has been made instructor in mathematics at Syracuse University.

The deaths are announced of DR. HERRMAN SCHAEFFER, honorary professor of mathematics at the University of Jena, of Professor CHARLES PIAZZI SMITH, the astronomer, and of Mr. LEANDER J. McCORMICK, founder of the observatory bearing his name at the University of Virginia.

Two errors in noticing appointments have occurred in recent numbers of the BULLETIN. Dr. H. LIEBMANN has become privat docent at Leipzig, not Königsberg, and Mr. F. E. Ross has been appointed instructor at the University of Nevada, not Nebraska.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. Vol. 2, zweiter Halbband: 1550–1668. 2te Auflage. Leipzig, Teubner, 1899. 8vo. 12 pp. and pp. 481–943. M. 12.00

FREDHOLM (I.). Sur une équation aux dérivées partielles du quatrième ordre. (*Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar*, Stockholm, 1899.) 8vo. Pp. 935–940.

FUHRMANN (W.). Beiträge zur Transformation algebraisch-trigonometrischer Funktionen. Teil II. Königsberg, 1899. 4to. 30 pp. M. 1.50

GERLACH (R.). Die Metrik in projektiven Koordinaten. (Diss.) Zürich, 1899. 8vo. 114 pp.

LING (G. H.) and MILLER (G. A.). Proof that there is no simple group whose order lies between 1092 and 2001. (*American Journal of Mathematics*, Vol. 22, pp. 13–26.) 4to.

MILLER (G. A.). Note on Netto's Theory of substitutions. (*Annals of Mathematics*, 2d series, Vol. 1, pp. 71–73.) 4to.

OTTI (H.). Eigenschaften der Besselschen Funktionen zweiter Art. (Diss.) Bern, 1899. 8vo. 55 pp.

ROTHROCK (D. A.). Invariants of the finite continuous groups of the plane. (Diss.) Leipzig, 1899. 8vo. 31 pp.

STENDER (H.). Invariante Flächen und Kurven bei konformen Gruppen des Raumes. (Diss.) Leipzig, 1899. 8vo. 59 pp.

II. ELEMENTARY MATHEMATICS.

ADAM (V.). Taschenbuch der Logarithmen. 28ste Auflage. Wien, 1899. 12mo. 9 + 99 pp. Cloth. M. 1.00

ANDRÉ (P.). Nouveau cours de géométrie à l'usage de tous les établissements d'instruction, contenant plus de onze cents problèmes résolus et à résoudre, trois traités complets (levé des plans, arpentage, partage des terres), des notions de nivellement et un grand nombre de questions usuelles. 24e édition. Paris, André, 1900. 18mo. 7 + 497 pp.

——. Algèbre élémentaire (No. 1), théorique et pratique, à l'usage des institutions, des écoles professionnelles et des écoles normales. 20e édition. Paris, André, 1899. 18mo. 216 pp. Fr. 1.60

Bertrand (J.). Algebra, translated by M. V. Pirozhinkov. Part I. St. Petersburg, 1899. 8vo. 581 pp. (Russian.)

Burenin (K.). See Malinin (A.).

Bychkov (F.). Collection of problems and examples in elementary algebra. 15th edition, revised. St. Petersburg, 1899. 8vo. 580 pp. (Russian.) M. 5.00

Dufailly (J.). Algèbre. 12e édition. Paris, Delagrave [1899]. 8vo. 286 pp. Fr. 4.00

Gauss (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 60ste Auflage. Halle, 1899. 8vo. 166 + 35 pp. Cloth. M. 2.50

Gazzaniga (P.). See Veronese (G.).

Goddard (E. C.). See Lyman (E. A.).

Lyman (E. A.) and Goddard (E. C.). Plane Trigonometry. Five-place logarithmic and trigonometric tables, adapted from Gauss's Tables. Boston, Allyn and Bacon, 1899. 12mo. 8 + 91 + 66 pp. Cloth.

Malinin (A.) and Burenin (K.). Text-book of algebra, with a collection of problems, for gymnasia, etc. 10th edition. Moscow, 1900. 8vo. 416 pp. (Russian.) M. 3.50

Nikultsev (P.). Algebra, with a collection of problems. Part I. Theoretical part. 4th edition. Moscow, 1900. 8vo. 377 pp. (Russian.) M. 5.00

Pincherle (S.). Gli elementi dell'aritmetica ad uso delle scuole secondarie inferiori. 9a edizione riveduta e migliorata. Bologna, Zanichelli, 1900. 16mo. 8 + 343 pp. Fr. 2.00

Pirozhinkov (M. V.). See Bertrand (J.).

Sagorski (E.). Anleitung zur Aufstellung der Determination geometrischer Konstruktionsaufgaben. Pforta, 1899. 8vo. 74 pp. M. 1.50

Seiffert (A.). Die Herstellung der Raumgebilde als Ausgangspunkt, Entwicklungsprincip und Endziel des geometrischen Unterrichts. Charlottenburg, 1899. 4to. 24 pp. M. 2.20

Simon (P.). Guide méthodique de résolution des problèmes de géométrie élémentaire. Paris, 1899. 8vo. Boards. Fr. 4.00

Skerry. Practical papers in mensuration. London, 1899. 8vo. 140 pp. Cloth. 2s.

Veronese (G.) e Gazzaniga (P.). Elementi di geometria ad uso dei ginnasi e licei. Parte I e Parte II. Verona, Drucker, 1900. 16mo. 18 + 119 and 6 + 240 pp.

Viola (F.). La planimetria indipendente dal concetto di misura. Padova, Prosperini, 1899. 8vo. 80 pp.

III. APPLIED MATHEMATICS.

Alasia (C.). Calcolo grafico ed applicazioni della statica grafica: Città di Castello, Lapi, 1899. 8vo. 97 pp., 7 plates. Fr. 4.00

Dessau (B.). See **Felici** (A. R.).

Felici (A. R.). Ueber die mathematische Theorie der electrodynamischen Induktion. Uebersetzt von B. Dessau. Herausgegeben von E. Wiedemann. Leipzig, Engelmann, 1900. 12mo. 121 pp. Cloth. (Ostwald's Klassiker der exakten Wissenschaften, No. 109.) M. 1.80

Gross (T.). See **Sturm** (C.).

Mascart. Eléments de mécanique, rédigée conformément au programme de l'enseignement scientifique dans les lycées. 7e édition. Paris, Hachette, 1900. 8vo. 200 pp. Fr. 3.00

Rayleigh. See **Strutt** (J. W.).

Strutt (J. W.) [Baron Rayleigh]. Scientific papers. Vol. I: 1869–1881. London and New York, Macmillan, 1900. 8vo. 15 + 562 pp. Cloth. (Cambridge University Press series.) $5.00

Sturm (C.). Lehrbuch der Mechanik (cours de mécanique). Uebersetzt von T. Gross. Vol. II. Berlin, Calvary, 1900. 8vo. 23 + 403 pp. M. 8.00

Wiedemann (E.). See **Felici** (A. R.).

Sec No....
Book No ....
LIBRARY.

THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, February 24, 1900, extending, as usual, through a morning and an afternoon session. The first part of the afternoon was devoted to a joint meeting with the American Physical Society at which Professor J. K. Rees presented the paper noted in the list below. The total attendance during the day exceeded fifty persons, and included the following thirty-five members of the Society:

Dr. E. M. Blake, Professor Maxime Bôcher, Professor E. W. Brown, Professor F. N. Cole, Professor E. S. Crawley, Dr. W. S. Dennett, Professor A. M. Ely, Professor T. S. Fiske, Mr. A. S. Gale, Dr. G. B. Germann, Mr. H. E. Hawkes, Dr. G. W. Hill, Dr. J. I. Hutchinson, Professor Harold Jacoby, Mr. S. A. Joffe, Mr. C. J. Keyser, Dr. G. H. Ling, Dr. Emory McClintock, Dr. Alexander Macfarlane, Dr. James Maclay, Dr. Emilie N. Martin, Dr. G. A. Miller, Professor E. H. Moore, Professor F. Morley, Professor James Pierpont, Dr. M. B. Porter, Professor M. I. Pupin, Professor J. K. Rees, Mr. C. H. Rockwell, Professor T. J. J. See, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Miss E. C. Williams, Miss R. G. Wood, Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor Anne L. Bosworth, Rhode Island College, Kingston, R. I.; Mr. H. L. Coar, University of Illinois, Urbana, Ill.; Dr. F. R. Moulton, University of Chicago, Chicago, Ill.; Mr. F. G. Radelfinger, U. S. Hydrographic Office, Washington, D. C. Two applications for membership were reported.

A revision of the By-Laws, simplifying their arrangement and removing certain obsolete features, was adopted. Notice was also given of a proposed amendment of the Constitution, recommended by the Council and ordered to be submitted to the Society for action at the April meeting. By this amendment it is provided that the Ex-Presidents of the Society shall be life members of the Council, and that the number of members of the Council, other than officers, elected annually shall be increased from three to four. The presidential term of office is also extended to two years.

The following papers were presented at this meeting:

(1) Dr. ALEXANDER MACFARLANE: "On the nabla of quaternions."

(2) Dr. M. B. PORTER: "On the enumeration of the roots of the hypergeometric series between zero and one."

(3) Mr. H. W. KUHN: "List of the imprimitive groups of degree fifteen."

(4) Dr. G. A. MILLER: "On the groups of isomorphisms."

(5) Dr. J. I. HUTCHINSON: "The Hessian of the cubic surface, II."

(6) Professor MAXIME BÔCHER: "Some theorems concerning linear differential equations of the second order."

(7) Professor J. K. REES: "The variation of latitude at New York, and a determination of the constant of aberration from observations at the latitude observatory of Columbia University."

(8) Dr. G. W. HILL: "On the extension of Delaunay's method in the lunar theory to the general problem of planetary motion."

(9) Professor E. B. VAN VLECK: "On linear criteria for determining the circle of convergence of a power series."

(10) Professor F. MORLEY: "The metrical geometry of the plane n line."

(11) Professor L. E. DICKSON: "Two triply infinite systems of non-isomorphic simple groups of equal order."

(12) Professor L. E. DICKSON: "Isomorphism between certain systems of simple linear groups."

(13) Dr. L. W. REID: "A table of class numbers for cubic number bodies, with the method of their calculation."

Mr. Kuhn's paper was presented to the Society through Dr. G. A. Miller. Dr. Reid was introduced by Professor H. B. Fine. In the absence of the authors, Mr. Kuhn's paper was read by Dr. Miller, and Professor Dickson's papers were read by title. The papers of Professor Bôcher and Dr. Porter are published in the present number of the BULLETIN; those of Dr. Miller and Dr. Hutchinson, and the second paper of Professor Dickson will appear in later numbers. Abstracts of the remaining papers are given below.

In a paper on "The principles of differentiation in space analysis," read before the Society, January 26, 1895, Dr. Macfarlane pointed out that in space analysis there are two essentially distinct ways of differentiating a product: one result is obtained when the factors are first differentiated and the product then formed; a different result is obtained when the product is first formed and then differentiated. The main features of the second kind of differentiation as regards scalars were there developed. In the present paper Dr. Macfarlane applies the second kind of differentiation to the space operator denoted by Nabla, ∇. The paper shows how to form perfectly general expressions for ∇ and for ∇^2, of which latter Laplace's operator is a special case.

Mr. Kuhn's paper is in abstract as follows: In constructing the imprimitive groups of any degree, the first problem is to find all the intransitive groups that may be used as heads. The systems of intransitivity of these heads are transformed according to a transitive substitution group whose degree is equal to the number of systems of intransitivity of the given heads. For degree fifteen the number of these systems is either three or five. Hence every imprimitive group of degree fifteen is isomorphic with at least one of the seven transitive groups of degree three or five. Where the imprimitive group contains no substitution besides identity that transforms each one of its systems of imprimitivity into itself, this isomorphism must be simple. There are just two such groups. They are simply isomorphic respectively to the alternating and the symmetric group of degree five.

There are 21 groups that may be used as heads for the groups which contain three systems of imprimitivity; for those with five systems there are 9 such groups (including identity). The heads having been found, the next problem is to find the remaining substitutions in the possible imprimitive groups. There are 55 distinct imprimitive groups of degree fifteen that contain three systems of imprimitivity, and 56 that contain five such systems. Of the latter, 13 also contain three systems. Hence the total number of imprimitive groups of degree fifteen is 98. Of this number, 64 are solvable and 34 are unsolvable.

The latitude observatory under Professor Rees's charge was first placed on the present site of Columbia University near the corner of 118th Street and Amsterdam Avenue, New York. Observations were made at this station from April 24, 1893, to December 5, 1895. When the new buildings were begun, the latitude observatory was moved to the corner of 120th Street and Broadway. Here the observations were made from January 9, 1895, to the present time. The two stations were connected by a careful survey which showed the second position to be north of the first 7″.337. All observations made at the second point were reduced to the first. The zenith telescope made by Wanschaff of Berlin was employed throughout. Its apersure is 80 millimeters, and its focal length one meter. The observers were Professors Rees and Jacoby and Dr. H. S. Davis. Four groups of stars were used, having mean right ascensions approximately as follows :

Group I, 6 hours ; Group II, 14 hours ;
Group III, 18 hours ; Group IV, 22 hours.

Each group contained seven pairs of stars, and the groups were observed both morning and evening whenever the weather permitted.

SERIES A.			
April 24, 1893, to July, 1894 :	818	pairs by	Rees,
	302	" "	Jacoby,
	654	" "	Davis.
SERIES B.			
July, 1894, to January, 1896 :	771	" "	Rees,
	310	" "	Davis.
SERIES C.			
January, 1896, to January, 1898 :	1065	" "	Rees,
	774	" "	Davis.
SERIES D.			
January, 1898, to December, 1899 :	951	" "	Rees,
	873	" "	Davis.
Total pairs :	6,518		

This table shows the individual observers to have measured : Rees, 3,605 pairs ; Jacoby, 302 pairs ; Davis, 2,611 pairs ; total, 6,518 pairs.

The record shows that observations were taken on 758 nights. The computations give the accompanying table and curve which show the variation of latitude. The curve required by Dr. S. C. Chandler's formula (*Astronomical Journal*, No. 446) is shown in the dotted line. From 1896 the observed epochs of maxima and minima follow the computed in time.

The four series of observations gave the following values of the aberration constant :

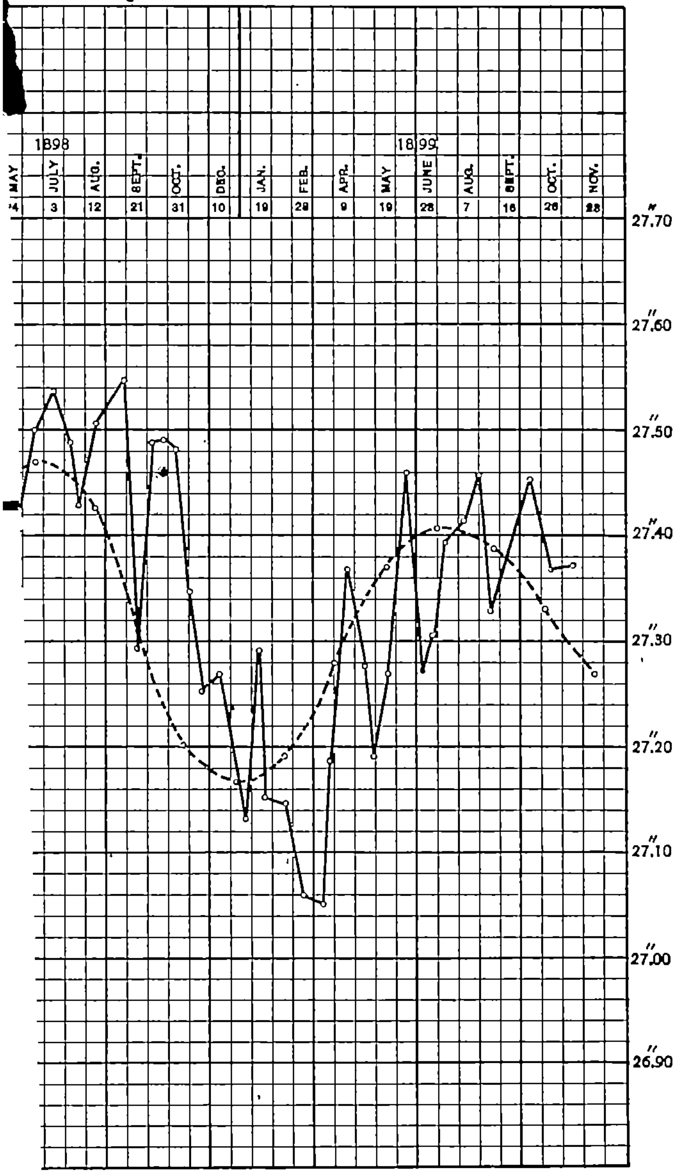
Series A.	20″4566	weight 18	Series C.	20″4695	weight 26
" B.	20 4525	" 16	" D.	20 4704	" 27

Taking the probable error of a single latitude observation as 0″.16, the constant of aberration is 20.464 ± 0″.006.

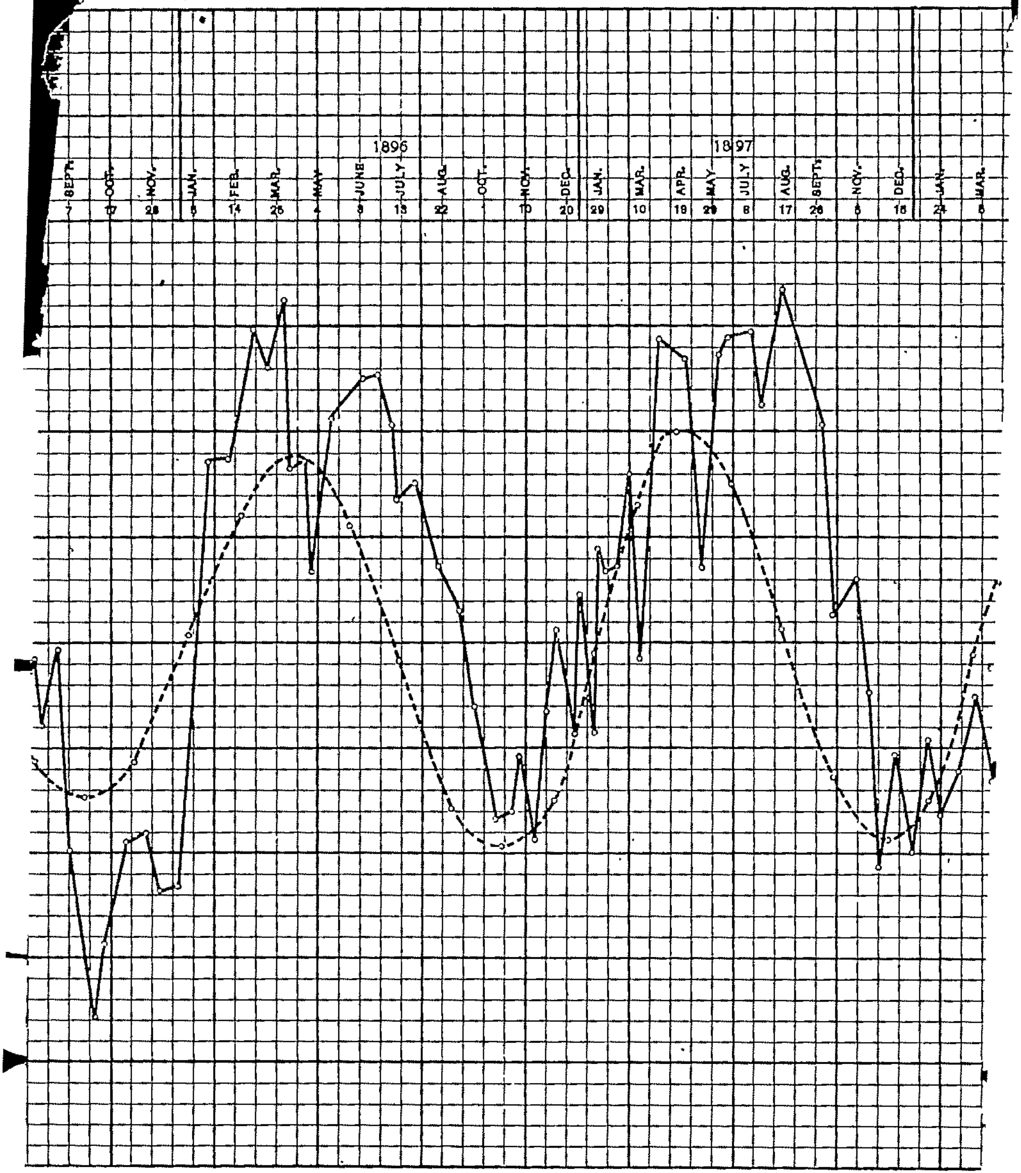
OBSERVATORY OF COLUMBIA UNIVERSITY.

Date.	40° 48′	$\phi - \phi_0$	No. pairs.
1893. May 9	27″.222	—.082	77
19	.144	—.160	44
27	.162	—.142	42
June 8	.178	—.126	59
17	.198	—.106	45
28	.259	—.045	32
July 8	.262	—.042	30
19	.177	—.127	38
26	.141	—.163	46
Aug. 6	.029	—.275	46
Oct. 6	.377	+.073	31
15	.244	—.060	43
30	.241	—.063	44
Nov. 9	.111	—.193	46
20	.204	—.100	48
Dec. 3	.232	—.072	18
11	.255	—.049	28
23	.302	—.002	43
1894. Jan. 2	.192	—.112	36
11	.168	—.136	65
23	.198	—.106	61
Feb. 2	.214	—.090	56
12	.148	—.156	59
24	.193	—.111	86
March 5	.234	—.070	68
21	.083	—.221	51
Apr. 1	.192	—.112	33
11	.199	—.105	65
23	.218	—.086	83
May 2	.153	—.151	89
12	.225	—.079	62
29	.216	—.088	50
June 8	.079	—.225	68
17	.193	—.111	64
July 1	.114	—.190	34
16	.120	—.184	38
Nov. 15	.226	—.078	39
29	.274	—.030	43
Dec. 18	.367	+.063	25
1895. Jan. 2	.305	+.001	20
15	.412	+.108	53
Feb. 2	.380	+.076	39
14	.332	+.028	40
26	.425	+.121	35
March 10	.500	+.196	18
21	.346	+.042	24
Apr. 8	.298	—.006	30
20	.353	+.049	48
May 7	.354	+.050	44
18	.213	—.091	47
June 4	.184	—.120	28
13	.257	—.047	51
July 5	.217	—.087	45
16	.286	—.018	23
28	.222	—.082	46

Date.		40° 48	$\varphi - \varphi_0$	No
1895.	Aug. 10	27″.295	—.009	
	23	.103	—.201	
	Sept. 15	26 .943	—.361	
	Oct. 1	.996	— 308	
	18	27 .111	—.193	
	Nov. 5	.119	—.185	
	18	.063	—.241	
	Dec. 2	.070	—.234	
1896.	Jan. 14	.474	+.170	
	28	.476	+.172	
	Feb. 11	.542	+.238	
	22	.600	+.296	
	March 5	.560	+.256	
	17	.627	+.323	
	28	.464	+.160	
	April 9	.477	+.173	
	19	.367	+.063	
	May 5	.513	+.209	
	18	.532	+.228	
	June 4	.551	+.247	
	20	.554	+.250	
	July 1	.501	+.197	
	12	.437	+.133	
	31	.458	+.154	
	Aug. 19	.379	+.075	
	Sept. 9	.331	+.027	
	26	.240	—.064	
	Oct 13	.135	—.169	
	25	.142	—.162	
	Nov. 10	.193	—.111	
	23	.118	—.186	
	Dec. 5	.232	—.072	
	15	.314	+.010	
	28	.219	—.085	
1897.	Jan. 8	.345	+.041	
	25	.214	—.090	
	Feb. 5	.387	+.083	
	18	.362	+.058	
	28	.378	+.074	
	March 10	.452	+.148	
	23	.283	—.021	
	April 25	.583	+.279	
	May 21	.565	+.261	
	June 1	.376	+.072	
	17	.573	+.269	
	July 2	.595	+.291	
	20	.597	+.293	
	Aug. 4	.522	+.218	
	22	.639	+.335	
	Sept. 30	.503	+.199	
	Oct. 16	.324	+.020	
	Nov. 4	.361	+.057	
	16	.254	—.050	
	26	.089	—.215	
	Dec. 10	.193	—.111	

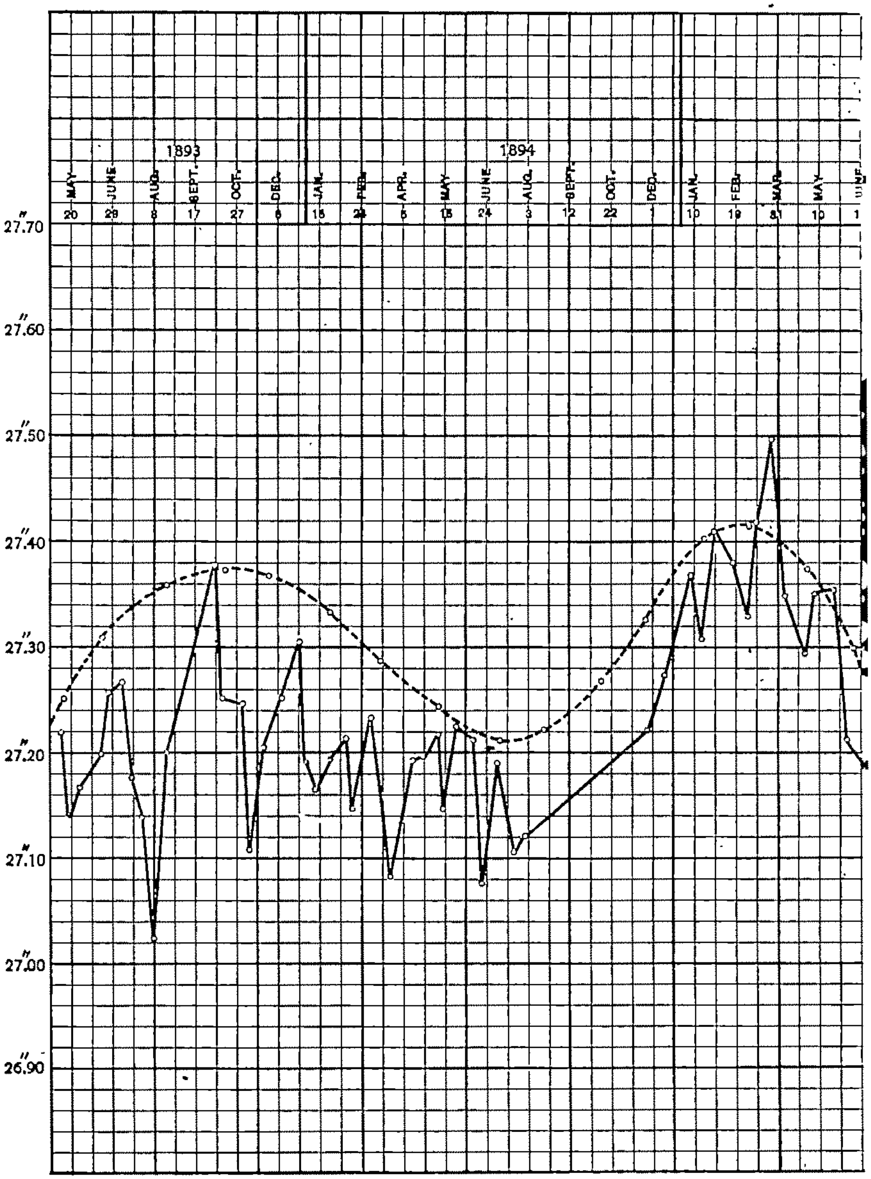

1898
1899
MAY 4
JULY 3
AUG. 12
SEPT. 21
OCT. 31
DEC. 10
JAN. 19
FEB. 28
APR. 9
MAY 19
JUNE 28
AUG. 7
SEPT. 16
OCT. 26
NOV. 28
27.70″
27.60″
27.50″
27.40″
27.30″
27.20″
27.10″
27.00″
26.90″

NIVERSITY OBSERVATORY, NEW YORK CITY. VARIATION OF LATITUDE.



dotted curve is constructed from Chandler's formula, Astronomical Journal No. 446.

COLU


1893
1894
MAY 20
JUNE 28
AUG. 6
SEPT. 17
OCT. 27
DEC. 6
JAN. 15
FEB. 23
APR. 5
MAY 15
JUNE 24
AUG. 3
SEPT. 12
OCT. 22
DEC. 1
JAN. 10
FEB. 19
MAR. 31
MAY 10
JUNE 1
27.70″
27.60″
27.50″
27.40″
27.30″
27.20″
27.10″
27.00″
26.90″

Date.	40° 48′	$\phi - \phi_0$	No. Pairs.
1897. Dec. 26	27″.100	— .204	22
1898. Jan. 10	.205	— .099	33
26	.139	— .165	43
Feb. 10	.177	— .127	53
March 1	.246	— .058	88
17	.169	— .135	25
April 4	.285	— .019	33
19	.461	+ .157	22
May 12	.365	+ .061	69
26	.429	+ .125	28
June 13	.501	+ .197	64
26	.539	+ .235	59
July 12	.488	+ .184	75
26	.430	+ .126	23
Aug. 12	.506	+ .202	32
Sept. 6	.549	+ .245	47
17	.295	— .009	28
Oct. 2	.488	+ .184	45
15	.490	+ .186	42
Nov. 1	.485	+ .181	59
15	.354	+ .050	40
Dec. 1	.251	— .053	48
14	.264	— .040	28
1899. Jan. 14	.138	— .166	36
23	.291	— .013	44
Feb. 3	.151	— .153	28
23	.146	— .158	35
March 13	.060	— .244	24
26	.055	— .249	16
April 6	.189	— .115	28
26	.366	+ .062	48
May 9	.275	— .029	29
14	.194	— .110	39
29	.267	— .037	61
June 16	.459	+ .155	11
30	.275	— .029	46
July 14	.309	+ .005	48
25	.391	+ .087	21
Aug. 12	.418	+ .114	51
22	.458	+ .154	42
Sept. 4	.322	+ .018	63
30	.397	+ .093	31
Oct. 17	.451	+ .147	35
Nov. 5	.361	+ .057	51
24	.369	+ .065	53
Mean	27″.304		

Probable error of one observation of unit weight ±0″.16

In Dr. Hill's paper, which will be published in the *Transactions*, Delaunay's method is explained as it would be applied to a planetary system, like the solar system. The attribution of greater generality to the method has the effect of rendering its establishment much easier. The limits of the domain in which the method is applicable are considered and are shown to be quite simple in their character. In the Delaunay transformation the special form of the equation of the conservation of energy constitutes a relation between the principal linear and angular variable. This relation is graphically exhibited by a plane curve. The general properties of this curve are noted. These curves are divisible into three classes. Delaunay's formulas of transformation are established by applying the theorem of Poisson to the deduction of the value of the Jacobian of the old variables with reference to the new. Some simplifications have been found for the expression of the partial derivatives of the general linear variable with reference to those conjugate to the angular variables employed. The subject is illustrated by making an application to the asteroid of the Hecuba type acted on by Jupiter; two cases are separately considered, viz., first, where the motion of the principal angular variable is continuous through the circumference, and second, where it executes a libration.

The criteria given by Professor E. B. Van Vleck for determining the radius of convergence of a power series $a_0 + a_1x + a_2x^2 + \cdots$ have a close connection with one of the two well known criteria which are based upon Cauchy's theorems concerning the convergence of an infinite series. According to one of these two criteria the radius of convergence is the limit of $1/|a_n^{1/n}|$ or, in case no limit exists, it is the greatest affix of a point of condensation of the sequence $1/|a_n^{1/n}|$, $(n = 1, 2, 3, \cdots)$. The other criterion gives the radius of convergence only when there is a limit for $|a_n/a_{n-1}|$. The reciprocal of this ratio is then the desired radius. But, though the former criterion is perfectly general and of the highest value for theoretical purposes, it nevertheless has little or no superiority as a practical test for determining the radius of convergence of a given series. For in most cases in which $|a_n/a_{n-1}|$ does not approach a limit, the determination of the limit of the sequence $1/|a_n^{1/n}|$ or of its points of condensation is extremely difficult. The object of Professor Van Vleck's paper is to establish criteria which apply in certain cases where the ratio test fails; namely, in the cases in which a number of coefficients of the power series are connected together by a linear relation which tends to take a limiting form as n increases.

The first of four theorems obtained in the paper is as follows: Given a series $a_0 + a_1x + a_2x^2 + \cdots$ in which, after some fixed term, $p + 1$ consecutive coefficients are connected by a linear relation

$$a_n = (b_1 + \epsilon_1^{(n)})a_{n-1} + (b_2 + \epsilon_2^{(n)})a_{n-2} + \cdots + (b_p + \epsilon_p^{(n)})a_{n-p};$$

if, for an arbitrarily assigned positive value ϵ, a positive integer m can be found such that for $n \geqq m$ every $|\epsilon_i^{(n)}|$ will be less than ϵ, the series will converge within a circle whose center is the origin and whose radius is the distance from the origin to the nearest root (or roots) of the polynomial $1 - b_1x - b_2x^2 - \cdots - b_px^p$. A special case of this theorem is that in which $p = 1$. The root of the polynomial $1/b_1$ is then also the limit of a_{n-1}/a_n, and we have Cauchy's ratio test for the determination of the radius of convergence.

The existence of a limiting relation between $p + 1$ coefficients of the series necessitates also the existence of other limiting linear relations containing a greater number of coefficients. Among all these linear relations there is, however, one and only one containing the minimum number of coefficients. If in the above theorem this relation be used, the circle will be, in general, the circle of convergence.

The second theorem of the paper is a restricted extension of the first to the case in which the number of terms in the linear relation increases without limit with increasing n. In place of the above polynomial an infinite series $1 - b_1x - b_2x^2 - \cdots$ is then to be introduced. The theory can also be extended to cover power series containing two or more variables. This extension is made for the case of two variables in theorems III. and IV., which may possibly be of special interest, since criteria for the convergence of series in two or more variables seem as yet to be very rare.

Professor Van Vleck's paper is to be published in the *Transactions*. The results of the paper will be applied in a subsequent article to the theory of linear differential equations.

Professor Morley's paper, which will be published in the April number of the *Transactions*, first establishes for n lines of a plane the existence of a circle, previously detected for 4 lines by Steiner and for 5 lines by Kantor; and shows that for all $n - 1$ lines included in the n lines, such circles pass through a point. The method used is a general analytic method of handling n lines of a plane, by means of certain characteristic constants. The circle and point form the beginning of a series of figures, of which Clifford's circle or point is the end; accordingly Clifford's theorem is proved by the method, and the coördinates of his chain found.

The curves considered are included in $x = rat.\ alg.\ t$, where t is a point on the unit circle; and it proves to be desirable to consider in connection with the curves those obtained by polarizing the expression in t, somewhat as is done in the theory of osculants. The curves so obtained are examined in the case of a limaçon, which first presents itself; and in the case of an ellipse, where it appears that, given $2p$ lines of an ellipse, the Clifford circle of any $2q - 1$ lines and that of the remaining lines are partners in a complex involution I_1^2.

The fact that the theory of 5 lines takes a peculiarly simple form when the lines touch a hypocycloid of class 3 suggests special examination of hypocycloids. The theory as given by Kantor for 5 lines is thus generalized at once, and further it is indicated how the theorems of the memoir itself, which concern merely the intersections of n lines, can be replaced by theorems in which the lines are taken 3, 4, or more at a time.

In a résumé of the known finite simple groups published in the BULLETIN, July, 1899, p. 470, Professor Dickson noted that for $p > 2$ the simple abelian group $A(2m, p^n)$ had the same order as the simple orthogonal group $O(2mH, p^n)$ and that the two were isomorphic for $m = 1$ and 2. [For the case $m = 1$, $p^n = 3$, the groups are not simple.] The present note announces* the truth of the conjecture there made, that the two groups were not isomorphic for $m > 2$. Besides these two triply infinite systems, there has been noted† but a single, isolated, case of non-isomorphic simple groups of equal orders, viz, for the order $\frac{1}{2}\ 8!$. By an elaborate investigation, Professor Dickson establishes the following theorems:

The general abelian group on $2m$ indices in the $GF[p^n]$, $p > 2$, contains exactly m sets of conjugate substitutions of period two, those of the rth set being conjugate with the product $T_{1,-1}\ T_{2,-1} \cdots T_{r,-1}$, where $T_{i,-1}$ merely changes the signs of ξ_i and η_i. It contains a single conjugate set of substitutions whose squares give $T \equiv T_{1,-1}\, T_{2,-1} \cdots T_{m,-1}$, the only self-conjugate substitution of the abelian group, the set being represented by $M_1\, M_2 \cdots M_m$.

---

* The complete investigation was offered January 25th to the *Quar. Jour. of Math.*

† Cf. BULLETIN, 1899, p. 473. An immediate proof is given by the writer in an article to appear in the *Amer. Jour. of Math.* for July, 1900.

From these theorems the following result is derived concerning the quotient group of the Abelian group by its maximal self-conjugate subgroup $\{1,\ T\}$, the case $m = 1$, $p^n = 3$ being excluded:

According as m is even or odd, the simple group $A(2m, p^n)$, $p > 2$, has exactly $\frac{1}{2}(m+2)$ or $\frac{1}{2}(m+1)$ distinct sets of conjugate operators of period two.

We readily see that the simple subgroup $O(2m+1, p^n)$ of the orthogonal group contains at least m substitutions of period two which are not conjugate, for example,

$$C_1 C_2, \quad C_1 C_2 C_3 C_4, \quad \cdots, \quad C_1 C_2 C_3 C_4 \cdots C_{2m-1} C_{2m},$$

where C_i denotes the orthogonal substitution altering ξ_i alone whose sign it changes. It follows that for $m > 2$, the orthogonal group contains a greater number of distinct sets of conjugate operators of period two than the simple group $A(2m, p^n)$ of equal order.

Dr. Reid's table gives the class number h, the discriminant $\triangle$, and a basis for each of 161 cubic number fields, and the factorization of certain rational primes into their prime ideal factors. When $h = 1$, the prime *number* factors of these primes are given. Units are also given for a majority of the fields. The method employed in the calculation of h is based upon the following theorem of Minkowski's: In every ideal class of an algebraic number field, k, there exists an ideal, whose norm is

$$< \left(\frac{4}{\pi}\right)^r \frac{\underline{|m}}{m^m} \left|\sqrt{\triangle}\right|,$$

where m is the degree and $\triangle$ the discriminant of the field, and r the number of pairs of imaginary fields which are found among the m conjugate fields $k, k^{(1)}, \cdots, k^{(m-1)}$. The process consists: 1° in obtaining all ideals satisfying the above condition; 2° in determining the number of ideal classes into which they fall. All prime ideals which satisfy the above condition are obtained by factoring the rational primes

$$< \left(\frac{4}{\pi}\right)^r \frac{\underline{|m}}{m^m} \left|\sqrt{\triangle}\right|.$$

This is effected in the case $\dfrac{d(\theta)}{\triangle} \not\equiv 0 \pmod{p}$ by means of the theorem: If p satisfy the condition $\frac{d(\theta)}{\triangle} \not\equiv 0$ (mod. p), and if $f(x)$ be resolved into its prime factors with respect to the modulus p:

$$f(x) \equiv \{P(x)\}^e\{P'(x)\}^{e'} \cdots \qquad (\text{mod. } p),$$

where $P(x)$, $P'(x)$, … are different prime functions with respect to p, of degrees f, f', … respectively, then

$$(p) = (p, P(\theta))^e (p, P'(\theta))^{e'} \cdots$$

is the required factorization of (p), where $(p, P(\theta))$, $(p, P'(\theta))$, … are different prime ideals of degrees $f, f', \cdots$ respectively. θ is an integer defining the field, $f(x) = 0$ the equation of lowest degree satisfied by θ, and $d(\theta)$ the discriminant of this equation. When $\frac{d(\theta)}{\triangle} \equiv 0$ (mod. p) the factorization of (p) can be effected in the case of cubic fields by a method due to Woronoj. The determination of the number of classes into which these ideals fall is made to depend upon the determination of the lowest power of a given ideal which is a principal ideal, and this upon the determination whether a given principal ideal (α) is the pth power of a principal ideal. Attention is called to the saving in reckoning effected by observing that, when we have $N - n + k < 2K$, then $h = K$, where N is the number of ideals to be investigated, n the number of these ideals whose classes have been determined, k the number of the known classes which have found representatives, and K the number of known classes. If the sth and tth are the lowest powers of $\mathfrak{j}$ and $\mathfrak{a}$ respectively, which are principal ideals, then a necessary condition that $\mathfrak{j}$ belong to one of the classes represented by a power of $\mathfrak{a}$ is $t \equiv 0$ (mod. s), in which case there are $\varphi(s)$ of these classes to one of which it is possible for $\mathfrak{j}$ to belong. They have as representatives $\mathfrak{a}^{t'i}$, where $t = t's$, and i is prime to s. The question whether $\mathfrak{j}$ belongs to a given class is decided by multiplying $\mathfrak{j}$ by an ideal $\mathfrak{b}$ belonging to the reciprocal class and determining whether $\mathfrak{j}\mathfrak{b}$ is a principal ideal by the method indicated. See Hilbert: "Bericht über die Theorie der algebraischen Zahlkörper," *Jahresbericht der deutschen Mathematiker-Vereinigung*, Vol. 4 (1894–5), §§ 11, 24; Minkowski: Geometrie der Zahlen; Woronoj: "The algebraic integers, which are functions of a root of an equation of the 3d degree," (translation of the Russian title).

F. N. COLE.

COLUMBIA UNIVERSITY.

SOME THEOREMS CONCERNING LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society, February 24, 1900.)

I WISH to communicate to the Society certain results at which I have arrived, reserving the proofs and also further developments for another occasion. We shall be concerned with differential equations whose coefficients are not assumed to be analytic, and we consider the case in which $x = 0$ is a *regular singular point** of the equation, that is, the case in which the equation is of the form:

$$(1) \qquad \frac{d^2y}{dx^2} + \left(\frac{\mu}{x} + p_1\right)\frac{dy}{dx} + \left(\frac{\nu}{x^2} + q_1\right)y = 0,$$

where μ and ν are constants, and $|p_1|$ and $x\,|q_1|$ can be integrated up to the point $x = 0$.† For the sake of simplicity I shall not here refer to the case in which the exponents of the point $x = 0$ are equal, although this case might easily be included. By methods analogous to those which I have sketched in a simpler case (BULLETIN, Second Series, Volume 6, p. 100) we obtain the

FIRST THEOREM OF COMPARISON: *If the exponents of $x = 0$ in (1) are real and unequal, and y_1 is the solution of (1) which corresponds to the larger exponent, and if y_1 vanishes when $x = x_1 > 0$, but does not vanish between 0 and x_1; if moreover we have a second equation of the same form as (1) which differs from it only in that throughout the interval $0 < x \leqq x_1$ the coefficient of y in the second equation $\geqq \nu/x^2 + q_1$ (the equality sign not holding throughout the whole interval); and finally if y_2 is the solution of this second equation which corresponds to the larger exponent of $x = 0$; then y_2 has at least one root in the interval $0 < x < x_1$.*

In order to compare the solutions corresponding to the smaller exponents, we must restrict our differential equation further by assuming that a constant $k < 1$ can be found such that $x^k p_1$ and $x^{k+1} q_1$ remain finite as x approaches zero. We will refer to this as *restriction* (A). We then have the

THEOREM: *The solution of (1) corresponding to the exponent*

---

* For the meaning of this and other terms used, see my paper in the January number of the *Transactions*, p. 40; or an earlier paper in the BULLETIN, 2d Series, vol. 5, p. 275.

† For a more precise statement see my papers just referred to.

with the smaller real part can be developed by the method of successive approximations exactly as the solution corresponding to the exponent with the larger real part is developed in my article in the Transactions, provided that $R\kappa < 1 - k$, *where* κ *is the difference of the exponents at* $x = 0$ *so taken that* $R\kappa > 0$.

By using this theorem we deduce the

SECOND THEOREM OF COMPARISON: *If the conditions of the first theorem of comparison are fulfilled, and if moreover both equations have the same exponents, and both satisfy restriction* (A); *and if* $\bar{y}_1$ *and* $\bar{y}_2$ *are the solutions corresponding to the smaller exponent; and if* $\bar{y}_1$ *vanishes when* $x = \bar{x}_1 > 0$, *but not in the interval* $0 < x < \bar{x}_1$, *then* $\bar{y}_2$ *will vanish at least once in this interval provided that* $\kappa < 1 - k$.

Finally I should like to mention a fact which had escaped my notice until after my paper in the *Transactions* was printed, namely that the class of singular points which I there discuss under the name *regular* can be brought into very close connection with the class of singular points previously studied by Kneser (*Crelle's Journal*, Volumes 116, 117, 120; *Mathematische Annalen*, Volume 49). This can be done by replacing the independent variable x, which I use, by z where $x = e^{-z}$. Although many of my results can be deduced by this method from those previously found by Kneser and *vice versa*, the results in the two cases are by no means coextensive, nor does either include the other. I shall come back to this matter more at length on a subsequent occasion. It may be noted that the method of successive approximations can also be applied directly to Kneser's case.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.

NOTE ON THE ENUMERATION OF THE ROOTS OF THE HYPERGEOMETRIC SERIES BETWEEN ZERO AND ONE.

BY DR. M. B. PORTER.

(Read before the American Mathematical Society, February 24, 1900.)

IN the May number of the BULLETIN for 1897, the writer gave a solution of the problem of enumerating the real roots of $F(\alpha, \beta, \gamma, x)$ between zero and one which depended on two well known theorems of Sturm—there referred to as [A] and [B]*—and Gauss' expression of $F(\alpha, \beta, \gamma, x)$ in terms of the two linearly independent solutions of the hypergeometric equation about the point $x = 1$. The investigation there given can be considerably simplified and Gauss' formula avoided by the use of the following theorem [C] : †

If $S(x) = \sum_0^\infty a_n x^n$ *is divergent for* $x = r$ *but convergent for* $|x| < |r|$, *and if* $S(x)$ *diverge to* $+\infty$ $(-\infty)$ *as* x *approaches* r, *then* $\lim_{n=\infty} S_n(r) = +\infty$ $(-\infty)$ *where* $S_n(r) = \sum_0^n a_n r^n$.

In the paper above mentioned it is shown that the problem of determining the number X of roots of the solution of the hypergeometric equation corresponding to the larger exponent of $x = 0$ can be reduced to the case where λ, μ, and ν ‡ (the exponent differences of 0, ∞, and 1 respectively) are not less than zero.

To determine X in this case it is only necessary to determine the number of roots lost by $F(\alpha, \beta, \gamma, x)$ when β, by the decrease of μ, increases from some initial negative value to zero.

By the first theorem of Sturm quoted, $F(\alpha, \beta, \gamma, x)$ is losing or certainly not gaining roots when, under the conditions laid down, μ decreases.

Moreover since $\nu \geqq 0$, $F(\alpha, \beta, \gamma, 1)$ is divergent; and by [C] $F(\alpha, \beta, \gamma, 1-0)$ changes from $+$ to $-$ ($-$ to $+$) when β increases through an odd (even) negative integer.

But when $F(\alpha, \beta, \gamma, 1-0)$ changes sign, β increasing (*i. e.*, μ decreasing), an odd number of roots must have been lost by $F(\alpha, \beta, \gamma, x)$ in the interval $0-1$ and it is easily seen (by considering the polynomials obtained when β is integral) that exactly one root is lost. Thus when λ, μ, and ν are not less than zero,

$$X = E(1-\beta). \qquad (\S)$$

In the same way it may be shown that in any case

$$X = E\left(\frac{\mu - |\lambda| - |\nu| + 1}{2}\right).$$

When $\lambda < 0$, the number N of roots of $F(\alpha, \beta, \gamma, x)$

---

[*] Theorem [B] is a slightly generalized form of the theorem given by Sturm, *Liouville's Journal*, vol. I.

[†] Cf. Stolz, Allgemeine Arithmetik, p. 279.

[‡] $\lambda = \gamma - 1$, $\mu = \alpha - \beta$, $\nu = \alpha + \beta - \gamma$.

[§] $E(n)$ is used to denote the largest positive integer less than n, so that $E(n) = 0$ if $n \leqq 1$.

between 0 and 1 is by [B] either X or $X + 1$, the even or odd value of N being chosen according as $F(\alpha, \beta, \gamma, 1 - 0)$ is greater or less than 0, viz., 1° If $\nu > 0$, according as $E(-\alpha) + E(-\beta) + E(-\gamma)$ is odd or even; 2° if $\nu < 0$, according as $E(\alpha - \gamma) + E(\beta - \gamma) + E(-\gamma)$ is odd or even. If, however, in 1° either $-\alpha$ or $-\beta$ is integral and positive $F(\alpha, \beta, \gamma, x)$ ceases to be linearly independent of the solution corresponding to the larger exponent of $x = 1$, so that, by [B], $N = X$. Similarly in 2°, if $\alpha - \gamma$ or $\beta - \gamma$ is a positive integer, $N = X$ for the same reason.

YALE UNIVERSITY,
February 12, 1900.

THE SUMMER MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG AT MUNICH, SEPTEMBER, 1899.

PROBABLY no mathematical society in the world brings together at its annual meetings so many illustrious mathematicians and exhibits an activity of such multifarious nature and far-reaching importance as the Deutsche Mathematiker-Vereinigung. Founded about a decade ago through the efforts of G. Cantor, W. Dyck and others, it has enjoyed the utmost prosperity. It is indeed to be regretted that its meetings are held so late in the summer that it is impossible, save in exceptional circumstances, for our own members to be present. Such a piece of good fortune befell me last year, and I hope the following personal reminiscences may prove interesting enough to justify their publication. For more details of scientific character I refer to the *Chronik* of the Vereinigung, the article by E. Lampe in the *Naturwissenschaftliche Rundschau*, (1899), p. 537, and the official *Verhandlungen der Gesellschaft Deutscher Naturforscher und Aertze*, Vogel, Leipzig, 1900.

The summer meetings of the Vereinigung are always held in connection with those of the Gesellschaft Deutscher Naturforscher und Aertze, which corresponds in Germany to our American Association for the Advancement of Science. This year the meeting was held September 17th to 23d at Munich, a city justly celebrated for its scientific institutions and collections of art, for its beautiful surroundings, and for the good nature and kindly spirit of its population.

Our first meeting was a general reception, of a purely social nature, held in the large Kaimsäle, Sunday evening. But large as these rooms were, the immense throng which had gathered in thousands from all parts of the world, but particularly from Germany, Austria and Switzerland, so filled them that it was all but impossible to move about. To find one's friends in such a crush was almost impossible, and I had almost given up the hope when I spied Wirtinger absorbed in an exchange of views on algebraic functions with Hensel. Near them were Gutzmer, Gordan, Czuber, Lampe, Pringsheim, Noether, and others. Some, like myself, were accompanied by their wives. Of course this first greeting after a long absence was animated and pleasant.

The business of the week began Monday morning and lasted till Friday afternoon. The Vereinigung held some of its meetings apart, and some in joint session with sections of the Gesellschaft. There were also general sessions in which everybody joined. The opening session of the Gesellschaft was of this last kind. This was held in the royal theater, which was literally crowded. Speeches of welcome were held by the first Geschäftsführer v. Winckel, by the honorary president, H. R. H. Prince Dr. Ludwig Ferdinand of Bavaria; by the Cultusminister v. Landmann; the Mayor, v. Borscht; the president of the Academy of Sciences, v. Zittel; the prorector of the University, v. Heigel; the director of the Polytechnicum, v. Hoyer; and finally by the president of the Gesellschaft, Wirkl., Geh. Admiralitätsrat, Dr. Neumayer of Hamburg. Then after the customary telegrams of courtesy had been read before being sent off to the Emperor of Germany, the Prinzregent Luitpold of Bavaria, and H. R. H. Duke Dr. Karl Theodor of Bavaria, the addresses of the day were in order. The lion of the occasion was Nansen who was greeted with prolonged and hearty applause. His theme was: "Meine Forschungsreise nach der Nordpolregion und deren Ergebnisse."

The first session of the Vereinigung took place the same afternoon in the Polytechnicum. Addresses of welcome were made by Bauer on the part of his colleagues in Munich and by Noether as president of the Vereinigung. These were followed by the only address of the day, an appreciative sketch by Engel of the life and labors of Sophus Lie.

All day Tuesday was devoted by the Vereinigung in separate session to the reading of papers. The first address was by Klein who made some extremely interesting and valuable remarks in addition to Engel's paper of the day before. Lie and Klein were fellow-students at Paris. At the outbreak of the Franco-Prussian war Klein returned to Germany but remained in correspondence with Lie, who at that time was deeply absorbed in his new Kugelgeometrie. As the Germans began to close in on Paris, Lie who was celebrated even in Norway, the home of such men as Nansen, for his powers as a pedestrian, proposed to make his way into Italy on foot. He got as far as Fontainebleau when he was arrested as a suspicious character. Suspicion was only heightened when his very unplausible intentions were learned. The examination of his scanty baggage confirmed the impression. His mathematical papers were supposed to be notes written in cypher on the military condition of the country he was traversing. Various mathematical terms were given a military meaning. The word Kugel was constantly occurring; these passages were thought to refer to the artillery. The word Complex was also frequently repeated. Through imperfect knowledge of German script this was read Lomley and was thought to refer to a certain Monsieur Lomley, perhaps another spy, at least to some officer or military agent. Lie protested his innocence, maintaining that his papers were not written in cypher, that they had nothing to do with military affairs, and in short that they were mathematics pure and simple. If this were indeed so, Lie's statements could be easily tested. As every one knows, the mathematics taught at the great military schools of France is of a very elevated nature. So an officer was summoned, doubtless one who was reputed to be expert in his mathematics and Lie was given a chance to prove his assertions. But, alas! Lie's Kugelgeometrie had never been taught at the École Polytechnique nor at St. Cyr, and so was utterly incomprehensible to his examinator, who became convinced that he had before him not only a spy, but a crazy one at that. So Lie was cast into prison until some weeks later, when at the instance of Darboux and other friends in Paris, he was released.

The two principal papers of the day were by Hensel, "Ueber die analytisch-arithmetische Theorie der algebraischen Functionen von zwei Variablen"; and by Hilbert, "Ueber das Dirichlet'sche Princip." The latter, treating an old and celebrated problem with striking originality, excited universal surprise and admiration. As everyone knows, Riemann's epoch making researches in abelian functions rest on a half-mathematical, half-physical principle which Riemann called Dirichlet's Principle and which relates to the minimum of a certain integral.

After Weierstrass's criticism it became necessary to give Riemann's theory a new foundation, which marred its original beauty and simplicity. Hilbert showed how the theory of ensembles, of partial differential equations, and the calculus of variations enable us to remove Weierstrass's objection and so return to Riemann's original treatment.

Another paper of general interest was by Noether, "Mittheilungen über Riemann's Vorlesungen von 1861–62 über Abel'sche Functionen." This was another step in the undertaking which has been some time on foot to collect and make accessible everything connected with Riemann's scientific activity.

On Tuesday evening there was a grand banquet given in the Odeon. High dignitaries and world renowned savants were present. There were toasts and speeches in profusion. I could not stay to the end, but had to hasten away to take part at a festivity of a more enjoyable kind. This came about so. The members of the Mathematische Vereinigung seem to consider great official banquets rather a tiresome superfluity. At any rate it was well known by past experience that hardly anyone would attend, and so Professor and Mrs. Pringsheim took this opportunity to extend to the members of the Vereinigung and their wives the hospitality of their house, or rather palace, for such it really is in the noble proportions of its rooms and its rare collections of objects of art. Here we were indeed royally entertained. After a rich collation, cigars and the genial flow of mathematical, postprandial speeches were in order.

Wednesday morning was spent in joint session with the naturwissenschaftliche Hauptgruppe of the Gesellschaft. The subject of the papers was the "Decimal division of the circle and unit of time." These gave rise to a lively and prolonged discussion. In the afternoon there were various excursions to points of interest in the surroundings of Munich. The most interesting was to Schleissheim, where a splendid mediæval pageant and an inspection of the treasures of the old castle were the attractions.

Thursday morning the two most interesting papers read were by Brill: "Ueber ein Beispiel des Herrn Boltzmann zur Mechanik von Hertz," followed by an instructive discussion in which Boltzmann took part; and by Study: "Geometrie der Dynamen." The word Dyname, is Plücker's term for a system of forces acting on a rigid body, which, as well known, can be replaced by a couple and a resulting force. Study gave an account of his new geometrical representation of such a system and illustrated its great value;

one paper by him on this subject has already appeared in the *Leipziger Berichte*.

In the afternoon a joint session with the Abtheilung für mathematischen und naturwissenschaftlichen Unterricht of the Gesellschaft was held. The papers by H. Weber and Hauck were of great interest. The subject of discussion was the "Ordnung des mathematischen Universitätsunterrichts auf Grund der neuen preussischen Prüfungsordnung." This refers to the new requirements in descriptive geometry, technical mechanics, graphical statics, geodesy, and the adjustment of observations in the state examinations for gymnasial teachers in mathematics. They have been imposed with the purpose of laying more emphasis on applied mathematics and so help fill up to some extent the great gap which is beginning to separate pure mathematics from the practical needs of present sociological conditions. Klein has been identified with this movement from the start and is perhaps its chief exponent.

From 11 to 1 o'clock the same day the business meeting of the Vereinigung took place. Various reports were made. In the election to replace Hauck and Voss, whose term of office on the council expired, A. Mayer (Leipzig) and Dyck (Munich) were chosen. Later, Mayer has begged to retire in favor of Minkowski, who had received the next highest number of votes. The president for the ensuing year is Hilbert.

I may mention here that a meeting of the academic commission and the editors of the Mathematical Encyclopedia took place some days later and the following arrangement was made for the remaining volumes: Vol. IV, Mechanics, editor, F. Klein; Vol. V, Physics, editor, A. Sommerfeld; Vol. VI, (*a*) Geophysics, Geodesy, editor, E. Wiechert; (*b*) Astronomy, editor, H. Burkhardt.

Thursday evening there was a gala presentation at the court theater. The members of the Vereinigung did not care to attend; they had a private entertainment. This was a Kneipe given by the Munich members. It took place in one of the halls of the Neuer Ratskeller, and, as usual, was graced by the presence of the ladies. It was a very pleasant and cordial occasion.

Friday there was again a general session in the royal theater. The address of chief interest to mathematicians was the remarkable paper by Boltzmann, of Vienna: "Ueber die Entwickelung der Methoden der theoretischen Physik in neuerer Zeit." The Vereinigung also had a meeting of its own the same day, at which various mathematical papers were read, and one by Rudel (Nürnberg) on "Die neue bayrische Prüfungsordnung für das Lehramtsexamen der Lehrer für Mathematik und Physik." This was followed by a long discussion on the questions involved in it and in the previous papers of Weber and Hauck.

Before closing, let me add that the mathematical papers mentioned here, together with many others, will appear ere long in the eighth volume of the *Jahresbericht* of the Vereinigung. The few remarks I have made will indicate sufficiently their importance and scope. I have, finally, the pleasure of thanking the amiable secretary of the Vereinigung, Prof. Dr. Gutzmer, for notes of the sessions I could not attend.

JAMES PIERPONT.

YALE UNIVERSITY,
March, 1900.

HILBERT'S FOUNDATIONS OF GEOMETRY.

Grundlagen der Geometrie. Von DR. DAVID HILBERT, o. Professor an der Universität Göttingen. (Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen. Herausgegeben von dem Fest-Comitee.) Leipzig, Teubner, 1899. 8vo, 92 pp.

THE committee in charge of the unveiling of the Gauss-Weber monument at Göttingen has published a memorial volume intended to commemorate the celebration and to serve as a worthy tribute to the genius of the two great men of science. Two professors of the University of Göttingen present in this volume their investigations concerning the foundations of the exact sciences: Professor Hilbert treats of the foundations of geometry; Professor Wiechert discusses the foundations of electrodynamics. The present notice deals only with the former of these memoirs.

It is the object of geometry to analyze and describe our space intuition. The abstraction from spatial intuition leads to three systems of objects: *points*, *straight lines*, and *planes*, which as elements of such intuition, must lie at the basis of any description of space. By means of definitions these elements are brought into certain correlations for which geometry tends to establish general laws. In order to obtain in this way a logically consistent system of propositions certain requirements, called *axioms*, must be satisfied by all imaginable mutual relations between the elements.

Among the axioms of geometry two types can be distinguished: axioms of position and axioms of magnitude. The axioms must have immediate general validity and form a system of propositions independent of each other, not further reducible, and not in contradiction one with any other. Only on the basis of such axioms is any geometrical *definition* possible; that is, a definition provided it be thinkable, gains its meaning only when it can be shown from the axioms that it has a real content. In addition to these requirements it can be demanded of a system of axioms that it be *simple*, in other words, that the least possible number of propositions be used to establish and sharply circumscribe the relations between the elements, none of the axioms being redundant, *i. e.*, deducible as a corollary from any of the others. The addition of the requirement of *completeness* of the system of axioms will have a meaning only with regard to the particular purpose in view. It is possible to detach (as Professor Hilbert repeatedly does in his researches) certain axioms from the system and build up by means of them alone a geometry forming a logically consistent system and leading to no contradictions. It is, however, perfectly legitimate to inquire what is the complete system of axioms required for laying the foundation of analytic geometry.

Euclid's system of geometry has always been open to objection on two points: the introduction of the axiom of parallels, and the doctrine of proportions and areas. While the latter point, since Euclid's times, has hardly been essentially improved, it is well known how numerous have been the investigations concerning the former. The question whether Euclid's eleventh axiom can be deduced from his other axioms has finally been answered in the negative, a non-euclidean geometry having been constructed by Gauss, Lobachevsky, and Bolyai. The new methods that brought about this final settlement of the old controversy over a much debated problem have led to entirely new views concerning the investigation of the axioms in general. They made it possible for Riemann, Helmholtz, and Lie to found geometry on an analytical basis, a method very different from Euclid's. By conceiving of space as a manifold of numbers these authors dispose at once of a number of geometrical axioms without the necessity of investigating them in detail. In sharp contrast to these analytical attempts we have the purely geometrical researches of Professor Veronese, and also the investigations of Professor Hilbert.

It is our author's aim to lay the proper foundations for euclidean geometry, and beyond this, for analytic geometry. His system thus finds its conclusion with the final recognition that space can be regarded as a manifold of numbers. Among the more important points in which Professor Hilbert's memoir marks a distinct advance I wish to call particular attention to the following:

(1) The introduction of the axioms of congruence, and the definition of motion as based on these; (2) the systematic investigation of the mutual independence of the axioms, this independence being proved by producing examples of new geometries which are in themselves interesting; (3) the principle of not merely proving a proposition in the most simple way but indicating precisely what axioms are necessary and sufficient for the proof; (4) the theory of proportions and areas without use of the axiom of continuity, and more generally, the proof that the whole of ordinary elementary geometry can be treated without reference to the axiom of continuity; (5) the various algebras for segments (Streckenrechnungen), in connection with the fundamental principles of arithmetic.

We now proceed to the discussion of particular points.

In the first chapter all the axioms are classed in five main groups. Group I, comprising the *axioms of combination* (Verknüpfung), contains two plane axioms, viz: (I, 1) Any two different points A and B determine a straight line a; (I, 2) any two different points of a straight line determine this line; and five space axioms (the only space axioms of the whole system), viz: (I, 3) any three points not on the same straight line determine a plane; (I, 4) any three points of a plane, not on the same straight line, determine this plane; (I, 5) if two points of a straight line lie in a plane, every point of the line lies in this plane; (I, 6) if two planes have a point in common, they have at least one other point in common; (I, 7) on any straight line there exist at least two points, in any plane at least three points not on a straight line, and in space at least four points not in a plane.

The group II, forming the *axioms of order* (Anordnung), contains four linear axioms, about the order of points on a straight line, such as (II, 3) among any three points on a straight line there always is one, and only one, that lies between the other two; and one plane axiom, viz: (II, 5) Let A, B, C be three points not on a straight line, and a a straight line in the plane ABC, not meeting any one of the points A, B, C; then, if the line a passes through a point within the segment AB, it will always pass through a point of the segment BC, or through a point of the segment AC.

These axioms are sufficient to show that a straight line contains an unlimited number of points, that it divides the plane into two regions, that any one of its points divides the line into two half-rays. It becomes possible to define polygons and it can be proved that a simple polygon divides the plane into two regions. The following very convenient definition of an *angle* might find its place here: an angle is the system of two half-rays issuing in a plane α from the same point O and belonging to different straight lines. The interior of the angle is the region that contains completely the segment joining any two internal points.

Euclid's *axiom of parallels* (group III), "whose introduction *simplifies* the foundations and *facilitates* the building up of geometry" is given in the form: In a plane α it is always possible to draw through a point A, not on the straight line a, one and only one straight line not meeting the line a.

In the fourth group we find, besides the axioms about the congruence of segments and angles, the following: (IV, 6) If for two triangles ABC and $A'B'C'$ the congruences

$$AB \equiv A'B', \quad AC \equiv A'C', \quad \measuredangle BAC \equiv B'A'C'$$

are true, then the congruences

$$\measuredangle ABC \equiv \measuredangle A'B'C' \quad \text{and} \quad \measuredangle ACB \equiv A'C'B'$$

are also satisfied. It is to be noted that, according to Professor Hilbert's definitions, congruence and symmetry are originally not distinguished.

Among the corollaries of the *axioms of congruence* attention may be called to the proof for the congruence of all right angles, a proposition which in Euclid appears as the fourth postulate. It is obvious how the four groups of axioms so far mentioned can serve to define motion in space, even in a non-euclidean space. The circle is defined in the usual way.

The *axiom of Archimedes* (or axiom of continuity), forming group V, completes the system, which contains in all 8 linear, 7 plane, and 5 space axioms. It is stated as follows: (V) Let A_1 be any point on a straight line between two arbitrarily given points A and B; construct the points $A_2, A_3, A_4, \cdots$ so that A_1 lies between A and A_2, A_2 between A_1 and A_3, and so on, and that the segments

$$AA_1, \quad A_1A_2, \quad A_2A_3, \quad A_3A_4, \cdots$$

are all equal; then among the points $A_2, A_3, A_4, \cdots$ there will always be a point A_n such that B lies between A and

A_n. While this formulation of the axiom enables us to define the equality of segments in the sense of general projective measurement, it does not involve the continuity of the straight line in the ordinary sense; it only furnishes a condition necessary for an algebra of segments. It would be best to avoid in this connection the use of the term continuity altogether; indeed, the axiom of Archimedes does not relieve us from the necessity of introducing explicitly an axiom of continuity, it merely makes the introduction of such an axiom possible. Thus, for the whole domain of geometry, Professor Hilbert's system of axioms is not sufficient. For instance, while it follows from this system that a circle and a straight line have in common two points, or one point, or no point, it would be impossible to decide geometrically whether a straight line that has some of its points within and some outside a circle will meet the circle; in other words, it remains undecided whether or not the circle is a closed figure. It also follows, for instance, that a right-angled triangle cannot in general be constructed from the hypotenuse and one side.

Is the system of axioms outlined above consistent in itself? Does it not contain any statement, or statements, whose application may finally lead to something unthinkable or self-contradictory? As geometry is built up by the indefinitely repeated application of the axioms, the possibility is not excluded that a contradiction might appear only after an unlimited repetition of such application. J. H. Lambert compares the axioms to as many equations that can be combined in innumerable ways. Professor Hilbert, to decide the question of consistency, imagines the domain of an enumerable ensemble of numbers and represents a point by two numbers of the domain, a straight line by the ratios of three numbers. With the aid of certain conventions about the order of the points on a line, etc., about translation and rotation, a geometry is thus defined for which all five groups of axioms hold. The question is thus transferred from the domain of geometry to that of arithmetic; any contradiction in the geometry must appear in the arithmetic of the imagined domain of numbers. But just because the question is merely transferred, the same problem remains open for arithmetic. It would seem desirable to find a decision in the geometrical domain itself and not to leave it to a lucky chance of future times. The importance of a final decision as to the absence of contradictions among the axioms is apparent; it is higher even than the question as to their mutual independence.

In the present memoir the author discusses the axiom of parallels, the congruence axiom for triangles, and the axiom of Archimedes, each as to its independence of the axioms of the other groups. An exhaustive investigation of the whole subject of the mutual independence of axioms was given by Professor Hilbert in a course of lectures on euclidean geometry in the University of Göttingen, 1898–99, which thus supplements the printed memoir. The method of proof in every case consists in showing that there exists a consistent system of postulates by means of which it is possible to construct a geometry in which the axiom under discussion does not hold.

Thus, to prove the independence of axiom (I, 5) of all other axioms of the first group, Professor Hilbert proceeds as follows. Let the points of the new geometry be those of euclidean space with the exception of a single one, O; let the planes be planes; and let us take as straight lines the circles through the point O, which does not exist in the new geometry. For this geometry all the axioms of the first group hold excepting the fifth. It should be observed that in this method of reasoning "the properties of euclidean space are used merely as abbreviating notations for certain *arithmetical* relations."

The proofs for the independence of the axioms become the more involved the greater the number of axioms regarded as valid. In his lectures Professor Hilbert enters upon a broader discussion of these questions, especially in regard to non-euclidean geometry, the proof of the congruence theorems in this geometry, etc.

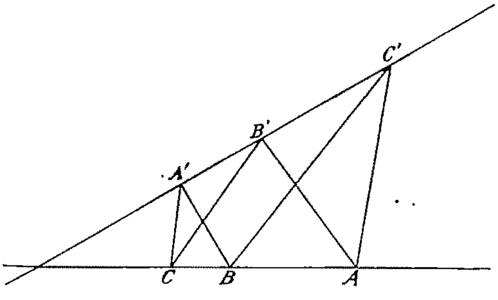


FIG. 1.

For the further development of *plane* geometry two well-known propositions appear to be of fundamental importance; the recognition of this fact must be regarded as an important advance. The author denotes these propositions briefly as Pascal's and Desargues's, formulating them as follows:

Pascal's proposition: Let A, B, C and A', B', C', respectively, be any three points on each of two intersecting straight lines, all different from the point of intersection; then, if CB' be parallel to BC' and CA' parallel to AC', BA' will be parallel to AB'.

Desargues's proposition: If two triangles be situated in a plane in such a way that any two corresponding sides are parallel, then the joins of the corresponding vertices pass through one and the same point or are parallel.

The proof of Pascal's proposition, as a theorem of plane geometry, is readily obtained by means of the axioms (I, 1), (I, 2), II, III, and IV (axioms of congruence), without the aid of the principle of Archimedes. The essential difference between the two proofs given by Professor Hilbert for this proposition lies in the fact that in the second proof not all the axioms of congruence are used, the axiom of congruence for triangles being replaced by one for isosceles triangles.

By devising an algebra of segments (Streckenrechnung) based on Pascal's proposition the true import of this theorem for the construction of the system of geometry is brought out very clearly. The sum of two segments on the same line being defined in the usual way, let the product be defined as follows: On one side of a right angle lay off from the vertex O the segment a, on the other the segments 1 and b; then draw $1a$ and through b the parallel to $1a$; this parallel will cut off on the other side a segment c (counted from O) which is defined as the product

$$c = a \cdot b$$

of the segment a into the segment b.

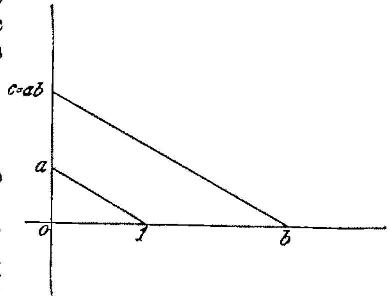


FIG. 2.

In this algebra of segments the commutative and associative laws hold of course for the sum; but they hold also, and this is the meaning of Pascal's proposition, for multiplication. Finally, it can be shown that the distributive law,

$$a(b + c) = ab + ac,$$

holds likewise.

It is obvious how closely this algebra is connected with the theory of proportions. Let the proportion

$$a : b = a' : b',$$

where a, b, a', b' are any segments, be defined as equivalent to the equation of segments

$$a \cdot b' = b \cdot a'.$$

If, moreover, similar triangles be defined in the usual way, it is easy to prove, with Professor Hilbert, on the basis of the algebra of segments, the general validity of the theorem of proportions, and it can further be shown that a straight line is represented by a linear equation.

While this use of Pascal's proposition for the theory of proportions is certainly an important advance it is still more surprising to see the great importance assumed by this proposition as basis for the theory of the areas of plane figures.

Two polygons are said to have *equal area* (flächengleich) if they can be resolved into a finite number of triangles that are congruent in pairs. They are said to have *equal content* (inhaltsgleich) when it is possible to add to them polygons of equal area so that the new polygons so arising have equal area. These two definitions are quite distinct because the investigation is to be carried on independently of the validity of the axiom of Archimedes. It can then be proved that rectangles, parallelograms, and triangles of equal bases and equal altitudes have *equal content*. The fundamental theorem formed by the converse of the last proposition, viz., *if two triangles of equal content have equal bases, they must have equal altitudes*, requires the introduction of the idea of *measure of area* (Flächenmass), which in the case of the triangle is simply half the product of base and altitude ; the theorem is then readily proved by very clear, though somewhat lengthy, considerations. As an elegant application of these results appears finally the proposition (previously discussed by other authors):

If, after cutting up a rectangle by means of straight lines into a number of triangles, any one of these triangles be omitted it will be impossible to make up the rectangle from the remaining triangles.

As regards Desargues's proposition it is known that it can easily be proved with the aid of all the axioms of group I (including the space axioms) as well as those of groups II and III. This fact can be expressed by saying that the existence of Desargues's proposition in the plane is a necessary condition if the plane geometry is to be part of a solid geometry, or the plane a part of space. Leaving out the space axioms it is impossible to prove Desargues's proposition with the aid of the remaining among the above-named axioms. Indeed, Professor Hilbert shows that this proposition cannot even be true in a plane geometry in which *all* axioms hold excepting the axiom of congruence for triangles. This proof will be read with great interest inasmuch as it leads to the construction of such a geometry which invites further investigation. It must, however, not be regarded as proved that the axioms of Hilbert's system are all necessary for the truth of Desargues's proposition; it is not impossible that its existence is independent of the axiom of parallels.

The importance of Desargues's proposition in the system of geometry and its relation to Pascal's proposition appear clearly from an algebra of segments based upon it. In this algebra, in which the constructions do not differ from those of the algebra referred to above except in so far as an arbitrary angle takes the place of the right angle, the associative and commutative laws hold for addition; for multiplication the associative and distributive laws are true, but not the *commutative* law.

The equation of the straight line is found to be of the form

$$ax + by + c = 0,$$

a, b, c being segments of the given system, x and y coördinates; in the products ax and by the order of the letters is essential. It can now be shown analytically that a solid geometry is possible in which all the axioms I to III hold. It follows that, the plane axioms of group I and the axioms of groups II and III being assumed, the existence of Desargues's proposition is the necessary and sufficient condition under which the plane geometry is part of the solid geometry based on those axioms.

To examine further the relations of Desargues's to Pascal's proposition we may proceed as follows, with the aid of the algebra of segments based on Desargues's theorem.

Let us take on one side of an arbitrary angle the segments

$$OA = a, \ OB = b, \ OC = c,$$

and on each side of the angle a unit segment

$$OE = 1, \ OE' = 1.$$

If through A, B, or C a parallel be drawn to the "unit line" EE', meeting the other side at A', B', C', then we put

$$OA = OA' = a, \ OB = OB' = b, \text{ etc.}$$

The sum of two segments b and c is formed by the usual rules: In the figure, $B'L$ is drawn parallel to OC, CL parallel to OB', and LD parallel to BB', which is parallel to EE'; hence

$$OB = CD$$

and

$$OD = b + c.$$

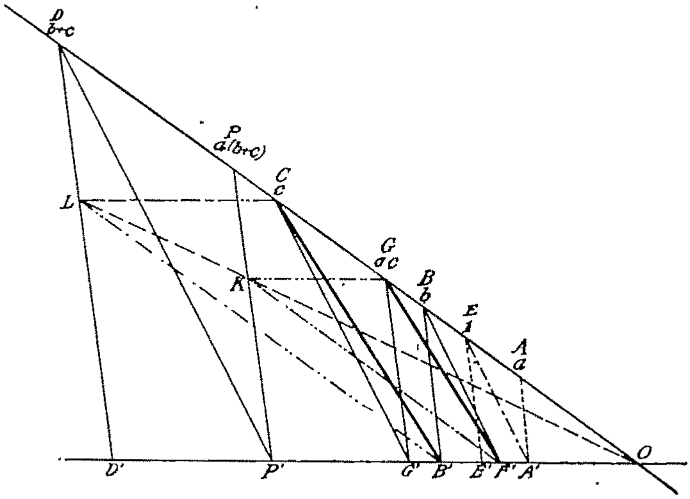


FIG. 3.

The product of the segment a into the segment b is defined simply by drawing through B a parallel BF' to EA' and putting

$$OF' = a \cdot b.$$

The commutative law does not hold for this product because we have not introduced the axioms of congruence, nor the axiom of Archimedes, so that these axioms cannot be used for the proof. Professor Hilbert has shown, however, that the distributive law holds for our multiplication, *i. e.*, we have for instance

$$a(b + c) = ab + ac.$$

The figure shows the construction of the two expressions $a(b + c)$ and $ab + ac$: we have

$$OF' = ab, \qquad OG' = OG = ac,$$
$$OD = b + c, \qquad OP' = OP = a(b + c) = ab + ac.$$

The distributive law requires that the line PP' which is parallel to AA' should pass through K.

It is now seen without difficulty that there exists one and only one segment x corresponding to the equation

$$x \cdot a(b + c) = ab.$$

At this stage we add explicitly to the axioms so far used, viz. (I, 1), (I, 2), II, III, the assumption that in our algebra the commutative law shall hold for multiplication on a straight line. Then the equation

$$ax(b + c) = ab$$

is true, and hence the segment x also satisfies the equation

$$x(b + c) = b,$$

which means for the figure the parallelism of the lines $F'P$ and $B'D$.

Thus the sides of the triangles $F'PK$ and $B'DL$ are parallel in pairs so that, according to the converse of Desargues's proposition, the points O, K, L lie on a straight line. From the triangles $F'GK$ and $B'CL$ it follows further that

$$F'G \text{ is parallel to } B'C.$$

In the hexagon $BF'GG'CB'$ the points B and G' can be selected arbitrarily, also the directions

$$BF' \text{ parallel to } CG', \text{ and } BB' \text{ parallel to } GG'.$$

We are thus led to Pascal's proposition for two straight lines; and it can now easily be shown that the commutative law of multiplication and Pascal's proposition hold for the whole plane.

Without introducing the new assumption it would not have been possible to show that

$$F'P \text{ is parallel to } B'D;$$

i. e., Pascal's proposition would not hold.

Let us sum up these results in the two propositions:

(1) Pascal's proposition can be proved by means of the axioms I, II, III, and the commutative law of multiplication, *i. e.*, without any use of the axioms of congruence.

(2) Pascal's proposition can *not* be proved by means of the axioms I, II, III alone, *i. e.*, without using the axioms of congruence and the axiom of Archimedes.

The algebras of segments based on Desargues's and Pascal's propositions, in which the totality of all segments is regarded as a number system, leads the author repeatedly to discussions concerning the principles of arithmetic.

The real numbers form a system of objects subject to certain laws of combination and order and to the proposition of Archimedes. The twelve laws of combination enumerated by Professor Hilbert refer to addition, multiplication, division, and involve the laws of association, distribution, and commutation. There are four propositions of order. The proposition of Archimedes is as follows :

If $a > 0$ and $b > 0$ be any two numbers, it is always possible to add a a sufficient number of times to itself so that the resulting sum shall have the property :

$$a + a + \cdots + a > b.$$

Any system of objects that satisfies at least some of these 17 propositions is called by Professor Hilbert a *complex number system.*

The two algebras of segments, which do not satisfy the 17th postulate, are called *non-archimedean* number systems ; the segments of the algebra based on Desargues's proposition, in particular, form an *arguesian* number system.

Of particular importance is the following result of the investigation as to the mutual interdependence of the 17 propositions :

In any number system in which the proposition of Archimedes holds, the commutative law of multiplication can be deduced from the other laws of operation ; and :

There exist non-archimedean number systems in which the commutative law of multiplication can not be deduced from the other laws.

In connection with the investigation of the principles of geometry Professor Hilbert discusses geometrical constructions. The old problem of constructions performable with the ruler or with the compasses alone is generalized to the question of indicating all problems solvable only by drawing straight lines and laying off segments. It appears that this condition is satisfied by the geometrical problems that can be solved by means of the axioms I to V. The closely related question as to whether any given geometrical problem can be solved by drawing straight lines and laying off segments alone, is answered by Professor Hilbert on the basis of profound speculations in the theory of numbers published by him elsewhere.

I conclude this review by expressing the hope that the important new views on the foundations of geometry opened up in this memoir may soon become generally known and be introduced into the teaching of elementary geometry.

J. SOMMER.

GÖTTINGEN, OCTOBER, 1899.

KOENIGS' LECTURES ON KINEMATICS.

Leçons de Cinématique professées à la Sorbonne par GABRIEL KOENIGS, avec des notes par M. G. DARBOUX et par MM. E. et F. COSSERAT. Paris, Hermann, 1897. 8vo., x + 499 pp.

WITH this book Professor Koenigs begins the publication of a treatise consisting of two or three volumes, which is to present the development of a course of lectures on kinematics delivered annually either at the École Normale or the Sorbonne for the last eight years. The first volume, the first ten chapters of which were printed in 1894, is devoted to theoretical kinematics; the rest of the work will be occupied with applied kinematics.

Kinematics as a distinct science is of comparatively recent origin. The formulæ which give the variations of the coördinates of the points of a movable solid in space were published by Euler in 1750. D'Alembert suggested the importance of studying the laws of movements separately. Ampère drew a definite demarcation between mechanics and the geometry of movement, but his object was to develop kinematical science solely for its use in the theory of mechanisms; the term kinematics is due to Ampère. Previously, in his geometry of position, Carnot predicted a much wider career for this science then to be, by calling attention to the fact that mechanics and hydromechanics would be infinitely simplified if the theory of geometrical motions were thoroughly investigated, since then the analytic difficulties encountered in the study of equilibrium and motion would be reduced to the general principle of the communication of motions, which is only another form of the principle of action and reaction. In 1838 Poncelet included the geometric properties of moving bodies in his course at the Faculty of sciences of Paris; with the exception of the notions of Chasles, we owe to Poncelet the theory of the continuous motion of a solid in space. Willis, of

Cambridge University, published a treatise on kinematics in 1841, and twenty-one years later Resal's treatise on pure kinematics appeared.

The present form assumed by this body of doctrine in the exposition of mathematical science in France is due largely to the lectures of Bouquet, Darboux, and Tannery, who have treated kinematical questions with that absolute precision and unflinching rigor demanded by mathematics. The influence of Tannery's course, given for several years at the Faculty of sciences of Paris, is especially notable; here we find the time relegated to its proper rôle as an auxiliary variable, and kinematics itself interpreted as the geometry of displacement.

Koenigs' treatise promises to be a classic. In facing the formidable array of researches and methods of Poinsot, Chasles, Bonnet, Ribaucour, Darboux, and a host of others, the author must have experienced no little difficulty in choosing a method of exposition of the subject matter. A treatise that is to be both scientific and didactic must consider the demands both of the student and of the scientist. Knowledge is of most use when accompanied by a kit of tools, and the instructor's art is a double one—he must not only present the facts of a subject with reasons for faith in them, but also employ those methods which promise results at the hands of the independent investigator. Further, the method should have the necessary breadth and unity, and be possessed of the clearness and directness of geometry and the power and generality of analysis. This multiplicity of demands Professor Koenigs has met admirably by basing his exposition on the geometry of the straight line and employing the mobile trieder of reference; the latter in the hands of Ribaucour and Darboux has proved itself to be the most certain and powerful implement yet used in infinitesimal geometry and it naturally lends itself with equal facility and elegance to the geometry of displacement.*

The first chapter is taken up with preliminary geometric notions, principally the theory of segments. The author regrets in his preface that the latter theory has not yet found a place in elementary mathematics. Its introduction in a work on kinematics is as annoying as that of kinematics in a treatise on dynamics. The theory belongs to geometry and, as Koenigs pertinently remarks, the business

---

* It may be remarked that the English writers on mechanics have also used the method of moving axes for a long time with success; see, for example, Routh's treatise on rigid dynamics.

of the geometer is not alone to discover new facts, but also to classify results already found and to group together ideas of the same kind. The chapter considers in turn projections, moments, systems of segments, couples, the theory of the right line, and concludes with a beautiful application of the theory of complexes to systems of segments.

The notions of motion, velocity, and acceleration are presented in the second chapter. The velocity is defined both as an algebraic and as a geometric function of the time. The hodograph is introduced to define the acceleration and deduce its properties. Among the applications are harmonic, uniform helicoidal, and uniformly varied rectilinear motions.

The third chapter is devoted to the change of the system of reference and the study of relative motion. The fundamental relation between the absolute velocity, the relative velocity, and the velocity of restraint is followed by the derivation of the fundamental formulæ of kinematics which express the projections of the absolute velocity on the axes of the mobile trieder. These are applied to the composition of velocities and lead directly to the theorems of Poinsot and Roberval for the construction of tangents and normals.

The author gives an interesting historical note relative to Roberval's method for constructing tangents. It was discovered in 1636 and communicated by Roberval to Fermat in 1640. It was published in the sixth volume of the memoirs of the Paris academy of sciences. Roberval's statement of the theorem is incorrect; this error was reproduced by Montucla in his history and by Monge in his descriptive geometry; the latter also gave faulty applications. These errors were rectified by Duhamel in a note inserted in the fifth volume of the *Savants étrangers*.

In concluding the chapter Koenigs shows how the composition of velocities may be utilized for the construction of tangents to conchoids of the most general form.

The fourth chapter analyzes in detail the nature of the motion of one invariable system with respect to another; it is one of the most successful parts of the volume. Its sections bear the captions: distribution of velocities in a moving solid, helicoidal motion, direct demonstrations of the helicoidal form of every infinitesimal displacement, relations between the theory of linear complexes and that of the displacement of a solid, determination of a continuous motion when the rotations are known, and inverse movement. The reviewer regrets that he has space only to call attention to the author's presentation of the geometric method of Chasles which has been treated with fulness in the work of Schoenflies, to the analogy between the ideas of Chasles and the method of infinitesimal transformations of Sophus Lie, and to the application to space curves of the method of the mobile trieder as developed analytically in the theory of surfaces of Darboux.

The fifth chapter defends Koenigs' choice of method against all comers; in less than seven pages we find the complete theory of acceleration in relative motion developed, the fundamental formulæ of Bour established and interpreted geometrically, the theorem of Coriolis demonstrated, and the distribution of accelerations in a moving solid studied.

The sixth and seventh chapters are devoted to the motion of a plane figure in its plane, the latter being taken up largely with examples. Among the many details of these two chapters we observe, with interest, Koenigs' generalization of Savary's formula, the study of trajectories in the vicinity of a point, the simple presentation of the principle of planimeters, and Koenigs' generalization of Steiner's theorem relative to roulettes.

Of the various movements of a body, two are particularly important: the first is that in which a plane of the body slides upon itself; the second is that in which a spherical shell of the body slides upon itself, or what amounts to the same thing, in which the body has a fixed point. The two preceding chapters considered the first case; the eighth chapter studies the second. The theorem of Rivals and the analogue of Savary's theorem are established by the same method employed in the case of plane motion; the formulæ of Euler and Olinde Rodrigues terminate the chapter.

The ninth chapter considers the most general continuous motion of a solid body. Koenigs studies first the curves connected with a moving figure which are possessed of envelopes and shows by the method of Darboux that these curves can be obtained by quadratures. In the most general motion of an invariable figure there is always a ruled surface Δ_m which joins at each instant a ruled surface Δ_f along a rectilineal generator Δ, which is found to be the axis of the tangent helicoidal motion; this particular rolling of Δ_m upon Δ_f, Reuleaux designated by the term viration; the author studies the distinctive character of the viration in the general rolling of ruled surfaces. He considers the applicability and deformation of ruled surfaces, the rolling of developables and space curves, and concludes with certain propositions relative to the acceleration in the motion of a solid body.

The heading: degrees of liberty of a moving system, movements with several parameters, gives the key to the tenth chapter. The various sections are occupied with the degrees of liberty of a solid body, the motion of a body subject to four conditions, and movements dependent on three parameters.

The eleventh chapter devoted to articulate systems is one of the most extensive of the volume. It is introduced with a valuable historical sketch recounting the researches* of Scheiner, Watt, Peaucellier, Kempe, Hart, Lipkine, Tchebicheff, Sylvester, Clifford, Roberts, Cayley, Saint-Loup, Laisant, Lemoine, and Darboux. The general reader may be interested in Neuberg's résumé of his elementary conferences on linkwork and Liguine's bibliography† to which the author refers. This chapter is admirably constructed and is enriched by Koenigs' own contributions which have been of both a practical and theoretical kind. The material is arranged in the following order—systems having four members, link transformers, exact or approximate determination of the rectilineal motion of a point, motions and transformations which can be realized by the same linkwork, applications of linkworks to the resolution of equations and the representation of functions, linkworks in space, general theorems on linkworks; among these last mentioned theorems there appears the very remarkable one of the author that every algebraic space curve or surface can be described by means of linkwork.

The last chapter, on displacement as a particular case of homography, introduces a new chapter in kinematics and is of itself a superb geometrical memoir. The reviewer must content himself however with a mere index of the sections —movement of a plane figure, displacement of a figure in space, imaginaries in the kinematics of the plane, linear substitutions of one variable and displacements about a fixed point.

The volume is happily supplemented by the notes of MM. Darboux, E. and F. Cosserat, and the author.

In his first note Professor Darboux presents a new demonstration of the formulæ of Euler and Olinde Rodrigues, which is direct and yields an immediate interpretation of the parameters.

---

* The names follow in the order taken up by Koenigs.

† *Bulletin des sciences mathématiques*, 2d series, vol. 7.

The second note of Darboux studies the reversions and plane inversions. By reversion (*renversement*) Darboux means the rotation of a figure through 180° about a straight line in space, and by plane inversion a symmetry with regard to a plane. These two transformations simplify the study and composition of displacements.

Darboux's third note is an elaborate memoir on algebraic movements. It consists of three divisions: movement of which all the trajectories are plane, use of the formulæ of Olinde Rodrigues in the study of algebraic movements, motions in which certain points of the body describe planes or plane curves.

The note of MM. Cosserat is an extract of a memoir published in the *Toulouse Annales* on the kinematics of a continuous medium. The introduction of this note is peculiarly fortunate for it is high time that kinematics should comprehend the study of deformation and of deformable spaces. The authors have included in their extract certain generalities on curvilinear coördinates, the deformation of a continuous medium in general, infinitely small deformation, use of the mobile trieder, and the case where the non-deformed medium is referred to any curvilinear coördinates.

Koenigs' own notes are eleven in number and on the following subjects: tetraedral coördinates of segments, the theory of Grassmann, infinitisimal properties of linear complexes, the expression of the virtual work of forces applied to a solid body, the volumes generated by a closed contour, the problem of centers of curvature in the movement of a plane figure, accelerations, Ball's theory of screws, the cylindroid, the composition of rotations and quaternions, and graphical representation.

The mechanical execution of the book is excellent and the marginal headings and leaders are especially useful. The mathematical public would welcome an announcement from M. Hermann that the second volume of Koenigs' work is ready.

E. O. LOVETT.

PRINCETON UNIVERSITY.

NOTES.

The British association for the advancement of science will hold its next annual meeting at Bradford opening September 5th. Sir William Turner is president of the association and Dr. J. Larmor is president of the mathematical section.

The annual meeting of the German mathematical association will be held simultaneously with the seventy-second meeting of the German association of scientists and physicians at Aachen, September 17–22, 1900. It is requested that the announcements of all mathematical papers intended for presentation be sent in time to reach Professor A. Gutzmer, Wildstrasse 2, Jena, before the first of May. Professor D. Hilbert is president of the association for the present year.

For the regular meeting of the London mathematical society to be held March 8th, a paper by Mr. W. F. Sheppard "On the use of the curve of error as an auxiliary curve in statistics," was announced.

At the meeting of the London royal society, held March 1st, Professor Karl Pearson presented, under the title, "On the correlation of characters not quantitatively measurable," the eighth of his series of papers of mathematical contributions to the theory of evolution.

The following papers were presented at the regular meeting of the Edinburgh mathematical society held March 9th: —"A note on change of coördinate axes," by Professor Steggall; "The condition for multiple roots of the equation in λ, $(a-\lambda, b-\lambda, \cdots)=0$," by Mr. C. Tweedie; "The analytical representation of a potential function by means of cylindrical and spherical harmonics, with applications to Green's problem," by Mr. J. Dougall.

At a meeting of the Royal Irish academy held at Dublin, January 22d, 1900, the Reverend W. R. Westropp Roberts read a paper "On the reduction of the integral $\int \frac{\varphi(z)\,dz}{\psi(z)\sqrt{f(z)}}$ to a number of other integrals of the form $\int \frac{dz}{(z-n)\sqrt{f(z)}}$, where $\varphi(z)$ and $\psi(z)$ are rational integral functions and $f(z)$ a polynomial of degree $2m$." At the meeting of this academy on February 12th, Professor Charles J. Joly read a paper on the place of the Ausdehnungslehre in the general associative algebra of the quaternion type.

Two mathematical papers were presented at the meeting of the Literary and philosophical society of Manchester, February 6th; both were by Mr. R. F. GWYTHER; they were entitled "On the conditions of propagation of a solitary wave" and "On the motion of the particles in certain cases of steady fluid motion."

THE Australasian association for the advancement of science met at Melbourne January 9th. The presidential address was delivered by Mr. R. L. J. ELLERY on "The beginnings and growth of astronomy in Australia"; and the presidential address of the mathematical section by Mr. G. H. KNIBBS on "The development of the atomic theory of matter." The next meeting of the association will be held at Hobart in January, 1902.

THE second part of Professor A. R. FORSYTH's Theory of differential equations, consisting of volumes II. and III. of his treatise, and devoted to non-linear ordinary equations, has recently been issued by the Cambridge University Press.

PROFESSOR É. PICARD presented the first part of the second volume of his work "Sur la théorie des fonctions algébriques de deux variables" to the Paris academy of sciences, March 5, 1900. This volume, as was the first, is published with the collaboration of M. SIMART. The part just issued deals principally with geometry on an algebraic surface. The point of view is chiefly that of algebra and analytical geometry, as adopted by Noether and the Italian mathematicians, although the transcendental position taken in the first volume of the treatise has not been entirely abandoned and will be resumed in the second part of the second volume. This second part will be occupied with the completion of the theory of double integrals of the second species; and a study of particular surfaces, especially those associated with abelian, hyperfuchsian and hyperabelian functions, will terminate the volume.

THE following volumes are announced as either in preparation or under the press for Teubner's series of textbooks in the mathematical sciences: P. BACHMANN, Elementary theory of numbers; M. BÔCHER, The real solutions of ordinary linear differential equations of the second order; G. BRUNEL, Analysis situs; G. CASTELNUOVO and F. EN-

RIQUES, Theory of algebraic surfaces; F. DINGELDEY, Conics and systems of conics; F. DINGELDEY, Collection of problems for the application of the infinitesimal calculus; F. ENRIQUES, Principles of geometry; J. HARKNESS, Elliptic functions; G. KOHN, Rational curves; A. KRAZER, Theory of theta functions; R. VON LILIENTHAL, Differential geometry; G. LORIA, The theory and history of special algebraic and transcendental plane curves; R. MEHMKE, Graphical and numerical methods of computing, including reckoning machines; E. NETTO, Theory of combinations; E. PASCAL, Theory and application of determinants; S. PINCHERLE, Functional equations and functional operations; A. PRINGSHEIM, Lectures on number theory and function theory (the elementary theory of infinite algorithms and the analytic functions of a complex variable, vol. I. theory of numbers, vol. II. theory of functions); C. SEGRE, Lectures on algebraic geometry, with especial references to spaces of many dimensions; D. SELIWANOFF, Calculus of differences; M. SIMON, Elementary geometry; P. STÄCKEL, Differential geometry of higher manifoldnesses; O. STAUDE, Surfaces of the second order and systems of such surfaces; O. STOLZ and J. A. GMEINER, Theoretical arithmetic; R. STURM, Theory of geometrical relations; R. STURM, Cubic space curves; K. TH. VAHLEN, History of the fundamental theorem of algebra; K. TH. VAHLEN, History of Sturm's theorem; A. VOSS, Representation and development of curved surfaces; E. VON WEBER, Lectures on Pfaff's problem and the theory of partial differential equations of the first order; A. WIMAN, Finite groups of linear transformations; H. G. ZEUTHEN, The enumerative methods of geometry.

THE inaugural academic lecture of Professor OTTO HÖLDER, delivered on recently assuming a full professorship of pure mathematics at the University of Leipzig, is promised at an early date from the press of B. G. Teubner. The lecture was a semi-popular one, entitled "Anschauung und Denken in der Geometrie."

MESSRS. Carré and Naud, of Paris, announce that a monograph on Interpolation by Dr. C. A. LAISANT, and one on Elimination by Dr. H. LAURENT, are in preparation for the physico-mathematical section of Scientia, a series of monographs published with the object of presenting the recent discoveries, leading ideas, and notable advances in the evolution of science. The editorial board of the physico-mathematical part of this undertaking consists of Professors

Appell, Cornu, d'Arsonval, Haller, Lippmann, Moissan, Poincaré, and Potier.

DURING the summer semester 1900 the several universities mentioned below offer the following mathematical courses:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Synthetic geometry, four hours; Applications of the theory of elliptic functions, four hours; Theory of analytic functions, two hours; Colloquia and seminar.—By Professor R. LEHMANN-FILHÉS: Analytical geometry, four hours; Theory of hypergeometric series, one hour.—By Professor J. KNOBLAUCH: Integral calculus, four hours; Theory of partial differential equations, four hours; Theory of elliptic functions, one hour.—By Professor E. R. HOPPE: Integral calculus, four hours; Analytical mechanics, four hours; Elementary questions of philosophy, two hours.—By Professor K. HENSEL: Differential calculus, four hours; Theory of surfaces of second order, two hours; Analytical mechanics, four hours.—By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar.—By Professor L. FUCHS: Introduction to the theory of functions, four hours; Seminar.—By Professor G. HETTNER: Infinite series, products and continued fractions, two hours.

UNIVERSITY OF BONN.—By Professor L. HEFFTER: Theory of functions, four hours; Theory of invariants, three hours; Exercises in function theory, one hour.—By Professor H. KORTUM: Elements of infinitesimal calculus, four hours; Theory of series, two hours; Seminar, two hours.—By Professor R. LIPSCHITZ: Elementary mechanics, four hours; Mathematical seminar, two hours.

UNIVERSITY OF BRESLAU.—By Professor J. ROSANES: Analytical geometry of the plane, four hours; Elements of theory of invariants, two hours; Seminar, one hour.—By Professor R. STURM: Differential geometry, three hours; Curves and surfaces of the third order, three hours; Seminar, two hours.—By Professor F. LONDON: Introduction to the theory of elliptic functions, four hours.

UNIVERSITY OF GIESSEN.—By Professor M. PASCH: Plane analytical geometry, four hours; Elliptic functions, four hours; Seminar, one hour.—By Professor E. NETTO: Elements of algebra, four hours; Definite integrals, two hours; Seminar, one hour.—By Professor R. HAUSSNER: Theory of invariants and forms, two hours; Solution of numerical equations, one hour; Descriptive geometry, five hours.

UNIVERSITY OF GREIFSWALD.—By Professor W. THOMÉ: Analytical geometry, two hours; Applications of the infinitesimal calculus to geometry, two hours; Seminar, two hours.—By Professor E. STUDY: Theory of functions, four hours; Infinitesimal calculus, four hours; Seminar, one hour.

UNIVERSITY OF HALLE-WITTENBERG.—By Professor G. CANTOR: The Galois theory of the solution of algebraic equations, four hours; Seminar, two hours.—By Professor A. WANGERIN: Differential calculus, five hours; Elliptic functions, five hours; Seminar, two hours.—By Professor V. EBERHARD: History of analysis, one hour; Theory of functions, four hours.—By Dr. GRASSMANN: Analytical mechanics, two hours; Descriptive geometry, two hours.—By Dr. NEUMANN: Selected chapters of analytical geometry, two hours; Synthetic geometry of conic sections, two hours.

UNIVERSITY OF JENA.—By Professor J. THOMAE: Elliptic functions, four hours; Projective geometry, four hours.—By Professor G. FREGE: Analytical geometry, four hours; Mathematical exercises, two hours; Fundamental notions of mathematics, one hour. —By Professor A. GUTZMER: Differential calculus, four hours; Exercises in the preceding, one hour; Introduction to higher algebra, four hours.

UNIVERSITY OF KÖNIGSBERG.—By Professor F. MEYER: Introduction to higher geometry, four hours; Seminar, one and a half hours.—By Professor A. SCHOENFLIES: Differential calculus, four hours; Seminar, one and a half hours.—By Professor L. SAALSCHÜTZ: Fourier's series, two hours; Bernoüilli functions, three hours; Exercises in Bernouilli numbers and functions, one hour.—By Dr. T. VAHLEN: Theory of linear differential equations, three hours.—By Dr. MÜLLER: Plane analytical geometry and exercises, four hours; Graphical statics, two hours.

UNIVERSITY OF LEIPZIG.—By Professors W. SCHEIBNER and A. MAYER: no courses are offered during the summer semester.—By Professor C. NEUMANN: Theory of curved surfaces, four hours; Seminar, one hour.—By Professor O. HÖLDER: Elliptic functions, four hours; Calculus of variations, two hours; Seminar, one hour.—By Professor F. ENGEL: Ordinary differential equations, four hours; Theory of continuous groups, two hours; Analytical mechanics, one hour; Seminar, one hour.—By Dr. F. HAUSDORFF: Introduction to higher analysis, four hours; Insurance mathematics, three hours.—By Dr. G. KOWALEWSKI: Analytical geometry of plane and space, with exercises, five hours; The quadrature of the circle, two hours.—By Dr. H. LIEBMANN: Synthetic geometry, three hours; Graphical statics, one hour; Definite integrals, two hours.

UNIVERSITY OF MARBURG.—By Professor F. SCHOTTKY: Curvature of lines and surfaces, four hours; Theory of representation, three hours; Seminar, two hours.—By Professor A. E. HESS: Differential calculus, five hours; Foundations of modern geometry, two hours; Seminar, two hours. —By Dr. F. VON DALWIGK: Definite integrals, three hours; Descriptive geometry, six hours.

UNIVERSITY OF STRASSBURG.—By Professor T. REYE: Synthetic geometry, two hours; Technical mechanics, four hours; Seminar, one hour.—By Professor H. WEBER: Theory of functions, four hours; Hydrodynamics, two hours; Seminar, two hours.—By Professor G. ROTH: Infinitesimal calculus, three hours; Plane analytical geometry, two hours.—By Professor A. KRAZER: Definite integrals, three hours; Graphical statics, two hours; Determinants, two hours.—By Dr. E. TIMERDING: Theory of algebraic curves, two hours.—By Dr. J. WELLSTEIN: Algebraic equations, three hours.

UNIVERSITY OF WÜRZBURG.—By Professor F. PRYM: Integral calculus, six hours; Exercises on the preceding, two hours; Selected chapters of function theory, two hours.—By Professor A. VOSS: Analytical and synthetical geometry of conics, four hours; Analytical geometry of space, four hours; Selected chapters of higher mathematics, two hours.

AT the University of Göttingen, from April 19th to May 1st, there is to be given a series of special lectures in mathematics and physics intended as an Easter vacation course for teachers in the high schools of Germany. The lectures in mathematics include lectures by Professor F. KLEIN, on the teaching of applied and technical mechanics in the schools; by Professor G. BOHLMANN, on the elements of the mathematics of life insurance; and by Professor F. SCHILLING, on descriptive geometry.

MESSRS. Macmillan and Bowes, 1 Trinity Street, Cambridge, England, have issued a new list of second-hand mathematical books, under date of February, 1900.

THREE letters written by DE MORGAN to SYLVESTER in

1856, when the latter was professor of mathematics at the Woolwich military academy, appear in the January number of the *Monist*, with an introduction by Professor G. B. HALSTED, of the University of Texas.

THE University of Oxford has recently instituted the degrees of doctor of letters and doctor of science as advanced research degrees.

THE London mathematical society has passed a resolution permitting the increase of the number of its members by further elections to three hundred and fifty. The previous limitation was two hundred and fifty.

PROFESSOR G. G. STOKES, of Cambridge University, has been elected a foreign associate of the Paris academy of sciences.

AT the University of Cambridge, the Hopkins prize for the period 1891–94 has been awarded to Mr. W. D. NIVEN, F.R. S., for his memoir on ellipsoidal harmonics, published in the *Philosophical Transactions* for 1891 ; Mr. NIVEN was formerly a fellow of Trinity College.

AMONG the foreign correspondents elected by the Berlin academy of sciences at the recent celebration of the two hundredth anniversary of its organization, are Professor J. WILLARD GIBBS, of Yale University, and Professor H. A. ROWLAND, of Johns Hopkins University.

THE Royal Irish Academy has elected Professor P. G. TAIT, of Edinburgh, and Professor J. J. THOMSON, of Cambridge, to honorary membership.

LORD KELVIN has been elected a foreign member of the Berlin academy of sciences.

DR. MAX ABRAHAM has been admitted as Privatdocent at the University of Göttingen. In the coming summer semester he will give a course on the electromagnetic theory of light.

DR. HARRIS HANCOCK, instructor in mathematics at the University of Chicago, has been elected professor of mathematics at the University of Cincinnati.

DR. G. W. MYERS, professor of astronomy in the University of Illinois, has resigned to take charge of the department of mathematics and astronomy in Chicago Institute.

PROFESSOR R. L. MORSE, of Hanover College, has been made professor emeritus of mathematics, after twenty-five years of service in that institution; the active professorship thus made vacant has been filled by the election of Dr. R. S. LAWRENCE, of Emporia College.

PROFESSOR H. WEBER has been elected rector of the University of Strassburg.

THE honorary degree of doctor of laws has been conferred on Professor A. R. FORSYTH by Glasgow University.

A STATUE of Professor CHARLES HACKLEY has been recently presented to Columbia University. Professor Hackley held the chair of mathematics in Columbia College from 1843–1861.

PROFESSOR E. BELTRAMI died at Rome, February 18, 1900.

PROFESSOR JOSEPH L. F. BERTRAND, of Paris, died recently at the age of seventy-eight years.

THE deaths are also announced of Miss CATHERINE WOLFE BRUCE, the well known patron of astronomical science; of Professor THOMAS PRESTON, fellow of the Royal University of Ireland; and of Mr. J. J. WALKER, one of the early members of the London Mathematical Society which he had served both as president and vice-president.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOLTZMANN (L.). See CLARK UNIVERSITY.

BOREL (E.). Nouvelles leçons sur la théorie des fonctions. Leçons sur les fonctions entières. Paris, Gauthier-Villars, 1900. 8vo. 6 + 126 pp.

BRANDENBURGER (C.). Anwendung der elliptischen Funktionen auf durch algebraische Funktionen vermittelte konforme Abbildungen. (Diss.) Zürich, 1899. 8vo. 49 pp., 4 plates.

BRAUN (J.). Das Fortschreitungsgesetz der Primzahlen durch eine transcendente Gleichung exakt dargestellt. (Progr.) Trier, 1899. 8vo. 34 pp., 6 tables.

Cahen (E.). Eléments de la théorie des nombres, congruences, formes quadratiques; nombres incommensurables, questions diverses. Paris, Gauthier-Villars, 1900. 8vo. 8 + 404 pp.

Cavallin (P.). Determination och Multiplikation. Logisk-matematiska Undersökningar. Lund, 1899. 8vo. 7 + 185 pp.

Cayley (A.). An elementary treatise on elliptic functions. 2d edition. Cambridge, Deighton, Bell, & Co., 1900. 8vo. 7s. 6d.

Chebyshov (P.). Works, collected and edited by A. Markov and N. Sonin. (In 2 volumes.) Vol. I. St. Petersburg, 1899. 4to. 6 + 714 pp., portrait. (Russian.) M. 17.50

Clark University 1889–1899. Decennial celebration. Worcester, Mass., printed for the University, 1899. 4to. 6 + 566 pp., 5 portraits. (Contains: W. E. Story, The department of mathematics, pp. 61–83; A. G. Webster, Department of physics, pp. 85–98; E. Picard, Lectures on mathematics: (1) Sur l'extension de quelques notions mathématiques, et en particulier de l'idée de fonction depuis un siècle; (2) Quelques vues générales sur la théorie des équations differentielles; (3) Sur la théorie des fonctions analytiques et sur quelques fonctions spéciales, pp. 207–259; L. Boltzmann, Ueber die Grundprincipien und Grundgleichungen der Mechanik, pp. 261–309.)

Cotton (E.). Sur les variétés à trois dimensions. (Thèse.) Paris, Gauthier-Villars, 1899. 4to. 61 pp.

Dölp (H.). Aufgaben zur Differential- und Integralrechnung nebst den Resultaten und den zur Lösung nötigen theoretischen Erläuterungen. 8te Auflage, neu bearbeitet von E. Netto. Giessen, 1900. 8vo. 4 + 216 pp. Cloth. M. 4.00

Forsyth (A. R.). Theory of differential equations. Part II: Ordinary equations, not linear. Vol. II: 11 + 344 pp.; Vol. III: 10 + 391 pp. London and New York, Macmillan, 1900. 8vo. Cloth. (Cambridge University Press series.) $6.00

Franck (P.) Ueber die Flächeninhalte und Bogenlängen von Fusspunktkurven und Rollkurven. (Diss.). Leipzig, 1899. 8vo. 57 pp.

Garaycochea (M. W.). Calculo binomial o analisis trascendente de binomio; obra original inedita, comentada por D. F. Villareal. Parte I. Lima, Gil, 1898. 12mo. 19 + 806 pp. Fr. 10.00

Gauss (C. F.). See Olbers (W.).

Guldberg (A.). Sur une classe particulière d'équations aux dérivées partielles du premier ordre. Christiania, Dybwad, 1899. 4to. 17 pp. (*Videnskabsselskabets Skrifter*, I. Math.-naturv. Klasse, 1899, No. 8.)

Hahn (R.). Die Entwicklung der Leibnizischen Metaphysik und der Einfluss der Mathematik auf dieselbe bis zum Jahre 1686. Halle, 1899. 4to. 35 pp.

Hedrick (E. R.). On three-dimensional determinants. 4to. (*Annals of Mathematics*, 2d series, Vol. 1, No. 2, 1900, pp. 49–67.)

Heidke (P.). Ueber Kreisteilungsgleichungen vom Primzahlgrad $p = p^{\pi_1} \cdot p_2^{\pi_2} \cdots p_\mu^{\pi_\mu} + 1 (\mu > 1)$. (Diss.) Greifswald, 1899. 8vo. 26 pp.

Heineck (C.). Invariante Kurvenintegrale bei infinitesimalen Transformationen in drei Veränderlichen x, y, z und deren Verwertung. (Diss.) Leipzig, 1899. 8vo. 75 pp.

Hessenberg (G.). Ueber die Invarianten linearer und quadratischer binärer Differentialformen und ihre Anwendung auf die Deformation der Flächen. (Diss.) Berlin, 1899. 8vo. 50 pp.

Liebmann (H.). Ueber die Verbiegung der geschlossenen Flächen positiver Krümmung. (Habilitationsschrift.) Leipzig, 1899. 8vo. 32 pp.

Markov (A.). See Chebyshov (P.).

Massau (J.). Mémoire sur l'intégration graphique des équations aux dérivées partielles. Fascicule 1: Intégration fausse; intégration par les caractéristiques; mouvement varié des eaux courantes; mascaret. Paris, Hermann, 1899. 4to. 144 pp. (Autogr.) Fr. 5.00.

Mehling (A.). Ueber diejenigen Flächen, die äquidistante infinitesimale Biegungen gestatten. (Diss.) Würzburg, 1899. 8vo. 62 pp.

Netto (E.). See Dölp (H.).

Olbers (W.), sein Leben und sein Werke. Im Auftrage der Nachkommen herausgegeben von C. Schilling. Vol. II: Briefwechsel zwischen Olbers und Gauss. Abteilung 1. Berlin, Springer, 1900. 8vo. 7+767 pp. M. 16.00

Picard (E.). See Clark University.

Portuondo (A.). Escuela especial de ingenieros de caminos, canales y puestos. 1a parte: Calculo de probabilidades; 2a parte: Teoria de los errores; 3a parte: Metodo de los minimos cuadrados. Madrid, Fortanet, 1898. 8vo. 282 pp., 4 plates.

Schilling (C.). See Olbers (W.).

Sonin (N.). See Chebyshov (P.).

Story (W. E.). See Clark University.

Tchébycheff. See Chebyshov (P.).

Timchenko (Q.). Elements of the theory of analytic functions. Part I: Historical account of the development of the ideas and methods on which the theory of analytic functions is based. Vol. I. St. Petersburg, 1899. 8vo. (Russian.) M. 9.00

Vecchi (S.). Sulle figure complete determinate da un numero qualunque di punti o da un numero qualunque di tangenti di una conica e sulle loro correlative nello spazio. Parma, Rossi-Ubaldi, 1899. 4to. 16 pp., 2 plates.

Villareal (D. F.). See Garaycochea (M. W.).

Webster (A. G.). See Clark University.

Westfall (J. v. E.). On a category of transformation groups in three and four dimensions. (Diss.) Leipzig, 1899. 8vo. 32 pp.

Wölffing (E.). Verzeichnis der Zeitschriften für die Gebiete der Mathematik, der Physik, der Technik und der verwandten Wissenschaften, welche auf Württembergischen Bibliotheken vorhanden sind. Stuttgart, 1899. 8vo. 18 pp. M. 1.50

Zindler (K.). Ueber Complexcurven. (*Monatshefte für Mathematik und Physik*, Jahrgang XI.) 8vo. 3 pp.

II. ELEMENTARY MATHEMATICS.

ANDRÉ (P.). Nouveau cours de trigonométrie, rédigé d'après le programme officiel, à l'usage des lycées, des collèges, des institutions et des aspirants au baccalauréat ès sciences. 9e édition. Paris, André [1900.] 8vo. 96 pp.

BOLTE (F.). Leitfaden für den Unterricht in der Arithmetik, zum Gebrauche an Navigationsschulen bearbeitet. 2te Auflage. Hamburg, Peuser, 1900. 8vo. 77 pp. Boards. M. 2.00

BOSSE (L.) und MÜLLER (H.). Algebra. Mit Aufgaben und fünfstelliger Logarithmentafel. Berlin, Parey, 1900. 8vo. 4 + 124 + 24 pp. M. 1.80

BRÉMANT (A.). Cours d'algèbre essentiellement pratique, avec de nombreuses applications, à l'usage des élèves de l'année complémentaire-Nouvelle édition. Paris, Hatier [1900]. 16 mo. 76 pp.

BRUHNS (C.). Neues logarithmisch-trigonometrisches Handbuch auf 7 Decimalen. 5te Ausgabe. Leipzig, 1900. 8vo. 24 + 610 pp. (The same with text in English, French, Italian.) M. 4.20

DEKKER (P.). See NIEMÖLLER (F.).

FRÈRES (LES) des écoles chrétiennes. Géométrie, cours moyen. Paris, Poussielgue [1900]. 16mo. 191 pp. (Enseignement primaire; livres classiques rédigés en trois cours gradués pour chaque série du programme officiel.)

FULCHERIS (P.). Elementi di geometria ad uso delle scuole tecniche e normali. Testo e tavole. 17a edizione, riveduta ed ampliata. Torino, Paravia, 1900. 8vo. 147 pp., 18 plates. Fr. 2.00

GEISS (L. G.). Raumlehre, für höhere Mädchenschulen und Lehrerinnenseminare methodisch bearbeitet. 3te Auflage. Bensheim, Ehrhard, 1900. 8vo. 8 + 132 pp. Boards. M. 1.80

GETRERO (N. J. A. R.). Principios y ejercicios practicos de aritmetica. 2a edicion, aumentada con algunas teorias y mas de 400 problemas. Cadiz, 1899. 8vo. 7 + 132 pp. Fr. 4.00

GIROD (F.). Cours d'algèbre élémentaire théorique et pratique (No. 4), à l'usage des lycées et des collèges, etc., renfermant près de seize cents exercices. 16e édition, revue et augmentée. Paris, André, 1900. 8vo. 472 pp. Fr. 4.00

——. Cours de géométrie théorique et pratique, à l'usage des lycées et des collèges, etc., contenant de nombreuses applications au dessin, à l'architecture, etc. 16e édition. Paris, André, 1899. 8vo. 432 pp. Fr. 4.00

——. Solutions raisonnées des problèmes énoncés dans le Cours et dans le Traité élementaire d'algèbre à l'usage des lycées et des collèges, etc. 3e édition, revue et corrigée. Paris, André [1900].

——. Traité élémentaire d'algèbre théorique et pratique (No. 3), à l'usage des lycées, des collèges, etc., renfermant un grand nombre d'exercices. 13e édition. Paris, André, 1900. 8vo. 288 pp.

——. Traité élémentaire d'arithmétique théorique et pratique (No. 3), à l'usage des lycées et des collèges, etc. 17e édition. Paris, André, 1900. 8vo. 224 pp. Fr, 2.50

———. Traité élémentaire de géométrie, à l'usage des lycées, des collèges, etc. 12e édition. Paris, André, 1899. 8vo. 279 pp. Fr. 3.00

Guérin (A.). Note di geometria per gli allievi della classe quarta elementare. 4ta edizione, secondo i programmi 29 novembre 1894. Roma, Vallardi, 1899. 16mo. 24 pp. Fr. 0.15

———. Note di geometria per gli allievi della classe quinta elementare 3a edizione, secondo i programmi 29 novembre 1894. Milano, Vallardi, 1899. 16mo. 24 pp. Fr. 0.15

Grévy (A.). Arithmétique, à l'usage des élèves de troisième classique et moderne et de rhétorique. 2e édition. Paris, Nony, 1900. 18mo. 199 pp. Fr. 2.00

Guilmin (A.). Cours complet d'arithmétique (No. 1), à l'usage des lycées et collèges. Nouvelle édition. Paris, Belin, 1899. 8vo. 8 + 376 pp.

Juel (C.). Trigonometri til brug i tekniske skoler. Kjöbenhavn, 1899. 8vo. 60 pp. M. 2.00

M. (F. G.). Cours d'algèbre élémentaire, conforme aux derniers programmes de l'enseignement classique et de l'enseignement moderne. 2e édition. Paris, Poussielgue [1900]. 8vo. 568 pp.

Maillard (L.). Eléments de géométrie. Lausanne, Viret-Genton, 1899. 12mo. 238 pp.

Mandoli (C.). Trattato di algebra elementare ad uso dei licei. 2a edizione, riveduta dall'autore. Napoli, Trani, 1900. 16mo. 11 + 234 pp. Fr. 3.50

Monnet (L. F.). Cours élémentaire d'arithmétique théorique et pratique, à l'usage des lycées, des collèges, etc., contenant un très grand nombre d'exercices résolus ou à résoudre. 20e édition. Paris, André, 1899. 18mo. 360 pp. Fr. 2.00

———. Solutions des exercices et des problèmes énoncés dans le cours élémentaire d'arithmétique théorique et pratique à l'usage des lycées, des collèges, des écoles normales primaires, etc. 2e édition. Paris, André [1900]. 18mo. 356 pp. Fr. 2.00

Mougin (E.). Etude pratique sur la preuve par 7, par 9 et la preuve ordinaire dans l'addition, la soustraction, la multiplication, la division, la racine carrée et la racine cubique. Laval, Barnéoud, 1900. 16mo. 16 pp. Fr. 0.50

Müller (H.). See Bosse (L.).

Niemöller (F.) und Dekker (P.). Arithmetisches und algebraisches Unterrichtsbuch. Für den mathematischen Unterricht in der Mittelstufe (4tes bis 6tes Schuljahr) höherer Lehranstalten nach den Bestimmungen der preussischen Lehrpläne von 1892 bearbeitet. Heft 2: Pensum der Obertertia und Untersekunda (Sekunda und Prima der Realschulen). Breslau, Hirt, 1900. 8vo. 136 pp. Boards. M. 1.60

Pendlebury (C.). Short course of elementary plane trigonometry. London, Bell, 1900. 12mo. 172 pp. 2s. 6d.

Schwering (K.). Trigonometrie für höhere Lehranstalten. 2te Auflage. Freiburg i. B., Herder, 1900. 8vo. 7 + 53 pp. M. 0.80

SMITH (D. E.). The teaching of elementary mathematics. New York, The Macmillan Company, 1900. 12mo. 15 + 312 pp. Cloth. (Teachers' professional library.) $1.00

WENG (A.). Geometrie der Kegelschnitte im Realschulunterricht. Heilbronn, 1899. 4to. 47 pp.

III. APPLIED MATHEMATICS.

BÄCKLUND (A. V.). Elektrodynamik. Efter Författarens Universitets-Föreläsningar. Lund, 1899. 8vo. 198 pp. M. 6.00

BAYER (A.). Theorie der Transformatoren für dreiphasige Wechselströme. (Diss.) Zürich, 1898. 8vo. 106 pp.

BJERKNES (V.). Vorlesungen über hydrodynamische Fernkräfte nach C. A. Bjerknes' Theorie. Vol. I. Leipzig, Barth, 1900. 8vo. 16 + 338 pp. M. 10.00

FITZGERALD (G. F.). See LORD KELVIN.

GROSSMANN (L.). Die Mathematik im Dienste der Nationalökonomie unter Rücksichtnahme auf die praktische Handhabung der Disciplinen der Finanzwissenschaft und Versicherungstechnik mit einigen, durch selbständige wissenschaftliche Errungenschaften auf dem Gebiete der reinen Mathematik begründeten, neuen Fundamenten der politischen Arithmetik. Lieferung 11. Wien, Selbstverlag, 1900. 8vo. 4 + 80 pp. M. 5.00

GURZHÉEV (S.). Practical mechanics for intermediate technical schools, with a collection of problems with solutions. 3d edition. St. Petersburg, 1900. 8vo. 456 pp. (Russian.) M. 6.50

HECHT (K.). Lehrbuch der reinen und angewandten Mechanik für Maschinen- und Bautechniker. Vol. II: Die Festigkeitslehre. Dresden, Kühtmann, 1900. 8vo. 7 + 385 + 31 pp. M. 9.00

HERTZ (H.). The principles of mechanics presented in a new form. With an introduction by H. von Helmholtz. Authorized English translation by D. E. Jones and J. T. Walley. London and New York, Macmillan, 1900. 8vo. 28 + 276 pp. Cloth. $3.25

JONES (D. E.). See HERTZ (H.).

JONES (H. C.). The theory of electrolytic dissociation and some of its applications. New York, The Macmillan Company, 1900. 12mo. 12 + 289 pp. Cloth. $1.60

LORD KELVIN, professor of natural philosophy in the University of Glasgow, 1846–1899. With an essay on his scientific work by G. F. Fitzgerald. Glasgow, 1899. 4to. 7s. 6d.

MACH (E.). Die Principien der Wärmelehre, historisch-kritisch entwikkelt. 2te Auflage. Leipzig, Barth, 1900. 8vo. 12 + 484 pp., 6 portraits. M. 10.00

RILLET (J. J.). Géométrie descriptive (première partie : ligne droite et plan), suivie de notions sur les surfaces et sur les projections cotées. 2e édition. Paris, 1899. 16mo. 8 + 214 pp. (Encyclopédie du dessin et de la construction.) Fr. 2.50

SACERDOTE (P.). Recherches théoriques sur les déformations électrique des diélectriques solides isotropes. (Thèse.) Paris, Gauthier-Villars, 1899. 4to. 116 pp.

TALLQVIST (H.). Grunderna af potentialteorin med användning pa elektrostatiken och magnetismen. Helsingfors, 1899. 8vo. 11+253 pp. M. 5.00

THOMPSON (S. P.). Polyphase electric currents and alternate current motors. Enlarged edition. London, Spon, 1900. 8vo. 512 pp. 21s.

VOLKMANN (P.). Einführung in das Studium der theoretischen Physik, insbesondere in das der analytischen Mechanik, mit einer Einleitung in die Theorie der physikalischen Erkenntnis. Vorlesungen. Leipzig, Teubner, 1900. 8vo. 16 + 370 pp. M. 14.00

WALLEY (J. T.). See HERTZ (H.).

WALLIN (K.). Elektriciteten och dess användning for belysning och arbetsöfverföring. 2te upplaga. Stockholm, 1899. 8vo. 106 pp. M. 2.50

WALLON (E.). Leçons d'optique géométrique, à l'usage des élèves de mathématiques spéciales. Paris, Gauthier-Villars, 1900. 8vo. 6 + 344 pp.

WOLF (F. E.). Berechnung und Versuch zum Nachweiss der ponderomotorischen Wirkung von veränderlichen magnetischen Feldern auf electrostatisch geladene Körper. (Diss.). Greifswald, 1899. 8vo. 42 pp.

ZINGER (N.). Treatise on astronomy. Theoretical part. St. Petersburg, 1899. 8vo. 373 pp. (Russian.)

Binder :—Pages 295–6 here attached are to be substituted for those issued in the April number.

L RARY

tion with the aid of the remaining among the above-named axioms. Indeed, Professor Hilbert shows that this proposition cannot even be true in a plane geometry in which *all* axioms hold excepting the axiom of congruence for triangles. This proof will be read with great interest inasmuch as it leads to the construction of such a geometry which invites further investigation. It must, however, not be regarded as proved that the axioms of Hilbert's system are all necessary for the truth of Desargues's proposition; it is not impossible that its existence is independent of the axiom of parallels.

The importance of Desargues's proposition in the system of geometry and its relation to Pascal's proposition appear clearly from an algebra of segments based upon it. In this algebra, in which the constructions do not differ from those of the algebra referred to above except in so far as an arbitrary angle takes the place of the right angle, the associative and commutative laws hold for addition; for multiplication the associative and distributive laws are true, but not the *commutative* law.

The equation of the straight line is found to be of the form

$$ax + by + c = 0,$$

a, b, c being segments of the given system, x and y coördinates; in the products ax and by the order of the letters is essential. It can now be shown analytically that a solid geometry is possible in which all the axioms I to III hold. It follows that, the plane axioms of group I and the axioms of groups II and III being assumed, the existence of Desargues's proposition is the necessary and sufficient condition under which the plane geometry is part of the solid geometry based on those axioms.

To examine further the relations of Desargues's to Pascal's proposition we may proceed as follows, with the aid of the algebra of segments based on Desargues's theorem.

Let us take on one side of an arbitrary angle the segments

$$OA = a, \; OB = b, \; OC = c,$$

and on each side of the angle a unit segment

$$OE = 1, \; OE' = 1.$$

If through A, B, or C a parallel be drawn to the "unit line" EE', meeting the other side at A', B', C', then we put

$$OA = OA' = a, \; OB = OB' = b, \text{ etc.}$$

The sum of two segments b and c is formed by the usual rules: In the figure, $B'L$ is drawn parallel to OC, CL parallel to OB', and LD parallel to BB', which is parallel to EE'; hence

$$OB = CD$$

and

$$OD = b + c.$$

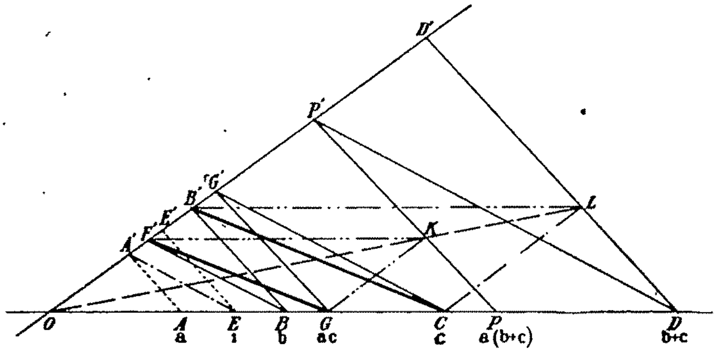


FIG. 3.

The product of the segment a into the segment b is defined simply by drawing through B a parallel BF' to EA' and putting

$$OF' = a \cdot b.$$

The commutative law does not hold for this product because we have not introduced the axioms of congruence, nor the axiom of Archimedes, so that these axioms cannot be used for the proof. Professor Hilbert has shown, however, that the distributive law holds for our multiplication, *i. e.*, we have for instance

$$a(b + c) = ab + ac.$$

The figure shows the construction of the two expressions $a(b + c)$ and $ab + ac$: we have

$$OF' = ab, \qquad OG' = OG = ac,$$
$$OD = b + c, \qquad OP' = OP = a(b + c) = ab + ac.$$

ON THE GEOMETRY OF THE CIRCLE.

BY DR. VIRGIL SNYDER.

(Read before the American Mathematical Society, December 28, 1899.)

LET $x_1, x_2, \cdots, x_5$ be any five numbers which satisfy the homogeneous quadratic identity $\omega(x) \equiv 0$; these numbers may be taken as the homogeneous coördinates of the circle.

The linear geometry of the circle can be readily interpreted from the paper of the author* on Dupin's cyclides; and the corresponding theorems for the quadratic configurations from another paper.† In the latter, one of the numbers x_6 was given a restricted interpretation, that of representing all the points of space; and the resulting theorems all referred to cyclides. A set of similar theorems exists for the bicircular quartic curves when, in the geometry of the circle, one coördinate equated to zero represents the points of a plane. All the known theorems regarding these curves, as given by Casey,‡ Darboux,§ and Loria,‖ can be very easily derived, and a number of new ones which are not contained in these memoirs. Another specialization is that obtained by taking the lines of a plane as one of the fundamental complexes. This case has not been systematically treated.

Now suppose that any five complexes mutually in involution be taken as fundamental complexes. In general, none of these complexes is orthogonal, and no circle belongs to them all. The quadratic identity now becomes

$$\omega(x) \equiv \sum_{r=1}^{5} x_r^2 = 0.$$

These coördinates now represent circles, and not points, as in the memoirs quoted. This theorem results immediately: The curve of singularities of a general quadratic complex is

---

*V. Snyder, "On the determination of nodes in Dupin's cyclides," *Ann. of Math.*, vol. 11, p. 137.

†V. Snyder, "Geometry of some differential expressions," BULLETIN, vol. 4, p. 144.

‡J. Casey, "On the bicircular quartics," *Trans. R. Irish Acad.*, vol. 24, p. 359.

§G. Darboux, "Sur une classe remarquable de courbes et de surfaces," Paris, 1873.

‖G. Loria, "Sur la géométrie analytique du cercle," *Quar. Jour., of Math.*, vol. 22, p. 44.

likewise curve of singularities of ∞^1 such complexes and is focal curve (or envelope) for five quadratic congruences.

The congruences whose envelopes coincide with the curve of singularities of $\sum_{\tau=1}^{5} a_\tau x_\tau^2 = 0$ are

$$x_\kappa = 0, \qquad \sum_{\tau=1}^{5} \frac{x_\tau^2}{a_\tau - a_\kappa} = 0 \quad (\tau \neq \kappa).$$

The locus of centers of circles belonging to any congruence is easily found to be a bicircular quartic, hence:

The curve of singularities of a general quadratic complex is the envelope of circles which cut a fixed circle at a constant angle and whose centers lie on a bicircular quartic.

This curve can be generated in five-fold manner and if φ_τ, φ_κ denote the angles of the τth and κth fundamental complexes, and $[\tau, \kappa]$ the angle between the defining circles, then

$$\cos\,[\tau, \kappa] = \cos\varphi_\tau \cdot \cos\varphi_\kappa$$

for all values of τ, κ from 1 to 5.*

If any coördinate complex is orthogonal, its defining circle belongs to all the other coördinate complexes; if four of the coördinate complexes are orthogonal, the locus of centers in each congruence is a conic section and this case reduces to the one treated before.

By employing the method given on p. 150 of volume 4 of the BULLETIN, one easily obtains this theorem:

Eight tangents can be drawn to the curve of singularities parallel to any given line.

One cannot conclude that the curve is of class 8; for some particular cases arise in which the line at infinity is a multiple factor in the degraded curve, yet eight parallel tangents can be drawn to the remaining curve in any direction.

The order of the curve is 24, as may be found as follows: The problem is to find the envelope of the circle

$$(\mu - \nu)(x^2 + y^2) - 2\xi xt - 2\eta yt + (\mu + \nu)t^2 = 0,$$

subject to the three conditions

$$\omega(\xi, \eta, \lambda, \mu, \nu) = 0, \quad F_2(\xi, \eta, \lambda, \mu, \nu) = 0,$$
$$a\xi + b\eta + d\lambda + e\mu + f\nu = 0;$$

---

*See V. Snyder, "Ueber die linearen Complexe der Lie'schen Kugelgeometrie," Göttingen, 1895, p. 37.

or, in terms of the new coördinates, to find the envelope of $\sum_{\tau=1}^{5} s_\tau x_\tau = 0$, subject to the conditions

$$\sum_{\tau=1}^{5} a_\tau x_\tau^2 = 0, \quad \sum_{\tau=1}^{5} x_\tau^2 = 0, \quad \text{and (say)} \quad x_1 = 0.$$

Putting $x_1 = 0$ in all the forms, and differentiating, regarding x_5 (say) as constant, we obtain

$$\Delta = \begin{vmatrix} s_2 & s_3 & s_4 \\ a_2 x_2 & a_3 x_3 & a_4 x_4 \\ x_2 & x_3 & x_4 \end{vmatrix} = 0.$$

It now remains to eliminate x_2, x_3, x_4, x_5 between the four homogeneous equations

$$\sum_{\tau=2}^{5} s_\tau x_\tau = 0, (1,1); \sum x_\tau^2 = 0, (2,0); \sum a\, x_\tau^2 = 0, (2,0);\ \Delta = 0, (2,1).$$

In each parenthesis, the first number denotes the degree in which x_τ enters the equation, and the second the degree in s_τ. Now by the theory of elimination, the s_τ from the first equation will enter the x eliminant to the degree 2.2.2, and those from Δ to the degree 2.2.1; hence the s_τ appear to degree 12. But $s_\tau = 0$ is the cartesian equation of a circle, hence the curve of singularities is of degree 24.

The curve passes 12 times through each circle point; its equation contains 12 constants.

Straight lines, conics, circular cubics, and bicircular quartics, as well as their parallels, evolutes, and the involutes of their caustics by reflection, can be exhaustively classified from this standpoint; they all appear as degraded forms of the curve of singularities.

The lines drawn perpendicular to the asymptotes of the bicircular quartic (locus of centers), and cutting the fundamental circle of x_τ at the angle of the complex φ_τ, are all bitangents to the curve of singularities. The bicircular quartic has 4 asymptotes and there are two double tangents associated with each; hence each fundamental circle gives rise to 8 double tangents, all of which are tangent to a circle concentric with the given one. There are five such sets or forty in all (besides the line at infinity which may be a multiple tangent). The asymptotes of bicircular quartics, however, are all isotropic and perpendiculars to them must coincide with the lines themselves. They therefore fall together in pairs and pass through the centers of the five fundamental circles; hence:

The bitangents of the curve of singularities are all isotropic; one pair passes through the center of each fundamental circle.

The four points in which each circle intersects its associated quartic are foci of the curve of singularities; hence the latter has 20 single foci. There are 12 double foci, but they may not all be distinct.

By varying c in the equations

$$x_\kappa = 0, \qquad \sum_{\tau=1}^{6} \frac{x_\tau^2}{a_\tau - c} = 0,$$

a family of cosingular congruences is obtained, having a series of focal curves all of which pass through a finite number of fixed points. These curves are distributed into five distinct families. Two curves of the system pass through every point in the plane, the angle of intersection being a determinate function of φ_τ, φ_κ obtained by considering the tangent congruences of the complexes defining the curve at the given point. Each of the five sets contains a pair of circles as a degraded focal curve.

In a similar manner the following theorem may be proved:

The envelope of circles which cut a fixed circle orthogonally and whose centers lie on a curve of class n is a curve of order 2n, which passes n times through each circle point.

Its foci and double tangents are easily determined as before.

A corresponding generalization can be made in the geometry of the sphere. The depiction of the Kummer surface in general sphere space is the envelope of spheres whose centers lie on a cyclide and which cut a fixed sphere at a constant angle. The same surface can be generated in six ways, though the loci of centers are not in general confocal nor are the fixed spheres orthogonal. Twelve tangent planes can be drawn parallel to any given plane. The cones of Kummer of bitangent planes which belong to the cyclide give rise to six developables of tangent planes to the imaginary circle at infinity. The lines of curvature on these developables are defined by pencils of spheres concentric with the fundamental spheres.*

This surface appears to be of order 64, contains the circle at infinity as a 32-fold line, and depends on 18 constants.

CORNELL UNIVERSITY,
December 28, 1899.

---

* See V. Snyder. "Lines of curvature on annular surfaces." *Amer. Jour. of Math.*, vol. 22, p. 96.

ISOMORPHISM BETWEEN CERTAIN SYSTEMS OF SIMPLE LINEAR GROUPS.

BY PROFESSOR L. E. DICKSON.

(Read before the American Mathematical Society, February 24, 1900.)

1. In an article in the Bulletin for July, 1899, giving the known finite simple groups, I made the conjecture that the simple quaternary hyperorthogonal group $HO(4, p^{2n})$ in the $GF[p^{2n}]$ was isomorphic with the second hypoabelian group $SH(6, 2^n)$, the orthogonal group $O(6, p^n)$, or the group $NS(6, p^n)$, according as p^n is of the form 2^n, $4l-1$, or $4l+1$ respectively. For the case* $p^n = 2$, and for the case† $p^n = 3$, I have proven the conjecture true by setting up abstract groups holoedrically isomorphic with the linear groups in question. The calculations were necessarily long, so that the method of procedure would scarcely be adapted to the case of general p^n. From the correspondence of generators established in these two special cases, I have been led to the short proof for the general case given in this paper. The proof is based upon the theory of the second compound of a linear homogeneous group, as developed in the Bulletin for December 1898, and in the *Transactions* for January, 1900, pp. 91–96. In place of the hyperorthogonal group $HO(4, p^{2n})$, I employ the holoedrically isomorphic hyperabelian group‡ $HA(4, p^{2n})$, which contains as a subgroup the simple abelian group $A(4, p^n)$. The calculations appear to me to possess considerable elegance.

2. Theorem.—*According as* -1 *is a square or a not-square in the* $GF[p^n]$ $(p > 2)$, $HA(4, p^{2n})$ *is holoedrically isomorphic with the group* $NS(6, p^n)$ *or* $O(6, p^n)$.

Corresponding to every substitution of the subgroup $A(4, p^n)$ there is a senary linear homogeneous substitution of the second compound H with the two invariants

$$(1)\qquad \psi \equiv Y_{12}Y_{34} - Y_{13}Y_{24} + Y_{14}Y_{23}, \qquad Z \equiv Y_{12} + Y_{34}.$$

To the hyperabelian substitution, in which I belongs to the $GF(p^{2n})$,

$$I \equiv \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I^{-p^n} & 0 & 0 \\ 0 & 0 & I^{-1} & 0 \\ 0 & 0 & 0 & I^{p^n} \end{bmatrix}$$

---

* *Proc. of the Lond. Math. Soc.*, vol. 31, pp. 30–68.

† *Transactions*, vol. 1, No. 3 (July, 1900).

‡ *Proc. Lond. Math. Soc.*, vol. 31, pp. 30–68.

there corresponds in the second compound G of $HA(4, p^{2n})$, the substitution

$$I' : \begin{cases} Y_{12}' = I^{-p^n+1}Y_{12}, & Y_{13}' = Y_{13}, \quad Y_{14}' = I^{p^n+1}Y_{14}, \\ Y_{23}' = I^{-p^n-1}Y_{23}, & Y_{24}' = Y_{24}, \quad Y_{34}' = I^{p^n-1}Y_{34}. \end{cases}$$

Introducing in place of Y_{12} and Y_{34} the new indices $(p > 2)$

$$(2) \qquad \xi_1 \equiv \tfrac{1}{2}(Y_{12} - Y_{34}), \quad \xi_6 \equiv \frac{-1}{2J}(Y_{12} + Y_{34}),$$

the reversion of which gives

$$(3) \qquad Y_{12} \equiv \xi_1 - J\xi_6, \quad Y_{34} \equiv -\xi_1 - J\xi_6,$$

the substitution I' takes the form

$$I'' : \begin{cases} \xi_1' = \tfrac{1}{2}\theta\xi_1 + \tfrac{1}{2}J(I^{p^n-1} - I^{-p^n+1})\xi_6, \\ \xi_6' = \frac{1}{2J}(I^{p^n-1} - I^{-p^n+1})\xi_1 + \tfrac{1}{2}\theta\xi_6, \\ Y_{13}' = Y_{13}, \ Y_{14}' = I^{p^n+1}Y_{14}, \\ Y_{24}' = Y_{24}, \ Y_{23}' = I^{-p^n-1}Y_{23}, \end{cases}$$

$$(4) \qquad \theta \equiv I^{p^n-1} + I^{-p^n+1}.$$

Since $\theta^{p^n} = \theta$, θ belongs to the $GF[p^n]$. The coefficient

$$\tfrac{1}{2}J(I^{p^n-1} - I^{-p^n+1}),$$

whose p^nth power is $\frac{1}{2}J^{p^n}(I^{1-p^n} - I^{-1+p^n})$, belongs to the $GF[p^n]$ if and only if $J \equiv -J^{p^n}$, or $J^{p^n-1} = -1$. With this condition satisfied, the substitution I' belongs to the $GF[p^n]$ and leaves invariant the new form of ψ, viz.,

$$\psi'' \equiv -\xi_1^2 + J^2\xi_6^2 - Y_{13}Y_{24} + Y_{14}Y_{23}.$$

By the relation just assumed to hold,

$$(J^2)^{p^n-1} = 1, \quad (J^2)^{(p^n-1)/2} = -1,$$

so that J^2 belongs to the $GF[p^n]$, but is a not-square in it. It follows * that the group G'' given by the extension of H''

---

* "Determination of the structure of all linear homogeneous groups in a Galois field which are defined by a quadratic invariant," *Amer. Jour. of Math.*, vol. 21, pp. 193–256.

(H when expressed in the indices ξ_1, ξ_6, Y_{13}, Y_{14}, Y_{23}, Y_{24}) by the substitution I' is a subgroup of the group Q of senary linear substitutions in the $GF[p^n]$ which leaves ψ'' invariant. The latter has a subgroup Q'' of index 2 which contains all the substitutions* Q_{16}. Further, Q'' is extended to Q by any substitution O_{16} which is not of the form Q_{16}. We proceed to prove that $G'' \equiv Q''$. In order that H'' shall contain the substitution

$$h: \quad Y_{14}' = I^{p^n+1} Y_{14}, \quad Y_{23}' = I^{-p^n-1} Y_{23},$$

it is necessary and sufficient that I^{p^n+1} be a square in the $GF[p^n]$, or

$$I^{(p^n+1)(p^n-1)/2} = +1,$$

i. e., that I be a square in the $GF[p^{2n}]$. If this condition be satisfied, G'' contains the product $K \equiv I'h^{-1}$, viz.,

$$\xi_1' = \tfrac{1}{2}\theta\xi_1 + \tfrac{1}{2}J(I^{p^n-1} - I^{-p^n+1})\xi_6,$$
$$\xi_6' = \frac{1}{2J}(I^{p^n-1} - I^{-p^n+1})\xi_1 + \tfrac{1}{2}\theta\xi_6,$$

having determinant unity and leaving invariant $-\xi_1^2 + J^2\xi_6^2$. Let

$$(5) \qquad \alpha \equiv \tfrac{1}{2}\left(I^{\frac{p^n-1}{2}} + I^{\frac{-p^n+1}{2}}\right), \quad \beta \equiv \tfrac{1}{2}J\left(I^{\frac{p^n-1}{2}} - I^{\frac{-p^n+1}{2}}\right).$$

We readily find that

$$(6) \qquad \alpha^2 - \frac{1}{J^2}\beta^2 = 1, \quad 2\alpha\beta = \frac{J}{2}(I^{p^n-1} - I^{-p^n+1}),$$
$$2\alpha^2 - 1 = \tfrac{1}{2}(I^{p^n-1} + I^{-p^n+1}).$$

Hence $K = Q_{1,\ 6}^{\alpha,\ \beta}$, α and β being marks of the $GF[p^n]$. Inversely, if α and β be any set of solutions in the $GF[p^n]$ of the first relation (6), we can determine a square I in the $GF[p^{2n}]$ which satisfies the last two relations (6). In fact, from them,

$$I^{p^n-1} = 2\alpha^2 - 1 + \frac{2\alpha\beta}{J} = \left(\alpha + \frac{\beta}{J}\right)^2, \quad I^{-p^n+1} = \left(\alpha - \frac{\beta}{J}\right)^2.$$

The second follows from the first, since

$$\left(\alpha + \frac{\beta}{J}\right)^2\left(\alpha - \frac{\beta}{J}\right)^2 = \left(\alpha^2 - \frac{\beta^2}{J^2}\right)^2 = 1.$$

---

* For the notations, see *Amer. Jour. of Math.*, vol. 21, p. 207.

To show that the first relation can be solved for a square I in the $GF[p^{2n}]$, we note that

$$\left(a+\frac{\beta}{J}\right)^{p^n+1}=\left(a+\frac{\beta}{J^{p^n}}\right)\left(a+\frac{\beta}{J}\right)=a^2-\frac{\beta^2}{J^2}=1.$$

We have therefore shown that G'' contains every $Q^{\alpha,\beta}_{1,6}$. For I a not-square in the $GF[p^{2n}]$, the substitution I' is the product of an $O^{\rho,\sigma}_{1,6}$ (not a Q_{16}) by h, neither of which separately belongs to G''. By introducing new indices ξ_2, ξ_3 in place of Y_{14}, Y_{23}, we may give h the form $O^{\kappa,\lambda}_{2,3}$ (not a Q_{23}). It follows that $G''\equiv Q''$. In order to obtain the quotient group $HA(4, p^{2n})$, we consider as identical the products S and $ST\equiv TS$, where T is a hyperabelian substitution of determinant unity multiplying every index by τ, so

that $$\tau^4=1,\quad \tau^{p^n+1}=1.$$

Hence, if $p^n=4l+1$, the second compound G'' of $HA(4, p^{2n})$ is simple, and therefore holoedrically isomorphic with $NS(6, p^n)$. If $p^n=4l-1$, G'' has a maximal invariant subgroup of order two, generated by the substitution changing the signs of all six indices, the quotient group being $O(6, p^n)$.

3. THEOREM.—*The simple groups $HA(4, 2^{2n})$ and $SH(6, 2^n)$ are holoedrically isomorphic.*

Let λ be a mark of the $GF[2^n]$ such that the equation

$$(6)\qquad \sigma^2+\sigma+\lambda^2=0$$

is irreducible in the $GF[2^n]$. The condition on λ for the irreducibility of (6) is readily given (*American Journal of Mathematics*, l. c., p. 224) and can always be satisfied. Its roots are then σ and σ^{2^n} where

$$(7)\qquad \sigma+\sigma^{2^n}=-1.$$

We apply to the substitution I' of §2 the transformation of indices

$$(8)\qquad Y_{12}\equiv\lambda^{\frac{1}{2}}\xi_1-\sigma\lambda^{-\frac{1}{2}}\xi_6,\quad Y_{34}\equiv\lambda^{\frac{1}{2}}\xi_1-\lambda^{\frac{3}{2}}\sigma^{-1}\xi_6,$$

which gives reciprocally, for $p=2$,

$$(9)\qquad \xi_1\equiv\lambda^{\frac{3}{2}}\sigma^{-1}Y_{12}+\lambda^{-\frac{1}{2}}\sigma Y_{34},\quad \xi_6=\lambda^{\frac{1}{2}}(Y_{12}+Y_{34}).$$

Multiplying I' by h^{-1} (h defined as in §2), we obtain the substitution :

$$T':\qquad Y_{12}'=\tau Y_{12},\quad Y_{34}'=\tau^{-1}Y_{34}\qquad (\tau=I^{-2^n+1}).$$

Expressed in terms of the new indices ξ_1, ξ_6 defined by (9), T' becomes

$$T'' : \begin{cases} \xi_1' = (\lambda^2\sigma^{-1}\tau + \sigma\tau^{-1})\xi_1 - \lambda(\tau + \tau^{-1})\xi_6, \\ \xi_6' = \lambda(\tau + \tau^{-1})\xi_1 - (\lambda^2\sigma^{-1}\tau^{-1} + \sigma\tau)\xi_6. \end{cases}$$

We readily prove that the coefficients of T'' belong to the $GF[2^n]$. Since $\tau \equiv I^{-2^n+1}$, we find that $\tau + \tau^{-1}$ equals its 2^nth power and is therefore in the $GF[2^n]$. Furthermore, by (6),

$$\lambda^2\sigma^{-1}\tau + \sigma\tau^{-1} \equiv \tau + \sigma(\tau + \tau^{-1}) \qquad (\text{mod } 2).$$

The latter equals its 2^nth power since, by (7), we have

$$\tau^{2^n} - \tau + (\sigma^{2^n} - \sigma)(\tau + \tau^{-1}) = (\tau + \tau^{-1})(\sigma^{2^n} - \sigma + 1)$$
$$[\equiv 0 \qquad (\text{mod } 2).$$

Changing τ into τ^{-1}, it follows that $\lambda^2\sigma^{-1}\tau^{-1} + \sigma\tau$ belongs to the $GF[2^n]$.

By the transformation (8), ψ becomes, on applying (6),

$$\psi_1 \equiv \lambda(\xi_1^2 + \xi_6^2) + \xi_1\xi_6 - Y_{13}Y_{24} + Y_{14}Y_{23}.$$

Hence the group Γ'' given by the extension of the group H'' by T'' is a subgroup of $SH(6, 2^n)$. Evidently H'' contains all the substitutions of $SH(6, 2^n)$ which leave ξ_6 (as well as ψ_1) invariant. But $SH(6, 2^n)$ is given by the extension of H'' by the substitutions*

$$O_{1,\,6}^{\alpha,\,\beta} : \begin{cases} \xi_1' = \alpha\xi_1 + \lambda(\alpha + \beta)\xi_6 \\ \xi_6' = \lambda(\alpha + \beta)\xi_1 + \beta\xi_6 \end{cases} \qquad [\alpha\beta + \lambda^2(\alpha^2 + \beta^2) = 1].$$

The totality of substitutions $O_{1,\,6}^{\alpha,\,\beta}$ is identical with the totality of substitutions T'' when τ takes all possible values for which $\tau^{2^n+1} = 1$. In fact, if we set

$$\alpha = \lambda^2\sigma^{-1}\tau + \sigma\tau^{-1}, \quad \beta = \lambda^2\sigma^{-1}\tau^{-1} + \sigma\tau,$$

we get by (6),

$$\alpha + \beta = \tau + \tau^{-1}, \quad \alpha\beta + \lambda^2(\alpha^2 + \beta^2) = 1.$$

Hence every T'' is of the form $O_{1,\,6}^{\alpha,\,\beta}$. Inversely, if we set

$$\tau = \alpha + \sigma(\alpha + \beta), \quad \text{whence} \quad \tau^{-1} = \beta + \sigma(\alpha + \beta),$$

we see that every $O_{1,\,6}^{\alpha,\,\beta}$ is of the form T''.

---

* *Amer. Jour. of Math.*, vol. 21, pp. 225–233. For uniformity of notation, I here use $O_{1,\,6}^{\alpha,\,\beta}$ in place of $O_1^{\alpha,\,\beta}$ of p. 226, replacing η_1 by ξ_6.

It follows that Γ'' is identical with $SH(6, 2^n)$. But Γ'' is holoedrically isomorphic with $G_{p=2}$ and therefore with $HA(4, 2^{2n})$, whose second compound is $G_{p=2}$.

4. NOTE.—It appears that the quaternary transformation group which naturally corresponds to the finite group $HO(4, p^{2n})$ is *not* continuous.

THE UNIVERSITY OF TEXAS,
January 27, 1900.

THE HESSIAN OF THE CUBIC SURFACE. II.

BY DR. J. I. HUTCHINSON.

(Read before the American Mathematical Society, February 24, 1900.)

THE aim of the following paper is to extend the results of a previous article on the same subject (BULLETIN, March, 1899, p. 282) by determining all the quintic and sextic curves on the Hessian of the cubic surface, and giving some theorems connected with them, and with the quartic curves already determined.

I will write the equation of the Hessian in the form

$$F \equiv xyzu + wyzu + wxzu + wxyu + wxyz = 0,$$

where w, x, y, z, u are connected by the relation

$$aw + bx + cy + dz + eu = 0,$$

in which a, b, c, d, e are arbitrary constants.

As already shown, the surface F contains three classes of biquadratic curves, viz.:

a_1. A class containing 15 families which lie on 30 families of cones, all the cones of the same family cutting F in two lines and tangent along a third.

a_2. A class containing 30 families of curves lying on 30 families of cones tangent to F along two lines.

a_3. A class containing 15 families of curves determined by as many families of quadrics each intersecting F in a gauche quadrilateral, and by 30 families of quadrics each meeting F in two lines and a conic.

Consider the family of a_1 determined by the cones

$$(1)\qquad \begin{aligned} A_1 &\equiv x(w + y) + \lambda wy = 0, \\ A_1' &\equiv x(z + u) + (1 - \lambda)zu = 0. \end{aligned}$$

The plane $w + y = 0$ is tangent to all of the cones A_1 and hence is a double tangent plane for each biquadratic curve of the family. The two points of tangency with each curve form pairs of points in an involution on the line wy. Similarly, the plane $z + u = 0$ is a double tangent plane for each curve of the family, the points of contact forming an involution on the line zu.

All the curves of the family pass through the four nodes of F which lie on the lines wx and xy excluding their point of intersection (or we may say, the four which lie in the plane x excluding the vertices of A_1 and A_1'). These are the only points common to two curves of the same family.

The 15 families of α_1 can be grouped in 5 sets of 3 each, any one set having the property that a cubic surface can be passed through three curves arbitrarily chosen, one from each family of the set. For example, with the family determined by equations (1) are associated the two others obtained by the permutations (wz) and (wu). The family of cubic surfaces which intersects F in these three families of curves has the equation

$$A_1[x + \lambda'(1 - \lambda'')z + \lambda''(1 - \lambda')u] + A_1'[(1 - \lambda')(1 - \lambda'')w + x + \lambda'\lambda''y] = 0,$$

since the left member is unaltered by the permutations $(wz)(\lambda\lambda')$ and $(wu)(\lambda\lambda'')$.

Consider the family α_8 determined by the intersection of the quadrics

$$(2) \qquad \begin{aligned} A_8 &\equiv wy + \lambda zu = 0, \\ A_8' &\equiv wx + wy + xy - \lambda x(z + u) = 0. \end{aligned}$$

Each curve of the family passes through the same four nodes of F which form the points of intersection of the curve of the family (1) of α_1. These two families of α_1 and α_8 are also associated in such a way that they form the complete intersection with F of the family of quadrics

$$A_1 - \lambda' A_1' \equiv (\lambda - 1)A_8 + A_8' = 0.$$

The four points in which a given curve of the family (2) meets the lines wz, wu, yz, yu lie in a plane whose equation is

$$y + w - \lambda(z + u) = 0.$$

The plane $w + u = 0$ intersects A_8 in two lines one of which is the line wu. The plane is accordingly tangent to

A_3 in the point whose coördinates are $w = u = 0$, $y = \lambda z$. These coördinates also satisfy the equation $A_3' = 0$. Hence, the plane $w + u$ touches each biquadratic of the family in that point of the line wu where it is tangent to the quadric A_3 containing the given curve. Similarly for the planes $w + z$, $y + z$, $y + u$.

By means of the permutations (wz) and (wu) applied to the equations (2) we determine two other families which together with the first are so related that a cubic surface can be passed through three curves arbitrarily chosen, one from each family. The equation of the cubic is

$$A_3[\lambda''w + (1 + \lambda')(1 + \lambda'')x + \lambda'y] - A_3'[\lambda''w + \lambda'y - \lambda'\lambda''z - u] = 0,$$

since the left member is unaltered by the permutations $(yu)(\lambda\lambda')$ and $(wu)(\lambda\lambda'')$.

The curves (1) of α_1, and (2) of α_3 have in common the property that they meet any of the conics $\{xw\}$ (in the plane $w + x = 0$), $\{xy\}$, $\{xz\}$, $\{xu\}$ in pairs of points forming an involution.

Any curve of (1) intersects any curve of (2) in 8 points, on account of the identity

$$A_1 - \mu A_1' + (1 - \lambda)A_3 - A_3' \equiv 0,$$

where A_1, A_1' contain the parameter λ, and A_3, A_3' contain μ. But since

$$(1 - \lambda)A_1 + \lambda\mu A_1' - \lambda(1 - \lambda)A_3 \equiv x[(1 - \lambda)(w + y) + \lambda\mu(z + u)],$$

it follows that the four movable points of intersection lie in the plane

$$(1 - \lambda)(w + y) + \lambda\mu(z + u) = 0,$$

the remaining four being in the plane $x = 0$ as already noticed.

Consider the family of α_2 determined by the cones

$$(3)\qquad \begin{aligned} A_2 &\equiv \lambda w^2 + wx + wy + xy = 0, \\ A_2' &\equiv \lambda(zu + wz + wu) + (x + y)(z + u) = 0. \end{aligned}$$

Two curves of the same family do not intersect.

The planes $w + x$, $w + y$, and $z + u$ are double tangent planes to each curve of the family, the pairs of points so determined forming involutions on the lines wx, wy, and zu respectively.

The four lines xz, yz, xu, yu are met by each curve in one point each, the four points so determined lying in the plane

$$\lambda w + x + y = 0.$$

If we apply to equations (3) the permutations (wxz) and (wzx), we obtain two other families of curves which together with (3) form the complete intersections of F and a family of cubic surfaces whose equation is

$$\begin{aligned} A_2[\lambda' x + y + \lambda''(1-\lambda')z - (1-\lambda')(1-\lambda'')u] \\ + A_2'[(\lambda'\lambda'' - \lambda'' + 1)w + \lambda'\lambda'' x + y + \lambda'' z] = 0. \end{aligned}$$

The function in the left member is unchanged by the permutation $(wxz)(\lambda\lambda'\lambda'')$. By means of the permutations (xy), (zu), and $(xy)(zu)$ which leave A_2 and A_2' unchanged we see that the family (3) is also associated in a similar manner with three other pairs of families.

There are four classes of quintic curves, all of the first species according to Salmon's classification. (Geometry of three dimensions, 4th edition, p. 318.)

β_1. Ten families of quintics formed by ten families of quadric cones passing through three lines of F which meet in a point. Such are the cones

$$(4) \qquad B_1 \equiv \mu_1 xy + \mu_2 wy + \mu_3 wx = 0.$$

The same family of quintics is also determined by the partial intersection of F and the three families of cubic surfaces

$$\begin{aligned} \Gamma_1 &\equiv \mu_1 xy(z+u) + zu[(\mu_1 - \mu_3)x + (\mu_1 - \mu_2)y] = 0, \\ \Gamma_2 &\equiv \mu_2 wy(z+u) + zu[(\mu_2 - \mu_3)w + (\mu_2 - \mu_1)y] = 0, \\ \Gamma_3 &\equiv \mu_3 wx(z+u) + zu[(\mu_3 - \mu_2)w + (\mu_3 - \mu_1)x] = 0, \end{aligned}$$

since

$$\mu_1 F \equiv w\Gamma_1 + zuB_1, \quad \mu_2 F \equiv x\Gamma_2 + zuB_1, \quad \mu_3 F \equiv y\Gamma_3 + zuB_1.$$

If Γ_i' denote the function Γ_i with μ_k' written in the place of μ_k, it is easy to see that the two curves whose parameters are μ_k and μ_k' lie on the cubic surface Γ, where

$$\Gamma \equiv \mu_1'\Gamma_1 + \mu_2'\Gamma_2 + \mu_3'\Gamma_3 \equiv \mu_1\Gamma_1' + \mu_2\Gamma_2' + \mu_3\Gamma_3'.$$

For $\mu_k' = \mu_k$ we obtain a family of cubic surfaces inscribed to F along the quintics of the family (4) and the line zu.

By taking $\mu_i = 0$ we see that this family of quintics includes three families of plane cubics, and by taking $\mu_i = \mu_k$ we find that it also includes three families of biquadratics of a_1.

Every curve of the family passes through the seven nodes lying on the lines wx, wy, and xy. Any two curves intersect in two other points whose joining line is evidently a common generator of the two cones on which the two curves lie.

Through any two curves of β_1, whether belonging to the same family or not, a cubic surface can be passed which meets F elsewhere in two lines.

β_2. The next class of quintics contains sixty families determined by sixty families of cones, each passing through intersecting lines of F and tangent along one of them. The equation

$$(5) \qquad B_2 \equiv ay^2 + \beta wy + \gamma x(w + y) = 0$$

leads to a family of this class which is also determined by the cubic surfaces

$$G \equiv ay(xz + xu + zu) + (a - \beta + \gamma)xzu + \gamma x^2(z + u) = 0,$$

since

$$(a - \beta)F \equiv G(w + y) - B_2(xz + uz + xu).$$

Every curve of the family passes through the three nodes which lie on the line xy. Any two of the curves intersect in two other points.

β_3. The third class contains sixty families determined by sixty families of quadrics passing through two non-intersecting lines of F and a third line cutting both. The two equations

$$(6) \qquad \begin{aligned} B_3 &\equiv y(\lambda w + \mu x) + wz = 0, \\ H_3 &\equiv xu(w + z) - (\lambda w + \mu x)(yu + xu + xy) = 0, \end{aligned}$$

determine such a family, since

$$F \equiv yH + B_3(yu + xu + xy).$$

Every curve of the family passes through the five nodes situated on the lines wz, wy, and yz excluding the two points of intersection. Any two curves have two other points of intersection.

If we apply to equations (6) the permutations $(wx)(zu)$, $(yz)(xu)$, $(wz)(yu)$, $(wu)(xy)$, we obtain together with (6) five families of β_3 having the property that if two curves be taken from any two different families of the set, they will lie on a cubic surface which either passes through a conic, or is tangent to F along a line. The 60 families of β_3 group themselves into 12 sets of 5 of the like character.

β_4. The fourth class contains 30 families determined by 30 families of quadrics meeting F in a line and a conic. Such a family is given by the equations

$$B_4 \equiv zu + yu + yz - (w + x)(\lambda y + \mu z) = 0,$$

$$J \equiv \lambda wxy + \mu wxz + yzu = 0,$$

since

$$F \equiv wxB_4 + (w + x)J.$$

Any two curves of the family intersect in five points, three of which are the nodes contained on the line yz.

There are six different classes of sextic curves on the Hessian.

γ_1. The first class contains 30 families, each of which is determined by the quadrics passing through two intersecting lines of F. The curves determined by the equation

$$C_1 \equiv wp + xy = 0,$$

where

$$p = \alpha w + \beta x + \gamma y + \delta z + \varepsilon u,$$

also lie on the cubic surfaces

$$K_1 \equiv -zup + xyz + xyu + xzu + yzu = 0,$$

since

$$F \equiv zuC_1 + wK_1.$$

Any two curves of this family with parameters $\alpha, \cdots$ and $\alpha_1, \cdots$ intersect in the four nodes lying on the lines wx and wy, excepting their common point, and in six other points lying on a conic in the plane $p - p_1 = 0$.

A second family generated by the equations

$$\overline{C}_1 \equiv wp' + zu = 0,$$

$$\overline{K}_1 \equiv -xyp' + xyz + xyu + xzu + yzu = 0$$

is associated with the preceding in such a way that a curve from each family lies on the cubic surface

$$K \equiv p'C_1 - K_1 \equiv p\overline{C}_1 - \overline{K}_1 = 0.$$

γ_2. The second class, containing 10 families, is determined by quadrics which touch F along a line. Such a family is

$$C_2 \equiv (w + x)p + wx = 0,$$

also determined by the surfaces

$$K_2 \equiv -yzu + p(yz + yu + zu).$$

In this case we have

$$F \equiv C_2(yz + yu + zu) - (w + x)K_2.$$

γ_3. The third class also contains 10 families. The quadric

$$C_3 \equiv (w + x)p + yz + yu + zu = 0,$$

which meets F in a conic, and the cubic surface

$$K_3 \equiv wxp - yzu = 0$$

determine such a family, since

$$F \equiv wxC_3 - (w + x)K_3.$$

Any two curves of the same family in either γ_2 or γ_3 intersect in six points lying on a conic.

A curve from each of the two families of γ_2 and γ_3 just mentioned lies on the cubic surface

$$K \equiv C_2 p' + K_2 \equiv C_3' p + K_3'.$$

γ_4. The fourth class contains 15 families, each determined by the intersection with F of a family of quadrics such as

$$\alpha wy + \beta wz + \gamma xy + \delta xz = 0,$$

passing through two non-intersecting lines wx, yz. A sextic of this class cannot lie on a cubic surface.

γ_5. The fifth class of sextics contains 60 families. These curves are formed by the partial intersection of two cubic surfaces such as

$$\triangle_1 \equiv y(wx + wz + xz) + z(\alpha wx + \beta xz + \gamma wz) = 0,$$
$$\triangle_2 \equiv u(\alpha wx + \beta xz + \gamma wz) - wx(y + u),$$

which intersect in the three lines wx, wz, xz, and a sextic on F, since $$F \equiv u\triangle_1 - z\triangle_2.$$

γ_6. The sixth class contains a single family, the Steiner sextics, determined by the equation

$$S \equiv \kappa xyz + \lambda wyz + \mu wxz + \nu wxy = 0,$$

or equally well by four other equations referred to the four other fundamental tetrahedra.

Let s and s' be any two curves of the family and let $T = 0$, $T' = 0$ be the two cubic surfaces which touch F along s and s' respectively. Also let $H = 0$ be the cubic surface containing both s and s'. Then an identity of the form

$$(\mathrm{I}) \qquad QF \equiv TT' + H^2$$

necessarily exists, where Q is a function of the second degree. A course of reasoning similar to that used by Humbert in connection with the Kummer surface enables us to draw a number of interesting conclusions from (I).

The two curves s and s' intersect in four points p_1, p_2, p_3, p_4 (besides the 10 nodes of F through which every curve of the family passes). The five surfaces Q, F, T, T', H are mutually tangent at each of these four points.

The surface T intersects H in s and a twisted cubic c. It is evident that c lies on Q and that T touches Q along this cubic. Similarly, T' touches Q along a cubic c' which also lies on H.

The 24 points of intersection of Q, H, and F are double points of the surface $TT' = 0$. Among these occur the points p_i, each counted four times. The remaining eight points, since they lie on the curves s and s', are not points of tangency of T and T'. They must therefore be nodes on either T or T'. It is evident that half of them are on the one, and half on the other surface. Let a_1, a_2, a_3, a_4 be the four nodes on T. The joining line of any two of these points lies entirely on T, and is accordingly tangent to F in a third point. Hence, *the tetrahedron whose vertices are the nodes of the cubic surface T which touches F along s is inscribed to F as to its vertices and circumscribed to F as to its edges.* There are evidently ∞^3 such tetrahedra.

The four points a_i, and the points p_i counted twice, form the complete intersection of the cubic c with F.

The cubic surfaces S and S' defining the sextics s and s' intersect in six lines and a twisted cubic k. The latter passes through the four points p_i and the four vertices of the tetrahedron of reference. Conversely, every twisted cubic k passing through the vertices of the tetrahedron of reference intersects F in four remaining points p_i, which form the four points of intersection of two sextics s and s' (and hence are the common points of a singly infinite system of sextics.)

The Hessian F is invariant for the birational transformation T_6 defined by the equations

$$aw' : bx' : cy' : dz' = \frac{1}{w} : \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

This transformation interchanges the family of cubics k with the lines of space, and the sextics s with the plane quartics of F.* Four other transformations of a similar character may be obtained by starting from any of the other four tetrahedra contained in the fundamental pentahedron. These are

$$T_1: \quad bx' : cy' : dz' : ew' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{u},$$

$$T_2: \quad aw' : cy' : dz' : eu' = \frac{1}{w} : \frac{1}{y} : \frac{1}{z} : \frac{1}{u},$$

$$T_3: \quad aw' : bx' : dz' : eu' = \frac{1}{w} : \frac{1}{x} : \frac{1}{z} : \frac{1}{u},$$

$$T_4: \quad aw' : bx' : cy' : eu' = \frac{1}{w} : \frac{1}{x} : \frac{1}{y} : \frac{1}{u}.$$

These five operations, each of period 2, generate a group G of infinite order, since the operation T_iT_j is of infinite period. As far as the points of F are concerned, each transformation T_i has exactly the same effect. In other words, *there exists a subgroup of index* 2 *under the group G for which each point of F is unchanged in position.*

Since a plane quartic has 28 bitangents it follows that the sextic s has 28 bitangent cubics k out of each of the five families of such cubics. Suppose the plane of the quartic q corresponding to s to be rotated about one of the bitangents of the curve. The intersections of the moving plane with F will form a single infinity of plane quartics having the same line for bitangent. Hence, for a given sextic s there are 28 subfamilies of ∞^1 sextics, each family being tangent to s in a pair of points, or in other words, the four points p_i fall together in pairs 28 times on a given sextic.

Two sextics do not in general touch each other at a node of F. There are ∞^2 sextics which touch each other and have a given generator of the tangent cone at the node for a common tangent line.

Suppose that we consider the ∞^1 of these curves which are also mutually tangent at a second node of F. If the joining line of the first and second nodes is a line of F, then these curves are also mutually tangent in two other nodes of F, the four nodes together forming the vertices of one of the five fundamental tetrahedra. If the first and second nodes do not have a line of F for joining line then the two remaining points of intersection determine a line which meets two of the lines of F. For example, if the two nodes be

---

* Cf. Salmon, Geom. of Three Dim., 4th ed., p. 495.

wxu and *yzu* then the joining line of the two remaining points of intersection of the sextics so determined will meet F elsewhere in the two lines *wx* and *yz*.

Cornell University,
February, 1900.

NOTE ON THE GROUP OF ISOMORPHISMS.

BY DR. G. A. MILLER.

(Read before the American Mathematical Society, February 24, 1900.)

Let $s_1, s_2, \cdots, s_g$ represent all the operators of a group G and let $t_\alpha s_\alpha$ correspond to s_α ($\alpha = 1, 2, \cdots, g$) in any given simple isomorphism of G with itself. It is evident that t_α is some operator of G. When G is abelian these t_α's must constitute a group T which is isomorphic with G.* In this isomorphism, t_α evidently can not be the inverse of s_α unless $s_\alpha = 1$. As this condition is sufficient as well as necessary, we have

Theorem I.—*Every simple isomorphism of an abelian group A with itself may be obtained by 1° making A isomorphic with one of its subgroups or with itself in such a manner that no operator corresponds to its inverse, and 2° making each operator of A correspond to itself multiplied by the operator which corresponds to it in the given isomorphism.*

The simplest case that can present itself is the one in which the subgroup of G, which corresponds to identity of T in the given isomorphism between G and T, includes T. The resulting simple isomorphism of G with itself must correspond to an operator in the group of isomorphisms of G, whose order is equal to the operator of highest order in T. When the order of T is an odd prime number p, or the double of an odd prime, only one other case can present itself; viz, the case in which T corresponds to itself, or to its subgroup of an odd prime order, in the given isomorphism between G and T. The resulting simple isomorphism of G with itself may clearly correspond to a cyclical group of order $p - 1$, or to any one of its subgroups in the group of isomorphisms of G. These results lead to the following

---

* When G is non-abelian, these t_α's need not constitute a group, as can be seen from the simple isomorphisms of the symmetric group of order 6 with itself.

THEOREM II.—*If we make an abelian group A simply isomorphic with itself by multiplying its operators by the operators of a subgroup whose order is an odd prime number p or the double of this prime number, the resulting simple isomorphism of A with itself will correspond to an operator of order p, $2p$, or $\frac{p-1}{a}$ (a being any division of $p-1$) in the group of isomorphisms of A.*[*]

The determination of all the possible orders of the corresponding operators in the group of isomorphisms of any abelian group, when T is a given subgroup, seems to be a problem of considerable difficulty. When the order of T is small the number of cases that have to be considered is also small. In addition to the orders included in the theorems given, we have the following when the order of T does not exceed 8: If T is the cyclical group of order four, the resulting isomorphism may correspond to an operator of order two in the group of isomorphisms, and when T is the non-cyclical group of this order, it may correspond to operators of orders 3 and 4. When T is the cyclical group of order 8, the orders of these operators may be 2, 4, and 8; when T is the direct product of the cyclical group of order 4 and an operator of order 2, the orders of the corresponding operators in the group of isomorphisms may be 2 and 4; finally, when T is the direct product of three operators of order 2, the given operators may be of orders 2, 3, 4, 6, and 7. While all of the possible cases for a given T may present themselves in the same group it is evident that this does not always happen.

For the sake of illustration we may consider the group of isomorphisms of the group of order 8 which is the direct product of three operators of order 2.[†] Each of its 7 subgroups of order 4 leads to three operators of order two, according to the second sentence below Theorem I. We thus obtain the 21 operators of order 2 of the required group of isomorphisms when we consider all the possible instances in which the order of T is 2. If the order of T is 4, two cases present themselves—in one case just two of the operators of T (including identity) correspond to operators of T, and in the other case each one of the operators of T corresponds to some operator of T. The former case leads to the 42 operators of order 4 and the latter to 56 operators of order 3 of the required group of isomorphisms. Finally, we obtain 48 operators of order 7 when we consider all the possible instances in which the order of T is 8. Hence we observe that the required group is the well known simple group of order 168.

---

[*] Cf. *Comptes rendus*, vol. 130, 1900, p. 316.

[†] Cf. Moore, BULLETIN, vol. 1, 1894, p. 63.

It is known that there are abelian groups which cannot be groups of cogredient isomorphisms.* Suppose that the group of cogredient isomorphisms H of a given group G is Hamiltonian. Each of the operators of G which corresponds to identity of H is self-conjugate in G. Since the non-identity commutator of H is some power of each one of a set of generators of H, all the operators of G which correspond to it must also be self-conjugate in G. As this is clearly impossible, it follows that *a Hamiltonian group cannot be a group of cogredient isomorphisms.* In fact, the preceding arguments apply to all groups which contain an operator different from identity, which is some power of each one of a set of generating operators. Hence such a group cannot be the group of cogredient isomorphisms of any group whatsoever.

CORNELL UNIVERSITY,
February, 1900.

LOBACHEVSKY'S GEOMETRY.

Nikolaj Iwanowitsch Lobatschefskij. Zwei Geometrische Abhandlungen aus dem Russischen übersetzt, mit Anmerkungen und mit einer Biographie des Verfassers von FRIEDRICH ENGEL. Leipzig, B. G. Teubner, 1899. 8vo. xvi + 476 pp.

This volume is the first in the series of the "Urkunden zur Geschichte der Nichteuklidischen Geometrie" planned by Friedrich Engel and Paul Stäckel in continuation of their "Theorie der Parallellinien von Euklid bis auf Gauss." The present work on Lobachevsky has been undertaken by Engel, while a volume on the two Bolyais by Stäckel is soon to appear. All geometricians will welcome the opportunity thus afforded of becoming fully acquainted with the writings of these men, who share with Gauss the honor of being the first to break loose from the authority of Euclid and to show that the latter's axiom concerning parallel lines is not only incapable of proof but entirely unnecessary in a logical geometry which fully explains all facts of experience. The earlier works of Lobachevsky have been until now locked up in the original Russian and their contents

---

Comptes rendus, vol. 130, 1900, p. 316.

are known to the public mainly through later abstracts in French and German. Of these the best known is the small pamphlet published in Berlin in 1840 with the title "Geometrische Untersuchungen zur Theorie der Parallellinien" and translated into English by Professor G. B. Halsted. This, though containing the outline of the author's theory, is too concise to do it justice. Professor Engel has rendered therefore a signal service to mathematicians in translating the two articles which form the first part of the present volume.

The titles of these articles are, in the German translation, "Ueber die Anfangsgründe der Geometrie" and "Neue Anfangsgründe der Geometrie mit einer vollständigen Theorie der Parallellinien." Both articles appeared in publications of the Kasan University, the former in 1829–30, the latter in 1836. A footnote by Lobachevsky himself informs us that the first article is taken from one presented to the Section for Mathematical-Physical Science in 1826 under the title, "Exposition succincte des principes de la Géométrie," but no copy of this earlier work is extant. The "Anfangsgründe" is thus the earliest of Lobachevsky's publications on this subject. It is comparatively short, only sixty-six pages in the translation, and correspondingly concise, the proofs of the earlier theorems not being given. It is to the "Neue Anfangsgründe" that one must turn for the best introduction to Lobachevsky's thought. We follow this in our review.

The first six chapters present a development of geometric theorems as far as these do not depend upon any theory of parallel lines, and can be read with approval by the most orthodox believer in the euclidean geometry. Lobachevsky's order of development is however entirely new. With him the distinctive geometric property of bodies is that of contact. This idea we get through the senses, back of it we can not go ; in fact, it is impossible to define it fully in words. By means of this property he discusses various kinds of sections of a body, and derives finally the conceptions of surface, line, and point, from the ways in which bodies or sections of bodies may touch each other. The distance of two points is their relative position and is determined by the contact of two bodies in the two points, independently of the form of these bodies outside of their points of contact. Hence, for example, a pair of dividers may determine distance and the idea of distance is independent of the conception of a straight line. The sphere is therefore the simplest solid, the spherical surface the simplest surface. A plane is determined by the intersections of equal spheres described about two fixed points as centers, and any two of these equal spheres define a circle. A straight line is defined as a line which remains in coincidence with itself in all positions when two points are fixed, and it is shown that in any circle there exist such lines, the diameters. With these ideas as the fundamental ones, it is clear that the geometry on the sphere is as simple as the geometry on the plane, and it is not surprising to find Lobachevsky carrying the two side by side. In one respect, at least, the spherical geometry is the simpler, for it may be shown that the sum of the angles of a spherical triangle is greater than π. In the plane geometry one must content himself with the theorems that the sum of the angles of a triangle cannot exceed π, and that if the sum for one triangle is equal to π the same is true of all triangles, while if the sum is less than π it is variable, decreasing as the lengths of the sides increase. At the outset Lobachevsky assumes that space is infinite. He is therefore able to prove that planes and straight lines are infinite in extent, that two straight lines can intersect in not more than one point, and that two perpendiculars to the same straight line cannot intersect. He misses, therefore, the elliptic geometry in which the sum of the angles of a triangle is greater than π. This geometry exists for him of course upon the sphere, and the formulas which he obtains later for plane triangles are shown by him to include the spherical formulas, but the possibility of the existence of this geometry on the plane is explicitly denied. It remained for Riemann to take the last step in the freeing of geometry from unnecessary hypotheses.

In chapter VII. begins the discussion of parallel lines on which the fame of the author rests. An axiom of Euclid asserts that if a straight line intersect two straight lines forming two interior angles on the same side of the first straight line which are together less than two right angles, then the two straight lines if sufficiently produced will meet. Many attempts have been made to justify this axiom by showing that its denial involves contradictions; but all such attempts have failed. Lobachevsky boldly denies the truth of the axiom and asserts that all lines radiating from a point fall into two classes with reference to any fixed line; namely, the converging lines which meet the fixed line, and the diverging lines which do not meet it. To the latter class belong two parallel lines which form the boundary between the two classes. More precisely, if C is a point at a perpendicular distance a from a line AB, then there exists a certain angle $\Pi(a)$ such that all lines which make with the perpendicular on the side toward AB an angle less than $\Pi(a)$ will meet AB, while lines which make an angle equal or greater than $\Pi(a)$ will not meet AB. The two lines which make the angle $\Pi(a)$ with the perpendicular are the parallel lines, and the angle $\Pi(a)$ is the angle of parallelism. If $\Pi(a)$ is identically equal to $\frac{1}{2}\pi$, the euclidean axiom results and it is readily shown that then the sum of the angles of a triangle equals π, and conversely. On the other hand, if the sum of the angles of a triangle is less than π, $\Pi(a)$ is variable, decreasing as a increases, becoming $\frac{1}{2}\pi$ only for $a = 0$, and being equal to any given acute angle for some value of a. There result therefore two systems of geometry, the usual one, and the other, called rather unfortunately by Lobachevsky the "imaginary" geometry. In the ordinary geometry a circle approaches a straight line when its radius is indefinitely increased, and the revolution of a straight line about an axis perpendicular to it generates a plane. In the imaginary geometry the limit of a circle is a boundary curve characterized by the property that all normals are parallel to a fixed line and hence to each other. The revolution of a boundary curve about a normal generates a boundary surface, which is cut by any plane in either a boundary curve or a circle. If now a triangle is formed on a boundary surface by means of three boundary curves it may be shown that the sum of its angles equals π. This triangle plays in the imaginary geometry the rôle of the plane triangle in the usual geometry.

This closes the synthetic and more elementary part of Lobachevsky's work. He turns now to the analytic treatment of the imaginary geometry. The usual trigonometric formulas hold for a boundary triangle, and a point of attack is thus found for the determination of $\Pi(a)$. It results that the angle of parallelism is an analytic function of its argument, expressed by the simple relation

$$\tan \tfrac{1}{2}\Pi(x) = e^{-x}.$$

Formulas connecting the sides and angles of a triangle are next deduced. These formulas include those of the usual plane and spherical geometry, and are in fact exactly those which one obtains by writing the formulas of spherical trigonometry for a triangle with pure imaginary sides. To derive the formulas of the euclidean geometry it is only necessary to assume that the sides of the triangle are so small that all higher powers may be neglected. This is to assume that the world of our experience is very small in comparison with the unit of length used in the imaginary geometry.

The foregoing is a brief sketch of the prominent ideas of the "Neue Anfangsgründe." Turning back to the "Anfangsgründe" we find in addition a large number of applications to the determination of the lengths of curves, the areas of plane and curved surfaces, and the volumes of solids. Many readers who are already familiar with Lobachevsky's theory of parallels will find this part of the work of great interest; for it is not contained in any other of the author's works. There is also a discussion based on the parallax of fixed stars to show that the discrepancy between the usual and the imaginary geometry is too small to be detected experimentally.

The second part of the volume before us is occupied with notes and a life of Lobachevsky by the translator. Lobachevsky's argument is in its large outline simple and clear and the same is true in general of his separate theorems, but he is in places obscure through too concise presentation or through suppression of intermediate reckoning. Occasionally also a demonstration is faulty. All such places have been carefully worked over by the annotator with the object and the result of making the work easily read by all. The notes are more copious in the case of the "Anfangsgründe" where the need is greater.

The biography is at the same time an interesting account of Lobachevsky's career as student, professor, and rector of the Kasan University, and an historical and critical discussion of his mathematical writings. Of particular interest is the chapter on the relation of Gauss, Lobachevsky, and Bolyai, and the conclusion reached by the author that each discovered the non-euclidean geometry independently of the others. Lobachevsky was connected with Gauss through his teacher Bartels who was a friend of Gauss, and from this fact it has been argued that Lobachevsky's discoveries were inspired by Gauss. Against this theory, Professor Engel presents facts to show that Lobachevsky could have learned from Bartels nothing more than that Gauss questioned the correctness of the euclidean axiom. Lobachevsky's own writings however do not bear out the hypothesis that he learned even this much from his teacher, for the gradual development of his thought is apparent in his unpublished works, from the time when in 1816 he attempted the proof of the disputed axiom, through the period when he recognized the failure of all attempts at such a proof but had not yet anything to take its place, until in 1826 he had found his way successfully through all his difficulties. He adds his own testimony as to the origin of his great theories in the opening sentences of the "Neue Anfangsgründe" in which he declares that the futility of the efforts made during two thousand years since Euclid to complete the theory of parallel lines aroused in him the suspicion that the ideas sought to be proved were not necessarily true. While it is remarkable that the solution of a two-thousand year old problem should be given almost simultaneously by three men, it should be remembered that these three were not the only mathematicians who had worked upon the problem. More than one had missed the solution by a hairsbreadth only; Lobachevsky, Bolyai, and Gauss succeeded in finding it.

FREDERICK S. WOODS.

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY.

VOGT'S ALGEBRAIC SOLUTION OF EQUATIONS.

Leçons sur la Résolution algébrique des Équations. Par H. VOGT, professeur adjoint à la Faculté des Sciences de Nancy. Paris, Nony et Cie., 1895. 8vo., viii+201 pp.

THE present work is, we suppose, intended to be an introduction to the *modern* theory of the algebraic solution of equations. It is true that the word *modern* does not appear in the title, but however elementary the character of a *new* book of this kind may be, it is natural to suppose that the author will present his material in accordance with modern points of view, as far as these are elementary and simple.

This, however, is not the case with the volume in hand, as we proceed to show. First and foremost we have the following serious criticism to make. The rockbed of the modern theory of the algebraic solution of equations is the principles of Galois. A text book on this subject which does not explain these with all detail and use them systematically from start to finish cannot be called modern.

That the present volume sins grievously in this respect can be shown at once. Galois' theory proposes a perfectly simple and uniform scheme for the solution of any given equation. In a work of this kind this scheme should be developed at the start and then undeviatingly employed throughout the work whenever the solution of a particular class of equations is being effected. Almost the first step in this scheme relates to the adjunction of a rational function of the roots of the given equation and the determination of the ensuing Galoisian group. Is it possible to fancy a modern treatment of the subject this book deals with, which takes up this problem, as fundamental as it is simple, just ten pages before its close! Such a fact means simply that the author throws Galois' theory to the winds.

The reason for this is nowhere explicitly given. It cannot be that Galois' principles are not fruitful; for they occupy a central position in the great field of modern mathematical speculation. It cannot be because they are too difficult to be treated in an elementary text book; for as we shall see the author treats questions much more abstruse than the elementary Galoisian theory offers. It cannot be that the problems it enables us to solve are uninteresting; for every one familiar with Galois' theory knows the contrary. It may be that M. Vogt thinks Galois' methods are not all rigorous. Chapter IX. at least lends color to this supposition. This chapter treats of the algebraic solution of equations and culminates in the theorem of Ruffini and Abel that the general equation of degree greater than four cannot be solved algebraically. The author frankly states that it is taken *en grande partie* from Chapter XIII. of Netto's book on Substitutionentheorie und ihre Anwendungen auf die Algebra. Now Netto in this chapter takes a stand which is either trivial or which is an impugnment of the correctness of those parts of Galois' theory which treat of problems of this character. As the question is altogether fundamental and as put by Netto may easily lead one astray, it is worth spending a few words on it. The question at issue is this: Does Galois' theory enable us to draw into the circle of our reasoning irrational resolvents, *i. e.*, resolvents whose roots are not rational functions of the roots of the given equation, or does it not? The statement of Netto's in this connection is this:* since the theory of substitutions treats only of *rational* functions of the roots it is impossible to employ this theory when dealing with irrational functions. To use the theory of substitutions, then, in a problem which requires the consideration of irrational resolvents would be a *petitio principii*. Such problems he declares can be settled only by algebraic reasoning. Strictly speaking this is doubtless quite correct, but taken in the strict sense these remarks are trivial. If taken in the sense that the demonstration, for example, of the Ruffini-Abel theorem by the Galoisian theory is unsound, and that it must be replaced by the laborious algebraical reasoning which Netto employs* and in which he is followed by Vogt, Netto's position is incorrect and pernicious. That irrational resolvents can be employed directly in Galois' theory follows from the simple fact that, whenever a reduction of the Galoisian group takes place on adjoining an accessory irrationality, the same effect can be produced by a rational function of the roots. The theorem which is fundamental in these questions, and which cannot be insisted on too much, is due to Jordan and given in his Traité, p. 269. The demonstration of this theorem is entirely substitution-theoretical. How does this agree with Netto's declaration: "treten daher * * * irrationale Funktionen der Wurzeln auf, so befinden wir uns auf einem Gebiet, in dem von Substitutionen überhaupt keine Rede mehr sein kann." It would certainly enlighten many if Professor Netto would explain how these remarks are to be put in harmony with §§ 230, 231 of his book, particularly with the statement made at the close of § 230.

---

*L. c., p. 235.

Let us look now at the selection of material M. Vogt has to offer the reader. In a work that treats Galois' theory so shabbily we are not surprised to find only some twenty pages devoted to the theory of substitution groups. This is certainly unfortunate. The theory of groups is every day winning in importance. One has only to think of the rôle they play in geometry and in the theory of differential equations. The Galoisian theory of equations offers a splendid opportunity to introduce the student to an important part of this great theory, namely, the theory of *finite* groups. At the very start of the Galoisian theory, substitution groups demand our attention. The notions of transitivity, primitivity, invariant subgroups, series of composition, and isomorphism present themselves simply and naturally at once. A little later when we begin to consider more carefully the groups of the resolvents we find it necessary to pass from the narrow notion of a substitution group to the broad and fertile notion of a group in the abstract. A little later still we come to the groups of the regular bodies, *i. e.*, to particular cases of the all important linear group. It is with regret that we see such an opportunity entirely ignored.

Instead then of a treatment of such questions, we find a wearisome reproduction of some of Kronecker's algebraical

---

*It is Kronecker's modification of Abel's proof. Cf. *Monatsber. d. Berl. Akad.*, 1879, p. 205.

theories. A chapter is devoted to cyclic and metacylic functions of n independent variables; another chapter presents Kronecker's celebrated method of decomposing a form in n variables for an arbitrary domain of rationality. A third chapter gives Kronecker's treatment of abelian equations; and a fourth, Kronecker's researches on metacylic equations of prime degree. We do not wish to be understood as underestimating Kronecker's methods; on the contrary, we are an ardent admirer of them. But the methods of Kronecker here given form but a small part of the equipment of this mathematical Hercules. To try to give what is necessary would utterly crush the reader; to give no more than M. Vogt has given seems to us wholly inadequate. With half the space, all the results of importance the author has given could be demonstrated by Galois' theory and with far greater ease to the reader.

I pass now to a few criticisms of detail. The systematic employment of indeterminates should certainly be preceded by a few words of explanation. The author does not bring out with sufficient emphasis the meaning of the very fundamental terms: valeur numérique, valeurs numériquement distinctes, algébriquement distinctes. The author seems to be influenced by Kronecker's dictum which forbids the use of purely logical definitions. At least in two important instances he has followed it, viz., in the decomposition of a form into irreducible factors, and in the actual determination of the Galoisian group for a given equation. A third equally important case he has not treated, viz., the problem of determining whether a given rational function of the roots belongs to a given group or not.

On p. 62 we are given a definition of a general equation, and the remark is made that equations whose coefficients are integers are special. On p. 80 special equations are said to be those whose Galoisian group is not the symmetric group. There is a confusion of terms here. As Hilbert showed for the first time there are an infinity of equations whose group is the symmetric group and whose coefficients are integers.*

On p. 146 the rule for forming cyclic functions is not universally applicable, as simple examples show.

Consider the Abelian equation for $R(1)$, $x^4 + 1 = 0$. Let $\varepsilon = e^{2\pi i/8}$; set $x_0 = \varepsilon$, $x_1 = \varepsilon^3$, $x_2 = \varepsilon^5$, $x_3 = \varepsilon^7$; also let $x_1 = \vartheta_1 x_0$, $x_2 = \vartheta_2 x$. Then Kronecker's scheme for the roots becomes

$$\vartheta_1^{h_1}\vartheta^{h_2}x_0 \qquad (h_1, h_2 = 0, 1).$$

---

* Crelle, vol. 110.

The functions y_α become here

$$y_0 = \sum_{k_2} \vartheta_2^{k_2} x_0 = x_0 + x_2 ; \quad y_1 = \sum_{k_1} \vartheta_1^{k_1} x_0 = x_1 + x_3 .$$

Now both y_0 and y_1 are zero; they are thus not cyclic, and the rule breaks down.

Finally we observe that the treatment in Chapters X. and XII. of Kronecker's problem, of finding the necessary and sufficient form of the roots of all algebraically solvable equations of prime degree n, is far too condensed for so abtruse a matter. It is also lacking in rigor in two essential points. The question whether the functions

$$\psi_\mu = \sum_{r=0}^{n-1} \omega^{\mu r} x_r \qquad (\omega^n = 1)$$

vanish or whether the functions

$$y_q = \psi_{g^q} \psi_{g^{q-1}}^{-g} \qquad (q = 1, 2, \cdots, n-1)$$

are distinct is not discussed.

Before closing we beg to have it clearly understood that our criticisms have been made on the supposition that the volume in hand is to serve as an introduction to the modern theory of the algebraic solution of equations. To one who is already familiar with the elements of this theory, the present work will give much interesting and valuable information, particularly in regard to the methods peculiar to Kronecker. It may then serve in some measure as a preparation toward studying the papers of this great master.

JAMES PIERPONT.

YALE UNIVERSITY,
March, 1900.

ELEMENTS OF THE CALCULUS.

The Elements of the Differential and Integral Calculus, based on the Kurzgefasstes Lehrbuch der Differential- und Integralrechnung, von W. Nernst und A. Schönflies. By J. W. A. YOUNG and C. E. LINEBARGER. New York, D. Appleton and Co., 1900. 8vo., xvii + 410 pp.

OF the various new text-books on the calculus, this recent joint publication by a teacher of mathematics and a teacher of physics and chemistry will doubtless attract much interest, based as it is upon the German work, intended primarily for chemists, which appeared in 1896 from the pens of Professor Nernst, the celebrated exponent of the new physical chemistry, and Professor Schönflies, widely known for his investigations in pure and applied mathematics. Having taught calculus to classes of technical students and having been formerly a chemist on a state geological survey, the reviewer finds himself fully in sympathy with such a work and ready to approve even a greater number of examples and illustrations from the fields of chemistry and physics.

To the students of mathematics, astronomy, advanced physics, or technology the book is intended as a "pre-view" or general survey of the field, giving a clear insight into the fundamental principles, methods, and results, rather than the more elaborate and complex parts of the subject. The book is also offered to the general student in the belief that an elementary knowledge of the calculus should round out the mathematical career of the students seeking liberal culture. In the opinion of the reviewer the work is well adapted to those teachers or students of the natural or exact sciences who would privately acquire through an interesting and rigorous text a fair knowledge of the calculus and a view of its wide application in all domains of science.

The book is in two senses an introduction to the subject. In the first place, it presupposes only a knowledge of geometry, plane trigonometry, and the very simplest parts of algebra. The first seventy-five pages are devoted to an introduction to analytic geometry, giving as much as is needed in the later chapters. Chapters on higher algebra are inserted when needed for their applications to the calculus. Twenty pages are devoted to limits, in which the pedagogical turn of mind of the author or authors is clearly discernible. Several definitions of limits are given, graded in difficulty up to the modern treatment by means of the standard "epsilon." Seven pages on logarithms, ten pages of illustrations of partial fractions, fifty pages on infinite series, of which fifteen are devoted to the elementary algebraic theory, are given at convenient places in the text.

In the second place, the elementary character of the book may be indicated by the remark that the authors avoid the subjects curvature and elliptic integrals, so that the problem of the rectification of the ellipse and the problem of the simple pendulum are not even mentioned.

The elementary parts of the calculus are treated with admirable clearness and pedagogical insight, and yet the proofs possess all the rigor demanded by the modern methods in analysis. Each fundamental process is first illustrated by simple examples chosen from geometry, mechanics, physics, or chemistry, and afterwards the general method is exhibited in mathematical formulæ.

The method of limits is used exclusively. In view of the strict avoidance of differentials, it would seem more pedagogical to replace the term differentiation by derivation. The earliest introduction (on p. 108) of the symbol dy/dx for the derivative might have been done with more care, giving temporarily the notation $\frac{d}{dx}(y)$ or the alternative notation D_xy. The reader would be spared the confusion usually attendant upon the first notation. That the authors themselves will admit the justness of this criticism is apparent from their remark on p. 172, "In the Differential Calculus we grew accustomed not to keep $\frac{d}{dx}$, the symbol for the operation of differentiation, invariably separated sharply from the quantity to which the operation was to be applied, but sometimes to write for compactness $\frac{dy}{dx}$, ⋯ instead of $\frac{d}{dx}y$, ⋯." As a matter of fact the former symbol occurs six times before the latter is (on p. 116) first introduced !

It is not easy to make students *feel* that dy/dx is not a fraction. Furthermore, of the two equivalent formulæ,

$$\frac{dy}{du}\cdot\frac{du}{dx}=\frac{dy}{dx},\quad \frac{d}{du}(y)\cdot\frac{d}{dx}(u)=\frac{d}{dx}(y),$$

the necessity of formal proof is not usually felt by the student in the case of the first form, but is apparent in the case of the second. Instead of the usual short proof of this formula (based upon the lemma that the limit of a product equals the product of the limits of the factors), the authors devote pp. 150–152 to a rather complicated proof. While the reviewer is accustomed to give both, he would rather omit the latter than the former.

The authors never define dx and dy as separately existing quantities,* but still retain the historic symbol $\int f(x)dx$ for the function whose derivative with respect to x is $f(x)$. The step involved is certainly logical, but rather severe logic. Here again the (temporary) use of an alternative symbol, as $D_x^{-1} f(x)$, coupled with a reference to the definite integral, would bring this heroic adherence to the method of limits somewhat nearer to the plane of the student. Indeed, what student will not stumble at the statement on p. 252: "The sign $\int$ represents a form of *s* now obsolete, standing for the word *sum*, and *ydx* represents the type of the terms of the sum." The refusal to define *dx* makes the formula (p. 192) for integration by parts quite cumbrous. A readjustment must certainly take place somewhere in a student's career, if he is to continue his mathematical or physical studies in the modern literature.

---

* This may be done with rigor by taking them to be *any* two quantities (infinitesimal or not) whose ratio is the derivative. For application in integral calculus, we may also define dx as an arbitrary infinitesimal increment of x and dy as that dependent function equal to the product of dx and the derivative y'. Geometrically, dx and Δy refer to the curve, while dx and dy refer to its tangent.

On p. 296 and in the examples on p. 297 and pp. 301–307, it is required to find dy/dx when y is given as an implicit function of x, say $\varphi(x, y) = 0$. It would seem desirable to explain the usual method of taking the derivative with respect to x of both members of the equation and then solving for dy/dx. This procedure is more practical (even if it involves the calculation of the same derivatives) since it does not require the retention of a formula somewhat difficult to remember.

On p. 300, formula (5), δ is used four times for ∂.

The last chapter, a literal translation of the corresponding chapter of the German text, is highly interesting and useful. It gives a new and successful method for the differentiation or integration of functions found empirically, *i. e.*, given by a table of experimental results. In it occurs (p. 390) a confusing change of notation. It would make for clearness to omit the x introduced, but not used, in the regular text, and to explain, elsewhere than in the example given in small print, the p and θ actually used.

By this time it will be seen that the reviewer has sought in vain for very serious grounds for criticism of the text. He may therefore be permitted to congratulate the authors upon the skill with which they have completed such an interesting and valuable introduction to the calculus.

L. E. DICKSON.

THE UNIVERSITY OF TEXAS.

PASCAL'S CALCULUS OF VARIATIONS.

Die Variationsrechnung. Von E. PASCAL. Deutsch von A. SCHEPP. B. G. Teubner, Leipzig, 1899. vi. + 146 pp.

THE German translation of the "Calcolo delle variazioni" published by Ernesto Pascal in 1897 gives to American mathematicians in convenient form the best book on the calculus of variations that has, to our knowledge, appeared up to the present time. The book consists of only 150 octavo pages and presents concisely the principal facts of the subject. A valuable feature of the work will certainly be found to be the very excellent and apparently complete bibliography given in connection with brief accounts of the development of the calculus of variations. No book with which we are acquainted could be better adapted to controvert the lay opinion that everything in mathematics is exact and beyond dispute. Again and again the author calls attention to errors made by writers in this field. He often calls attention, too, to gaps which remain still to be filled in the theory, and makes the reader sometimes feel that the results which we already have rest on a rather precarious foundation. The chief fault of the book, from our point of view, is that it sacrifices simple and natural discussion to the pursuit of the end so dear to Italian mathematicians, the greatest possible generality.

The apparent purpose of the author is to give an account, absolutely rigorous as far as it goes, of the present condition of the science. That such an end is in the calculus of variations especially difficult to attain appears from the fact that the proofs are not always precise and that the author prefers often to tell us that the work given is not rigorous rather than to attempt to make it so. A serious deficiency is the almost entire lack of reference to the work of Weierstrass. This lack is, of course, intentional and doubtless due to the author's courtesy in hesitating to refer to what is to appear in Schwarz's edition of Weierstrass's works. It seems to us, however, that Pascal, with entire justice to Weierstrass, might have absorbed a little more of Weierstrass's spirit, and distinguished more carefully among the different sorts of variations employed, laying a proper emphasis on the function of each kind. The most unsatisfactory part of the book is the treatment of Jacobi's criterium. This condition is abstruse and difficult to grasp in all its beauty and ingenuity even in the simplest case. Pascal gives a purely analytical treatment of the general case. We venture to say that few readers will care to follow him through these difficult generalities. Nevertheless, this part of the book is of value, presenting as it does results which are to be found, we believe, only in various periodicals. But of the simple geometrical meaning of Jacobi's criterium, of the fact that it is a necessary as well as a sufficient condition for a maximum or a minimum, of the fact that from this criterium we may discover how long a curve continues to satisfy the conditions of the problem—not a word is said. From this we may see then that the book is not perhaps so readable as it might have been made.

We mention some points that to us seem to require criticism. A complete list would be of considerable length and would perhaps lay an undue emphasis on the faults of the work. In the opening chapter (p. 15) the author defines a variation as something "unendlich klein" and similar to a differential. This is an unsatisfactory and confusing definition, and is amusing in juxtaposition to Pascal's comment on Lagrange (p. 5). On page 16 we find the following: "In der That, wenn die beiden unendlich nahen Curven das Bestreben haben, mit einander zusammenzufallen, so ist es selbstverständlich, dass z. B. auch ihre Tangenten zusammenzufallen suchen." Such a primeval statement is utterly out of place in a book purporting to be rigorous. In the italics preceding this quotation the "variation of the derivative" is used in place of the "derivative of the variation." This slip would seem less worthy of notice if the author considered it evident, as it really is, that the two are identical. He does not, however, but proves at length two pages afterwards that they are so. The discussion of the restrictions of the "regular variation," a common term which by the way Pascal does not use, is on the whole, gratifying. He omits, however, to state what seems to us the point of chief interest, that any special variation may with perfect freedom be used to obtain a *necessary* condition for a maximum or a minimum. In pages 17–19 the author does not always make clear his assumptions as to the nature of the variation. On page 21 he says F must be assumed continuous, while two pages before he assumed that F had finite partial derivatives, an assumption certainly implying continuity. On page 39 the reasoning is purely formal, precisely that which he condemns most severely. His statement here that his process leads to the problem of maxima and minima of ordinary analysis is not accurate. His result he justifies by a footnote which admits that the text is inconclusive.

The treatment of multiple integrals though brief and incomplete is, on the whole, very good. We are glad to see several problems mentioned which are sometimes omitted in a course on this subject, such as the general problem of the calculus of variations. This Pascal calls, without sufficient reason we think, "Mayer's problem." The book closes with a discussion of the most famous problems of the calculus, *e. g.*, Newton's problem and the brachistochrone, with shorter notices of many others. The translator has added to the book some references to articles that have appeared since 1897, and indexes.

J. K. WHITTEMORE.

HARVARD UNIVERSITY.

NOTES.

THE second number of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY, which has just appeared, consists of 162 pages and contains the following articles:—"On the metric geometry of the plane n–line," by F. MORLEY; "On relative motion," by ALEXANDER S. CHESSIN; "Plane cubics and irrational covariant cubics," by HENRY S. WHITE; "A purely geometric representation of all points in the projective plane," by JULIAN LOWELL COOLIDGE; "The decomposition of the general collineation of space into three skew reflections," by EDWIN B. WILSON; "A new method of determining the differential parameters and invariants of quadratic differential quantics," by HEINRICH MASCHKE; "On the extension of Delaunay's method in the lunar theory to the general problem of planetary motion," by G. W. HILL; "On the types of linear partial differential equations of the second order in three independent variables which are unaltered by the transformations of a continuous group," by J. E. CAMPBELL.

THE INTERNATIONAL MATHEMATICAL CONGRESS AT PARIS: At the two general sessions of the congress, the following addresses will be delivered:—August 6th: "On the historiography of mathematics," by Professor M. CANTOR.—"Three Italian analysts, Betti, Brioschi, Casorati, and three ways of considering the questions of analysis; their influence," by Professor V. VOLTERRA.—August 11th: "A page from the life of Weierstrass," by Professor G. MIT-

TAG-LEFFLER.—"The rôle of intuition and logic in mathematics," by Professor H. POINCARÉ.

The work of the congress will be carried on in six sections which will meet at least once daily. These sections are:—1° Arithmetic and algebra; 2° Analysis; 3° Geometry; 4° Mechanics, mathematical physics, and celestial mechanics; 5° Bibliography and history; 6° Teaching and methods. The members are requested to submit the subjects of their communications and the section to which they will be referred, before May 1st, to Professor H. Poincaré, president of the programme committee, rue des Grands-Augustins, Paris. The committee further requests that the time for the presentation of any communication be not allowed to exceed twenty minutes. Papers may be written and read in any of the four languages, German, English, French, and Italian.

THE final international conference on a catalogue of scientific literature has been postponed from Easter to June 12th. The conference takes place in London.

To the list of works on special branches of mathematics to be issued by B. G. Teubner, as announced in the April number of the BULLETIN, there should be added the following: W. F. OSGOOD, General theory of functions; W. WIRTINGER, Algebraic functions and their integrals, and Partial differential equations.

THE eighth edition of the logarithmic tables constructed by Professor GEORGE WILLIAM JONES, of Cornell University, has recently appeared bearing the imprint of Macmillan and Company, London, and G. W. Jones, Ithaca, N. Y., 1900.

CATALOGUES of new and second-hand mathematical works have recently been published by F. Muller and Company, Doelenstraat 10, Amsterdam, Holland, and B. Pellerano, Via Gennaro Serra 20, Naples, Italy.

H. WELTER, of Paris, has issued an elaborate catalogue of periodical publications, containing a number of broken and complete sets of the various mathematical journals of the world.

THE UNIVERSITY OF CHICAGO. The following advanced courses (four or five hours weekly) in pure mathematics have been announced for the summer quarter of 1900:—By Associate Professor H. MASCHKE: Elliptic functions; Modern analytic geometry; Advanced integral calculus, I.—

By Assistant Professor J. W. A. Young: Theory of numbers; Pedagogy of mathematics (one-half course).—By Mr. J. A. Smith: Theory of equations (first half of quarter, ten hours weekly.)

The mathematical club, with fortnightly meetings, is under the direction of the departmental faculties of mathematics and mathematical astronomy.

Columbia University. The following advanced courses are offered during the academic year 1900–1901 by the departments of mathematics and mechanics, each course extending throughout the year:—By Professor F. N. Cole: Riemann's theory of functions, including elliptic functions, three hours.—By Professor T. S. Fiske: Advanced calculus, three hours; Theory of abelian functions, three hours.—By Professor M. I. Pupin: Electromagnetic theory of light, two hours; Spherical harmonics, two hours.—By Professor R. S. Woodward: Advanced analytical mechanics, two hours; Theory of the potential function, two hours.—By Mr. C. J. Keyser: Modern theories of geometry, three hours.—By Dr. J. Maclay: Analytical theory of curves of double curvature and curved surfaces, three hours.—By Mr. H. B. Mitchell: Differential equations, three hours.—By Mr. J. C. Pfister: Theoretical mechanics, two hours.

The mathematical colloquium is held fortnightly.

Cornell University. During the academic year 1900–1901, the following advanced courses will be offered by the department of mathematics, each course occupying either two or three hours a week, as indicated, and continuing through the year:—By Professor L. A. Wait: Advanced analytic geometry, (a) plane, three hours, (b) solid, two hours; Advanced differential calculus, three hours.—By Professor J. McMahon: Theoretical mechanics, two hours; quaternions, two hours; Potential and spherical harmonics, two hours; Mathematical theory of sound, two hours.—By Professor J. H. Tanner: Algebraic invariants, two hours. By Dr. D. A. Murray: Differential equations, three hours; Finite differences, two hours.—By Dr. J. I. Hutchinson: Advanced integral calculus, two hours; Calculus of variations, three hours.—By Dr. V. Snyder: Projective geometry, three hours; Algebraic curves, two hours; General function theory, three hours; Theory of surfaces, two hours.—By Dr. G. A. Miller: Theory of groups of a finite order, three hours; Theory of numbers, two hours.

The Oliver mathematical club will meet every other week.

Harvard University. During the year 1900–1901 Professor Bôcher will be absent in Europe on leave. The following advanced courses in mathematics are offered :—By Professor J. M. Peirce : Tetrahedral coördinates, with applications to the study of points, lines, planes, and quadric surfaces; Quaternions (second course); Quaternion imaginaries and other selected topics in quaternions ; † Linear associative algebra.†—By Professors Byerly and B. O. Peirce : Trigonometric series, spherical harmonics, and potential function.—By Professor B. O. Peirce : Elasticity.†—By Professor Osgood : Infinite series and products ; † Galois's theory ; † either Calculus of variations, or Elliptic functions.—By Dr. Bouton : Modern geometry; Theory of functions (first course); Introduction to Lie's theory of groups.—By Mr. Whittemore : Theory of equations and introduction to invariants ; † Calculus (second course); Differential geometry.—By Mr. Coolidge : Geometry of position.

These courses will each involve three lectures a week throughout the year, except those marked †, which involve about half this number of lectures. Professors Asaph Hall and Osgood also offer courses in reading and research on Selected topics in celestial mechanics and Higher analysis, respectively.

During the summer semester, 1900, the several universisities below offer the following mathematical courses :

University of Innsbruck. By Professor Otto Stolz : Differential and integral calculus, with exercises in seminar, four hours ; General arithmetic, second part, introduction to the theory of functions of complex variables, with exercises in seminar, three hours.—By Professor W. Wirtinger : Analytical geometry of space (continued), three hours ; Theory of numbers (continued), two hours ; Seminar, two hours.

University of Prague. By Professor G. Pick : Differential and integral calculus, three hours ; Elements of the calculus of variations, two hours ; Geometrical exercises, two hours.—By Professor F. Lippich : Theoretical mechanics, three hours ; Theory and applications of potential, two hours.

University of Vienna. By Professor G. von Escherich : Elements of differential and integral calculus II, five hours ; Exercises on the preceding course, two hours ; Seminar, two hours ; Proseminar, one hour.—By Professor L. Gegen-

BAUER : Algebra (continued), four hours ; Elements of the theory of invariants, one hour ; Mathematical statistics, three hours ; Proseminar, one hour ; Seminar, two hours. —By Professor F. MERTENS : The laws of reciprocity (continuation), five hours ; Theory of probabilities, three hours ; Seminar, two hours ; Proseminar, one hour.—By Professor G. KOHN : Analytical geometry of space, four hours ; Exercises on the preceding, one hour.—By Dr. V. SERSAWY : The mathematics of insurance, I, three hours, II, four hours.—By Dr. A. TAUBER : Elements of perspective, two hours ; Insurance mathematics (continued), four hours ; Exercises on the latter course, two hours.—By Dr. K. ZINDLER : Kinematics, two hours.—By Dr. E. BLASCHKE : Introduction to mathematical statistics, II, three hours.—By Dr. K. ZSIGMONDY : The transcendentality of the numbers e and π, one hour.—By Dr. R. D. v. STERNECK : Elementary theory of numbers, two hours.

THE following subject has been announced by the Madrid academy of sciences for its mathematical prize for the year 1901 : "Didactic exposition of the modern theories of non-euclidean geometry, or a detailed analysis of the principal works upon this branch of science from the time of Gauss to the present." Competing memoirs should be written in Spanish or Latin and will be received by the secretary of the academy before December 31st, 1901. The first prize consists of fifteen hundred pesetas and a gold medal.

AMONG the corresponding members elected by the Berlin academy of sciences at its recent celebration commemorating the two hundredth anniversary of its foundation are the mathematicians Professor P. GORDAN, of Erlangen, Professor F. MERTENS, of Vienna, and Professor F. SCHOTTKY, of Marburg.

PROFESSOR P. DRUDE, of Leipsic, has become professor of mathematical physics at the University of Giessen.

IT is reported that Professor L. BOLZMANN, of the University of Vienna, has received a call to the University of Leipzig and is considering it favorably.

DR. J. HORN, formerly docent at the Polytechnic Institute at Charlottenburg, has been called to the School of Mines at Clausthal as professor of mathematics.

DR. L. E. DICKSON has resigned an associate professorship in mathematics at the University of Texas to become assistant professor at the University of Chicago. DR. H.

E. SLAUGHT has been promoted to an assistant professorship of collegiate mathematics in the latter university.

UNDER the recent readjustment of its relation to Columbia University, Barnard College is hereafter to have a distinct department of undergraduate mathematics. Professor F. N. COLE has been made head of this department, retaining however his position in the university graduate school of pure science. Dr. EDWARD KASNER (Ph.D., Columbia) has been appointed tutor, and Miss GRACE ANDREWS (M.A., Columbia) assistant in the department at Barnard.

DR. J. B. CHITTENDEN, tutor in mathematics at Columbia University, has been appointed professor of mathematics in the Brooklyn Polytechnic Institute, Brooklyn, N. Y. Mr. H. B. MITCHELL, assistant in mathematics at Columbia, has been promoted to a tutorship.

MR. A. D. FOX has been made professor of mathematics at Whitman College, Walla Walla, Wash.

MR. E. A. PATTINGILL, recently a graduate student at Cornell University, has been appointed instructor in mathematics in the Iowa State Agricultural College.

THE death is announced of Professor B. CHRISTOFFEL, professor of mathematics at the University of Strassburg, aged seventy years; Professor Christoffel had offered no lectures since 1894 on account of ill health.

THE death is also announced of Professor WILLIAM M. THRASHER, for many years professor of mathematics at Butler University, Irvington, Ind.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BENNER (H.). Bestimmung der Coefficienten, welche bei der Berechnung der Integrale $\int \frac{x^n dx}{\sqrt{1 + ax + bx^2}}$ und $\int \frac{x^n dx}{\sqrt{1 + ax + bx^2 + cx^3}}$ auftreten. (Diss., Erlangen, 1897.) Boston, Ginn, 1899. 8vo. 10 + 59 pp.

BIBLIOTHECA MATHEMATICA. Zeitschrift für Geschichte der mathematischen Wissenschaften, herausgegeben von G. Eneström. 3te [erweiterte] Folge. Vol. I: 1900. Leipzig, Teubner, 1900. Per annum M. 20.00

BÖGER (R.). Ebene Geometrie der Lage. Leipzig, Göschen, 1900. 8vo. 10 + 289 pp. Cloth. (Sammlung Schubert, No. VII.) M. 5.00

BROWN (E. W.). See TRANSACTIONS.

BULLART (W. G.) On the general classification of plane quartic curves. (Diss.) Worcester, Mass., 1899. 8vo. 16 pp., 2 plates.

DELAUNAY (N.). See WASSILIEF (A.).

DUDENSING (W.). Ueber die durch eine allgemeine dreigliedrige algebraische Gleichung definierte Funktion und ihre Bedeutung für die Auflösung der algebraischen Gleichungen von höherem als viertem Grade. Leipzig, Teubner, 1900. 8vo. 8 + 52 pp. M. 2.40

ENESTRÖM (G.). See BIBLIOTHECA.

FELDBLUM (M.). Ueber elementar-geometrische Konstruktionen. (Diss.) Göttingen, 1899. 8vo. 48 pp., 1 plate.

FISKE (T. S.). See TRANSACTIONS.

FREGE (G.). Ueber die Zahlen des Herrn H. Schubert. Jena, Pohle, 1900. 8vo. 6 + 32 pp. M. 1.20

GANTER (H.) und RUDIO (F.). Die Elemente der analytischen Geometrie. Zum Gebrauche an höheren Lehranstalten, sowie zum Selbststudium. Mit zahlreichen Uebungsbeispielen. Teil I: Die analytische Geometrie der Ebene. 4te Auflage. Leipzig, Teubner, 1900. 8vo. 8 + 180 pp. Cloth. M. 3.00

GASCO (D. L. G.). See MANSION (P.).

GRAF (J. H.) und GUBLER (E.). Einleitung in die Theorie der Besselschen Funktionen. Heft 2: Die Besselsche Funktion zweiter Art. Bern, Wyss, 1900. 8vo. 7 + 156 pp. M. 3.20

HAGEN (J. G.). Synopsis der höheren Mathematik. (In 4 Bänden.) Vol. III: Differential- und Integralrechnung. (In circa 5 Lieferungen.) Berlin, 1900. 4to. Lieferung 1: pp. 1–64. M. 5.00.

JAHRBUCH über die Fortschritte der Mathematik, begründet von C. Ohrtmann. Herausgegeben von E. Lampe. Vol. 28, Jahrgang 1897 3tes (Schluss-) Heft. Berlin, Reimer, 1900. 8vo. 68 pp. + pp 609–935. M. 12.40

KNESER (A.). Lehrbuch der Variationsrechnung. Braunschweig, Vieweg, 1900. 8vo. 15 + 311 pp. M. 8.00

KÖLMEL (F.). Bewegungen und Umlegungen der Ebene bei projektiver Massbestimmung. Untersuchungen zur nichteuklidischen Geometrie. Lahr, Schauenburg, 1900. 8vo. 7 + 99 pp., 1 plate. M. 2.50

LAMPE (E.). See JAHRBUCH.

LAURENT (H.). Cours de mathématiques, professé à l'Institut agronomique. Paris, Carré et Naud, 1900. 8vo. 224 pp.

MANSION (P.). Introduction à la théorie des déterminants, avec de nombreux exercices, à l'usage des établissements d'instruction moyenne. 3e édition. Gand, Hoste, 1899. 8vo. 40 pp. Fr. 1.00

——. Eléments de la théorie des déterminants, avec de nombreux exercices. 6e édition, revue et augmentée. Paris, Gauthier-Villars, 1900. 8vo. 6 + 91 pp. Fr. 3.00

——. Elemente der Theorie der Determinanten. Mit vielen Uebungsaufgaben. 3te, vermehrte Auflage. Leipzig, Teubner, 1900. 8vo. 103 pp. M. 2.60

——. Theoria sucinta de las funciones hiperbolicas. Traducido por D. L. G. Gasco. Valencia, Aguilar, 1899. 8vo. 44 pp. Fr. 1.00

Meyer (W. F.). Rapporto sui progressi della teoria proiettiva degli invarianti nell'ultimo quarto di secolo. Traduzione dal tedesco da G. Vivanti, con aggiunte dell'autore. Napoli, 1899. 4to. 240 pp. Fr. 8.00

Moore (E. H.). See Transactions.

Netto (E.). Vorlesungen über Algebra. Vol. 2, 2te (Schluss-) Lieferung. Leipzig, Teubner, 1900. 8vo, 12 pp.+pp. 193–519. M. 10.00.

Rice (H. L.). The theory and practice of interpolation, including mechanical quadrature, and other important problems concerned with the tabular values of functions. With the requisite tables. Lynn, Mass., Nichols, 1899. Folio. 10 + 234 pp. Cloth.

Rudio (F.). See Ganter (H.).

Simon (M.). Analytische Geometrie der Ebene. Leipzig, Göschen, 1900. 8vo. 7 + 372 pp. Cloth. (Sammlung Schubert, No. VIII.) M. 6.00

Transactions of the American Mathematical Society. Edited by E. H. Moore, E. W. Brown, T. S. Fiske. Published quarterly by the Society with the coöperation of Harvard University, Yale University, Princeton University, Columbia University, Haverford College, Northwestern University, Cornell University, The University of California, Bryn Mawr College, The University of Chicago. Vol. I, No. 1, January, 1900. New York, The Macmillan Company, 1900. 4to. 96 pp. Per annum $5.00

Vivanti (G.). See Meyer (W. F.).

Wassilief (A.) und Delaunay (N.). P. L. Tschebyschef. (P. L. Tschebyschef und seine wissenschaftlichen Leistungen, von Wassilief; Die Tschebyschefschen Arbeiten in der Theorie der Gelenkmechanismen, von Delaunay.) Leipzig, Teubner, 1900. 8vo. 3 + 70 pp., portrait. M. 4.00

II. ELEMENTARY MATHEMATICS.

Bagnoli (E.). Geometria rettilinea e curvilinea, trattata con metodo preeuclidico, e cronogoniometria. Roma, Loescher, 1899. 8vo. 295 pp., 24 plates. Fr. 5.00

——. Trattato delle corde nel circolo. Roma, Loescher, 1899. 8vo. 81 pp., 10 plates. Fr. 3.00

Behr (A.). Die Planimetrie (ebene Geometrie). Leipzig, Schäfer, 1900. 8vo. 8 + 104 pp. (Die Schule des Bautechnikers, herausgegeben im Verein mit Lehrern an Bau- und anderen technischen Fachschulen von F. Stade, Vol. 2.) M. 2.00

Bohnert (F.). Ebene und sphärische Trigonometrie. Leipzig, Göschen, 1900. 8vo. 8 + 160 pp. Cloth. (Sammlung Schubert, No. III.) M. 2.00

Chailan (E.). Eléments de trigonométrie, à l'usage de la classe de seconde moderne. Paris, Poussielgue, 1900. 16mo. 64 pp.

EHRHARDT (H.). Neues System der Flächenberechnung und Flächenteilung mit Hülfe einer planimetrischen Tafel, welche zugleich als Produkten- und Quadrattafel dient, nebst einer Sinustafel. Stuttgart, 1900. 8vo. 91 + 9 pp., 3 plates. M. 3.00

EUCLID, Books 1 and 2. Proofs simplified and re-arranged by T. Varley. London, Allman, 1900. 12mo. 156 pp. Cloth. 1s.

EVANS. Euclid riders fully worked out. London, Simpkin, 1900. 12mo. 115 pp. Boards. 1s. 6d.

FRANZ (R.). See UTH (K.).

GIROD (F.). Nouvelle arithmétique des écoles primaires, suivie de notions élémentaires d'algèbre et de comptabilité (troisième année, destinée spécialement au élèves qui se préparent aux examens du brevet élémentaire et du brevet supérieur). 6e édition. Paris, André, 1900. 18mo. 460 pp. Fr. 3.00

HALL (J. L.). Tables of squares. New York, Engineering News Publishing Company, 1900. 16mo. Cloth. $2.00

KÖHLER (E. T.). Manuale logaritmico-trigonometrico, contenente i logaritmi volgari o di Brigg di tutti i numeri fino à 108000 con 7 decimali, i logaritmi di Gauss, i logaritmi delle funzioni trigonometriche di 10 in 10 secondi, ecc. 11a edizione della versione italiana. Lipsia, 1900. 8vo. 38 + 388 pp. Fr. 4.00

KUHN (K.). Lehrbuch der Elementar-Arithmetik. Teil 1. 2te Auflage. Hildburghausen, Pezoldt, 1900. 8vo. 4 + 48 pp. (Technische Lehrhefte: Mathematik, Heft 1.) M. 1.50

LEVI (F.). Problemi elementari relativi ad alcuni casi di equivalenza di figure piane. Lodi, Dell'Avo, 1899. 8vo. 23 pp., 2 plates. Fr. 0.50

LÖSUNGEN der Absolutorial-Aufgaben aus der Mathematik an den humanistischen Gymnasien Bayerns seit dem Jahre 1867. Nachträge 1895–1898. München, Pohl, 1900. 8vo. 4 + 4 + 6 + 7 pp. M. 0.40

METZIG (C.). Lehrbuch der Arithmetik und Algebra nebst Aufgabensammlung für Baugewerkschulen und verwandte technische Lehranstalten, sowie zum Selbstunterricht. Auflösungen und Lösungsanleitungen. Breslau, Morgenstern, 1900. 8vo. 48 pp. M. 1.50

MÜLLER (O.). Tavole di logaritmi con cinque decimali. 6a edizione, aumentata delle tavole dei logaritmi d'addizione e sottrazione, per cura di M. Rajna. Milano, Hoepli, 1900. 16mo. 36 + 191 pp. (Manuali Hoepli.)

PÜNCHERA (J.). Der Geometrie-Unterricht in der I. und II. Klasse der Kantonschule und in Realschulen. Chur, Hitz, 1899. 8vo. 195 pp.

RAJNA (M.). MÜLLER (O.).

SCHUBERT (H.). Mathematische Mussestunden. Eine Sammlung von Geduldspielen, Kunststücken und Unterhaltungsaufgaben mathematischer Natur. 2te Auflage. 3 Bände. (Vol. I: Zahl-Probleme. 8 + 199 pp. Vol. II: Anordnungs- und Wahrscheinlichkeitsprobleme. 4 + 247 pp. Vol. III: Reise-Probleme und geometrische Probleme. 3 + 265 pp.) Leipzig, Göschen, 1900. 8vo. Cloth. M. 12.00.

Soluciones y respuestas de los ejercicios y problemas contenidos en las Lecciones elementales de aritmética, por los Hermanos de las escuelas cristianas. Libro del maestro. 3a edicion, corregida. Paris, Oudinot, 1900. 16mo. 136 pp.

Tödter (H.). Anfangsgründe der Arithmetik und Algebra. Für den Schul- und Selbstunterricht in entwickelnder Lehrform bearbeitet. Teil I. Ausgabe B. 4te Auflage. Bielefeld, Velhagen & Klasing, 1900. 8vo. 2 + 70 pp. M. 0.80.

Trognitz (R.). Mathematische Aufgaben nebst Lösungen aus dem Pensum der Prima der höheren Lehranstalten. Meiningen, 1899. 8vo. 34 pp.

Uth (K.). Planimetrie. Leitfaden mit Konstruktionsaufgaben und Uebungssätzen. 6te Auflage, von R. Franz. Kassel, Hühn, 1900. 8vo. 8 + 157 pp. Cloth. M. 2.00.

Varley (F.). See Euclid.

Vidal (C.). Pour la géométrie euclidienne ; étude critique élémentaire sur les fondements de la géométrie. Paris, Croville-Morant, 1900. 8vo. 3 pp.

Zeissig (E.). Praeparationen für Formenkunde (Raumlehre, Geometrie) als Fach an Volksschulen. Teil 2 : Betrachtung, Darstellung und Berechnung der krummflächigen Körperformen und krummlinigen Flächen. Langensalza, Beyer, 1900. 8vo. 14 + 167 pp. M. 2.00

III. APPLIED MATHEMATICS.

Almy (J. E.). Ueber die Entladungspotentiale in festen und tropfbarflüssigen Dielektricis. (Diss.) Berlin, Mayer & Müller, 1900. 8vo. 35 pp. M. 1.20

Baumgartner (E.). Encyclopédie des assurances. Vol. I : A–Bil. Paris, 1899. 4to. Prix de souscription pour l'ouvrage complet (5 vol.) Fr. 100.00

Bertini (G. B.). Breve studio sulle casse pensioni a dividendi incogniti e sulle progressioni nella mortalità. Torino, 1899. 12mo. 78 pp. Fr. 1.00

Boulanger (J.) et Ferrié (G.). Les ondes électriques et la télégraphie sans fil. Paris, 1899. 8vo. 96 pp. Fr. 1.50

Bruno (K.). Der Stoss elastischer Kugeln. (Progr.) Klagenfurt, 1899. 8vo. 22 pp., 1 plate.

Choura (J.). Leitfaden für den Unterricht in der darstellenden Geometrie an den k. und k. Militär-Akademien. Parallel- und Central-Perspektive ; cotierte Projektionen (cotierte Ebenen). Wien, Seidel, 1900. 8vo. 3 + 247 pp., 1 plate. M. 4.80

Cotton (A.). Le phénomène de Zeemann. Paris, 1899. 8vo. 100 pp. Fr. 2.00

Ferrié (G.). See Boulanger (J.).

Geigenmüller (R.). Die Mechanik. Leipzig, Schäfer, 1900. 8vo. 8 + 136 pp. (Die Schule des Bautechnikers, herausgegeben im Verein mit Lehrern an Bau- und anderen technischen Fachschulen von F. Stade, Vol. 6.) M. 2.50

König (W.). See Lommel (E. von).

KUGLER (F. X.). Die babylonische Mondrechnung. Zwei Systeme der Chaldäer über den Lauf des Mondes und der Sonne. Auf Grund mehrerer von J. N. Strassmaier, S. J., kopierten Keilinschriften des britischen Museums. Mit einem Anhang über chaldäische Planetentafeln. Freiburg i. B., Herder, 1900. 8vo. 15 + 215 pp., 13 plates. M. 24.00

LINEHAM (W. J.). Text-book of mechanical engineering. 4th edition, enlarged. London, Chapman, 1900. 12mo. 1002 pp. Cloth. 12s. 6d.

LOMMEL (E. VON). Lehrbuch der Experimentalphysik. 6te Auflage. Herausgegeben von W. König. Leipzig, Barth, 1900. 8vo. 10 + 574 pp., portrait, 1 colored plate. M. 6.40

LORENTZ (B.). Grösse und Masse der Weltkörper. (Progr.) Kalksburg, 1899. 8vo. 33 pp.

MASONI (U.). Corso di idraulica teorica e pratica. 2a edizione, notevolmente ampliata. Napoli, 1899. 8vo. 24 + 680 pp. Fr. 9.50

REINGANUM (M.). Theorie und Aufstellung einer Zustandsgleichung. (Diss.) Göttingen, 1899. 8vo. 117 pp.

WARBURG (A.). Referat über die Wärmeeinheit, erstattet in der gemeinschaftlichen Sitzung der Sektionen für Physik und angewandte Mathematik am 22. September auf der Naturforscher-Versammlung in München. Leipzig, 1900. 8vo. 19 pp. M. 0.60

WÜST (A.). Anleitung zum Gebrauch des Taschen-Rechenschiebers für Techniker. 4te Auflage. Mit einem Rechenschieber. Halle, Hofstetter, 1900. 12mo. 16 pp. Boards. M. 1.25

LIBRARY

THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, April 28, 1900. The American Physical Society was in session at the same time, and a part of the afternoon was devoted to a joint meeting, at which the papers noted in the list below were presented by Professors E. W. Brown and R. S. Woodward. The attendance during the day was about forty, including the following twenty-four members of the Society:

Professor E. W. Brown, Professor F. N. Cole, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. A. S. Gale, Dr. G. B. Germann, Dr. A. A. Himowich, Mr. S. A. Joffe, Mr. C. J. Keyser, Dr. James Maclay, Dr. Emilie N. Martin, Dr. G. A. Miller, Mr. C. A. Noble, Professor M. I. Pupin, Professor J. K. Rees, Mr. C. H. Rockwell, Professor Charlotte A. Scott, Dr. Virgil Snyder, Professor E. B. Van Vleck, Professor L. A. Wait, Professor A. G. Webster, Dr. J. Westlund, Miss E. C. Williams, Professor R. S. Woodward.

The President of the Society, Professor R. S. Woodward, occupied the chair, yielding it during the joint meeting to Professor William Hallock of the Physical Society. The amendments of the Constitution outlined in the report of the February meeting were adopted. The Council announced the election of the following persons to membership in the Society: Professor R. D. Ford, St. Lawrence University, Canton, N. Y.; Dr. L. W. Reid, Princeton University, Princeton, N. J. Eight applications for membership were reported.

The following papers were read at this meeting:

(1) Dr. VIRGIL SNYDER: "On some invariant scrolls in collineations which leave a group of five points invariant."

(2) Mr. A. S. GALE: "Note on four theorems of Chasles."

(3) Professor CHARLOTTE A. SCOTT: "A theorem on quadrilaterals in space" (preliminary communication).

(4) Mr. F. H. LOUD: "Sundry theorems concerning n-lines in a plane."

(5) Dr. E. J. WILCZYNSKI: "Transformation of systems of linear differential equations."

(6) Professor FLORIAN CAJORI: "Semi-convergent and divergent series whose product is absolutely convergent."

(7) Professor E. W. BROWN: "A possible explanation of the eleven year period of sunspot activity."

(8) Professor R. S. WOODWARD: "An elementary method of integrating certain linear differential equations."

(9) Dr. G. A. MILLER: "On a certain class of abelian groups."

(10) Professor H. B. NEWSON: "On singular transformations and continuous groups."

(11) Professor E. O. LOVETT: "Group theory and geometry of four dimensions."

(12) Professor E. O. LOVETT: "The condition that a linear total differential equation be integrable."

(13) Professor C. H. HINTON: "Observations on the principle of duality."

Professor Hinton's paper was presented to the Society through Professor Lovett. In the absence of the authors, Mr. Loud's paper was read by Professor C. A. Scott, and the papers of Dr. Wilczynski, Professor Cajori, Professor Newson, Professor Lovett, and Professor Hinton were read by title. Abstracts of the papers are given below.

Dr. Snyder's paper is in abstract as follows: There are six cyclical collineations of space whose orders are less than seven: T_2, the harmonic homology; T_3, the axial involution; $T_4 \equiv (123)(4)(5)$, in which the elements of space are in triads; $T_5 \equiv (123)(45)$, in sextuples; $T_6 \equiv (1234)(5)$, in quadruples; and $T_7 \equiv (12345)$, in quintuples. A cyclical collineation of order six and having five elements chosen arbitrarily does not exist. Systems of scrolls contained in a linear congruence which are left invariant by these transformations are discussed, with particular reference to their asymptotic lines. The transformations T_3, T_4, and T_7 leave a [3, 1] scroll invariant in such a way that each asymptotic line is transformed into itself. The quartic surface, locus of the vertex of a quadric cone which passes through two triads of T_4, goes into itself; it touches a [3, 1] scroll in every point of the twisted cubic through the six points. This cubic is an asymptotic line on both surfaces. Every [3, 1] scroll having a pair of double imaginary pinch points has asymptotic lines of order 3.

In 1860 Chasles published the theorems that, if a right line be displaced, the middle points of the chords joining congruent points lie on a line or are identical, and that the planes perpendicular to these chords at their middle points pass through a line or are identical; and also corresponding theorems relating to the displacement of a plane figure.

These theorems form the basis of the geometrical consideration of the theory of finite displacements in space as developed by Study. Mr. Gale's note, which will appear in the *Annals of Mathematics*, shows how the theorems of Chasles follow from Wiener's theorem that every displacement in space may be resolved in ∞^2 ways into the product of two axis reflections. By means of this theorem, displacements are resolved into the product of two transformations such that the points of an arbitrary line or plane are invariant with respect to the first transformation. The theorems of Chasles follow immediately from properties of the second transformation.

The following is a summary of Professor Scott's preliminary communication: If corresponding vertices a, a'; b, b'; c, c'; d, d'; e, e'; f, f' of two quadrilaterals in different planes are joined, an interesting configuration of 6 lines $\alpha, \beta, \gamma, \delta, \varepsilon, \varphi$ is obtained. It is cut by a singly infinite system of planes in complete quadrilaterals; these planes, being common tangent planes to two ruled quadrics with a common generator, are tangent planes to a cubic torse, and consequently also to a twisted cubic. The configuration is intimately associated with the theory of cubic curves and torses, yet there seems to be no recognition of the theorem in the literature of this subject. There is no difficulty in obtaining a proof in various forms; but to exhibit the theorem as fundamental in the treatment of projective forms, a proof that shall rely solely on the formal properties of points, lines, and planes is desirable. It appears that such a proof must be obtainable by means of triangles in perspective; but it still remains to be found.

The point of departure for Mr. Loud's paper is furnished by Professor Morley's article in the April *Transactions* "On the metric geometry of the plane n-line." Professor Morley, having discussed a certain class of theorems resulting from combining n-lines by twos, had pointed out that they might also be combined three or more at a time with analogous results. The present paper considers the case of combination by threes. After briefly noting those of his results which are strictly analogous to some of Professor Morley's, the author has indicated a means of deducing one series of theorems from the other. A distinction between the two series is pointed out in the fact that in the present case each line is restricted to a single direction. An inquiry into the relation of two figures, derived one from the other by the reversal of one given line, follows, and occupies the greater part of the paper. The general nature of the theorems discussed is exemplified in the following: The theorem which in the parallelism of the two series becomes the analogue of Miquel's asserts that the twenty points, which are centers of circles tangent to six given directed lines taken by threes, lie by fours on fifteen circles, whose circumferences meet by fives in six points, and these points lie on a single circle. Again, as an illustration of the effect of reversal, we have the theorem: If all but one of an even number of given lines be turned, each through a right angle, about the point at which it meets the one remaining line, and if the Clifford circles be constructed for these rotated lines in both their original position and that to which they are turned, these circles intersect orthogonally. By making each line in succession the one upon which the others are turned, as many orthogonal pairs of circles are obtained as there are given lines, and all these circles pass through a common point, which is the focus of the p-fold parabola tangent to all the given lines.

Dr. Wilczynski's paper furnishes a proof of the following theorem: The most general transformation which converts a general system of linear homogenous differential equations into another system of the same form and order is

$$x = f(\xi), \quad y_k = a_{k1}(\xi)\eta_1 + \cdots + a_{kn}(\xi)\eta_n \quad (k = 1, 2, \cdots, n),$$

where f and a_{kl} are arbitrary functions of ξ. This theorem is the basis of a theory of invariants of such systems upon which the author is now engaged.

Pringsheim states in one of his articles that he sees no *a priori* reason why one might not be able to construct semi-convergent series possessing properties which would render the product of two series absolutely convergent. But he was unable to produce examples of such series. In the present paper Professor Cajori gives not only pairs of semi-convergent series, but also pairs of series, one semi-convergent, the other divergent, whose product converges absolutely. The process of constructing the series is explained. In all cases brought under consideration, one of the factor series has zero for its sum. Example: Of the following two series, T_1 is semi-convergent, T_2 is divergent; their product is absolutely convergent:

$$T_1 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{6p+1}-\frac{1}{6p+2}+\frac{1}{6p+3}-\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}\right),$$

$$T_2 \equiv \sum_{p=0}^{p=\infty}\left(\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}+\frac{1}{6p+1}+\frac{1}{6p+2}-\frac{1}{6p+3}\right).$$

The following differential equations are of frequent occurrence in mechanics and mathematical physics :

$$\frac{dx}{dt}+\alpha y=0, \quad \frac{dy}{dt}-\alpha x=0, \tag{1}$$

and

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+\alpha^2x=0, \tag{2}$$

where α and n are constants. The integrals of (1) are well known to be

$$x=a\cos(\alpha t+\theta), \quad y=a\sin(\alpha t+\theta), \tag{3}$$

where a and θ are constants. The integral of (2) is well known to take different forms according as $n^2>\alpha^2$, $n^2=\alpha^2$, or $n^2<\alpha^2$. Equation (2) represents the phenomena of damped vibrations. When $n=0$, it represents simple harmonic motion in a straight line ; while equations (1) represent the rectangular components of uniform motion in a circle. The usual methods of integrating these equations require some knowledge of the theory of differential equations in general. It is the object of Professor Woodward's to show how to integrate them without the aid of such knowledge. To this end, observe that (1) and (2) are comprised in the following equations of the first order :

$$\frac{dx}{dt}+\alpha y+2nx=0, \quad \frac{dy}{dt}-\alpha x=0. \tag{4}$$

These equations give

$$\frac{dx}{2nx+\alpha y}=-\frac{dy}{\alpha x}=-dt,$$

whence, by the introduction of two factors β, γ,

$$\frac{\beta dx}{2nx+\alpha y}=-\frac{\beta\gamma dy}{\alpha\gamma x}=-\beta dt,$$

or

$$\frac{\beta dx-\beta\gamma dy}{(2n+\alpha\gamma)x+\alpha y}=-\beta dt. \tag{5}$$

The first member of this equation will be an exact differential if

$$\beta = 2n + a\gamma, \quad \text{and} \quad -\beta\gamma = a, \tag{6}$$

whence

$$a\gamma = -n \pm \sqrt{n^2 - a^2}.$$

Thus the two values of β derived from (6) are

$$\beta_1 = n + \sqrt{n^2 - a^2}, \quad \beta_2 = n - \sqrt{n^2 - a^2}. \tag{7}$$

Writing $\omega^2 = n^2 - a^2$, it is seen that

$$\begin{aligned} \beta_1 &= n + \omega, & \beta_2 &= n - \omega & (n^2 &> a^2)\ ; \\ \beta_1 &= n, & \beta_2 &= n & (n^2 &= a^2)\ ; \\ \beta_1 &= n + i\omega, & \beta_2 &= n - i\omega & (n^2 &< a^2). \end{aligned} \tag{8}$$

Introducing β_1 and β_2 in (4), and observing the second of (6), there result

$$\frac{\beta_1 dx + a dy}{\beta_1 x + ay} = -\beta_1 dt, \quad \frac{\beta_2 dx + a dy}{\beta_2 x + ay} = -\beta_2 dt.$$

Integrating these and calling the values of x and y for $t = 0$, x_0 and y_0 respectively,

$$\begin{aligned} \beta_1 x + ay &= (\beta_1 x_0 + a y_0) e^{-\beta_1 t}, \\ \beta_2 x + ay &= (\beta_2 x_0 + a y_0) e^{-\beta_2 t}, \end{aligned}$$

whence by elimination

$$\begin{aligned} (\beta_2 - \beta_1) x &= (\beta_2 x_0 + a y_0) e^{-\beta_2 t} - (\beta_1 x_0 + a y_0) e^{-\beta_1 t}, \\ a(\beta_2 - \beta_1) y &= (\beta_1 \beta_2 x_0 + a\beta_2 y_0) e^{-\beta_1 t} - (\beta_1 \beta_2 x_0 + a \beta_1 y_0) e^{-\beta_2 t}. \end{aligned} \tag{9}$$

Substituting in these the first pair of values from (8), there result

$$\begin{aligned} x &= e^{-nt}\left(x_0 \cosh \omega t - \frac{n x_0 + a y_0}{\omega} \sinh \omega t\right), \\ & \qquad\qquad\qquad\qquad\qquad\qquad (n^2 > a^2). \\ y &= e^{-nt}\left(y_0 \cosh \omega t + \frac{a x_0 + n y_0}{\omega} \sinh \omega t\right), \end{aligned} \tag{10}$$

Similarly, using in (9) the last pair of values in (8), or replacing ω in (10) by $i\omega$,

$$\begin{aligned} x &= e^{-nt}\left(x_0 \cos \omega t - \frac{n x_0 + a y_0}{\omega} \sin \omega t\right), \\ & \qquad\qquad\qquad\qquad\qquad\qquad (n^2 < a^2). \\ y &= e^{-nt}\left(y_0 \cos \omega t + \frac{a x_0 + n y_0}{\omega} \sin \omega t\right), \end{aligned} \tag{11}$$

For the intermediate case in which $n^2 = a^2$, or $\omega = 0$, it is seen at once from (10) or (11) that

$$x = e^{-nt}\{x_0 - a(x_0 + y_0)t\}, \qquad y = e^{-nt}\{y_0 + a(x_0 + y_0)t\}, \qquad (n^2 = a^2). \qquad (12)$$

For the special case in which $n = 0$, and hence $\omega = a$, or for the case of simple harmonic motion specified by (1), (11) give

$$x = x_0 \cos at - y_0 \sin at, \qquad y = y_0 \cos at + x_0 \sin at,$$

which become identical with (3) by means of the substitution $x_0 = a \cos \theta$, $y_0 = a \sin \theta$.

The following is an abstract of Dr. Miller's paper: The $\varphi(g)$ numbers which are less than g and prime to g constitute a group with respect to multiplication if the smallest positive residues mod g are taken in place of the products. This group G_1 is the group of isomorphisms of the cyclical group of order g. When G_1 is cyclical its order g_1 must be $p^\alpha(p-1)$, where p is some odd prime number. From this it is clear that g_1 must be even whenever G_1 is cyclical. In fact g_1 is always even. If $g_1 = 2^\beta$, G cannot have more than two independent generators of the same order except for the special case when $g = 24$. In this case G_1 is clearly of type (1, 1, 1).

Professor Lovett's first paper is in summary as follows: Speculations relative to n-dimensional space have followed several fairly well-defined trends: 1° A direct extension of the cartesian geometry, which extension is to be regarded as nothing more than a convenient form of phraseology; in this form n-dimensional space sprang forth from the minds of Grassmann, Cayley, Gauss, and Cauchy, and the idea was likely familiar to Euler and Lagrange. 2° The transformation of the ordinary visualizable spaces of two and three dimensions into manifoldnesses of higher or lower dimensions by introducing space elements other than the point or its dual element; for example, the line geometry of Plücker; the sphere geometry of Lie, the five-dimensional manifoldness of all conics in the plane as an auxiliary to Ball's theory of screws; this category becomes more concrete perhaps than any other. 3° The absolute geometry of space; here would appear the famous dissertation of Riemann, the well-known memoirs of Helmholtz, Klein, and

Lie, and the elaborate treatise of Veronese. 4° The extension of the methods and results of ordinary differential geometry and analysis situs to spaces of many dimensions; to this class belongs the works of Christoffel, Beltrami, Bianchi, Cesàro, and Ricci, and the quite recent contributions of Darboux and his followers. 5° The direct extention of the problems and concepts of metrical and projective geometry of ordinary space, as exemplified in the memoirs of Jordan, d'Ovidio, and Veronese. 6° The theory of birational correspondences between n-dimensional aggregates as studied by Noether, Kantor, and Brill. 7° The descriptive geometry of space of n dimensions as begun in the papers of Veronese, Stringham, Schlegel, and Segre. 8° The kinematics of higher spaces as developed by Jordan, Clifford, and Beltrami. 9° The interpretation of n-dimensional geometry in the light of the theory of groups as exhibited by Klein, Lie, and Poincaré.

It is proposed in the present paper to make an expository contribution to the ninth and fourth of the above categories, constructing ordinary four-dimensional space by the method of Lie's theory of continuous groups and studying curves of triple curvature by the intrinsic analysis developed by Cesàro in his Lezioni di geometria intrinseca.

In Professor Lovett's second paper, which will appear in the *Annals of Mathematics*, the well-known property $(U, V) \equiv 0$ of the Jacobian system

$$Uf \equiv \frac{\partial f}{\partial x} + \zeta_1 \frac{\partial f}{\partial z} = 0, \qquad Vf \equiv \frac{\partial f}{\partial y} + \zeta_2 \frac{\partial f}{\partial z} = 0,$$

is employed to derive the ordinary criterion for the integrability of the equation

$$Pdx + Qdy + Rdz = 0.$$

The result extends itself immediately to the case of n variables.

Professor Hinton's paper investigates the principle of duality as it would exist in a fictitious geometrical world in which the distance of a point is measured by means of the difference, instead of the sum, of the squares of its abscissa and ordinate.

F. N. Cole.

Columbia University.

THE APRIL MEETING OF THE CHICAGO SECTION.

THE seventh regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at Northwestern University, Evanston, Ill., on Saturday, April 14, 1900. The morning session was called to order at ten o'clock, by Professor E. H. Moore, Vice-President of the Society.

The following members were present:

Professor E. W. Davis, Professor Thomas F. Holgate, Mr. H. G. Keppel, Dr. Kurt Laves, Professor H. Maschke, Professor E. H. Moore, Professor James B. Shaw, Professor E. B. Skinner, Dr. H. F. Stecker, Professor H. S. White, Professor J. W. A. Young.

Professor James B. Shaw was elected chairman for the meeting, which extended through a forenoon and an afternoon session.

Papers were presented as follows:

(1) Professor E. H. MOORE: "Evaluation of an important determinant of special form."

(2) Professor H. MASCHKE: "A new method of determining the differential parameters and invariants of quadratic differential quantics."

(3) Mr. ROBERT E. MORITZ: "On the transcendency of π."

(4) Dr. H. F. BLICHFELDT: "Note on the functions of a given class which in a given interval differ the least possible from zero."

(5) Dr. KURT LAVES: "Maupertuis's principle of least action for forces involving the coördinates and their first and higher differential quotients."

(6) Professor MARTIN KRAUSE: "On systems of differential equations which are satisfied by quadruply periodic functions."

(7) Professor JAMES B. SHAW: "Multiple algebra."

(8) Professor E. W. DAVIS: "A graphic method for the solution of systems of linear equations."

(9) Professor L. E. DICKSON: "Determination of an abstract group of order $2^7.3^6.5.7$ holoedrically isomorphic with a certain orthogonal group and with a certain hyperabelian group."

(10) Professor E. H. MOORE: "On the generational determination of abstract groups."

(11) Professor H. B. NEWSON: "On the group of 216 collineations of the plane."

(12) Mr. JAMES ARCHY SMITH: "Definition and periodicity of the abstract group holoedrically isomorphic with a certain group of algebraic transformations."

(13) Professor E. W. DAVIS: "On the use of color in the representation of crystal groups."

Mr. Moritz's paper was presented to the Society and read by Professor Davis; Dr. Blichfeldt's and Mr. Smith's papers were presented through Professor Moore, and the former was also read by Professor Moore; Professor Krause's paper was presented to the Society and read by Professor Maschke. Professor Newson's paper was read by title. Abstracts of the papers are given below.

Professor Moore's first paper, which has been offered for publication to the *Annals of Mathematics*, contains, after certain generalities on determinantal notation, four proofs of the theorem that the determinant

$$| a_{gi,hj} | \qquad \begin{pmatrix} g, h = 1, \cdots, m \\ i, j = 1, \cdots, n \end{pmatrix}$$

of order mn, where throughout

$$a_{gi,hj} = b^{(i)}_{gh} \, c^{(h)}_{ij},$$

is the continued product of the n determinants $B^{(i)}$ of order m and the m determinants $C^{(h)}$ of order n

$$B^{(i)} = | b^{(i)}_{gh} |, \quad C^{(h)} = | c^{(h)}_{ij} |.$$

This theorem, stated in a different notation, was first discovered by Professor W. H. Metzler.

Professor Maschke's paper presents a symbolic method for the treatment of differential parameters and differential invariants. If F and Φ are any two invariants (*i. e.*, differential parameters or differential invariants) of the binary quadratic differential quantic

$$A = \sum_{r,s=1}^{2} a_{rs} dx_r dx_s,$$

then the expression

$$\frac{1}{\sqrt{a_{11}a_{22} - a_{12}^2}} \begin{vmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} \\ \frac{\partial \Phi}{\partial x_1} & \frac{\partial \Phi}{\partial x_2} \end{vmatrix} = \{F, \Phi\}$$

is again an invariant of A. Let $f, \varphi, \cdots, u, v, \cdots$ be arbitrary functions of x_1, x_2; as such they are invariants of A, and consequently the expressions

$$\{f, u\}, \quad \{f, \{\varphi, u\}\}, \quad \{\{f, u\}, \{\varphi, v\}\}, \quad \text{etc.}, \tag{1}$$

will be invariants of A. If now we agree to set

$$\frac{\partial f}{\partial x_r} \cdot \frac{\partial f}{\partial x_s} = a_{rs},$$

we obtain

$$\left(\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2\right)^2 = \Sigma a_{rs}\, dx_r\, dx_s = A,$$

and we call f a *symbol* of A. It follows then, as in algebra, that every product of expressions (1) containing each symbol $f, \varphi, \cdots$ precisely twice (with certain restrictions concerning the higher derivatives) will be an actual invariant of A. Thus the following invariants are obtained:

$$\{f, u\}^2 = \triangle_1 u, \ \ \{f, u\}\{f, v\} = \nabla(u, v), \ \ \{f, \{f, u\}\} = \triangle_2 u,$$
$$\{f, \varphi\}\{\{f, u\}, \{\varphi, u\}\} = 2\triangle_{22} u, \ \ \{\varphi, \psi\}\{\{f, \varphi\}, \{f, \psi\}\} = 2K.$$

The extension of the method from the case of binary differential quantics to that of n variables, from quadratic quantics to those of higher orders, from invariants to covariants, and also from invariants and covariants of one quantic to those of two or more quantics, is obvious. The full paper will appear in the *Transactions.*

To prove that π is transcendental it is sufficient to show that e cannot satisfy an equation of the form

$$C_0 + e^{a_1} + e^{a_2} + \cdots + e^{a_n} = 0, \tag{A}$$

where C_0 is an integer and the a's are associated algebraic numbers. Hurwitz has shown that, for every rational function $f(x) = 0$, another function $F(x) = 0$ exists, such that

$$F(x) - e^x F(0) = - x e^{(1-\theta)x} f(\theta x) \quad (0 < \theta < 1).$$

Mr. Moritz points out that by putting

$$f(x) = \frac{x^{p-1}}{(p-1)!} \cdot a_0^{np-1} \cdot \prod_{i=1}^{n} (a_i - x)^p$$

the assumption [A] leads to the equation

$$C_0 F(0) + \sum_{i=1}^{n} F(a_i) = \sum_{i=1}^{n} [- a_i e^{(1-\theta)a_i} f(\theta a_i)],$$

in which $C_0F(0)$ is not divisible by the integer p, $\sum F(a_i)$ is divisible by p, and the right member of the equation approaches zero as a limit with increasing values of p. This is absurd; hence $[A]$ does not exist.

A real function of a real variable and certain real parameters being given, Tschebycheff has treated the problem of determining the parameters so that in a given interval the maximum absolute value of the function shall be as small as possible. In Dr. Blichfeldt's paper, which has been offered to the *Transactions* for publication, the necessary conditions on the parameters which Tschebycheff exhibited are found, when supplemented by certain others, to be also sufficient conditions.

Dr. Laves shows that the theorem of Maupertuis can be proved to exist for certain forces of the general nature indicated in the title of his paper which possess an effective potential W. The potential W is an arbitrary function of the coördinates and velocities of the bodies involved, not containing the time t explicitly. The differential equations of motion for effective potential forces are, in Lagrange's second form,

$$\frac{d}{dt}\left(\frac{\partial(T+W)}{\partial q_i'}\right)-\frac{\partial}{\partial q_i}(T+W)=0 \quad (i=1,\cdots,k),$$

where T is the kinetic energy of the system. Starting from Jacobi's partial differential equation of the first order

$$\frac{\partial v}{\partial t}-K\left(q_1,\cdots,q_k,\frac{\partial v}{\partial q_1},\cdots,\frac{\partial v}{\partial q_k}\right)=0,$$

we assume the complete integral

$$v=-ht+\bar{v}(q_1,\cdots,q_k,a_1,\cdots,a_{k-1})+a_k$$

and define the action of the system during the interval of time t_1-t_0 by

$$A=\int dv=\int_{t_0}^{t_1}\Sigma\frac{\partial v}{\partial q_i}q_i'\,dt=\int_{t_0}^{t_1}\Sigma p_i q_i'\,dt$$

where
$$p_i=\frac{\partial(T+W)}{\partial q_i'}.$$

Calling $T+W=\varphi$, we make the special assumption that $\varphi=\varphi_0+\varphi_1+\varphi_2$, where φ_0, φ_1, φ_2 are homogeneous functions of q_i' of the degree indicated by the subscript in each case. The integral of the conservation of vis viva is

$$K \equiv \Sigma q'_i \frac{\partial (T+W)}{\partial q_i'} - (T+W) \equiv \varphi_2 - \varphi_0 = h.$$

The expression for A becomes

$$A = \int_{t_0}^{t_1} (2\varphi_2 + \varphi_1)\, dt.$$

We may eliminate dt by means of the equation $\varphi_2 = \varphi_0 + h$ by considering

$$\varphi_2 = \sum a_{ij} \frac{dq_i dq_j}{dt^2} \equiv \frac{d\sigma^2}{dt^2},$$

where we have put $d\sigma^2 \equiv \sum a_{ij} dq_i dq_j$. Thus we obtain

$dt = \frac{d\sigma}{\sqrt{\varphi_0 + h}}$. Putting $2\varphi_2 = 2(\varphi_0 + h)$ and $\varphi_1 \equiv \Sigma \frac{b_i dq_i}{dt} \equiv \frac{d\tau}{dt}$.
we obtain $A = \int_{\sigma_0}^{\sigma_1} 2\sqrt{\varphi_0 + h}\, d\sigma + \int_{\tau_0}^{\tau_1} dt$. It is easy to show that the condition $\delta A = 0$ is fulfilled for the actual path described by the system. In fact, putting $q_1 = f_1(\lambda), \cdots, q_k = f_k(\lambda)$, we can express A as a function of the parameter λ as follows:

$$A = \int_{\lambda_0}^{\lambda_1} \{2\sqrt{\varphi_0 + h}\sqrt{\Sigma a_{ij} \cdot \dot{q}_i \cdot \dot{q}_j} + \Sigma b_i \dot{q}_i\}\, d\lambda,$$

where we have put $\dot{q}_i = \frac{dq_i}{d\lambda}$.

Performing the variation of A, and remembering that the coordinates are given for $\lambda = \lambda_0$ and $\lambda = \lambda_1$, we obtain after an integration by parts and re-introduction of the variable t

$$\delta A = \int_{\tau_0}^{\tau_1} \Sigma L_i \delta q_i\, dt.$$

The coefficients L_i present themselves in the form

$$L_i = \frac{d}{dt}\left(\frac{\partial (T+W)}{\partial q_i'}\right) - \frac{\partial}{\partial q_i}(T+W).$$

They vanish in accordance with our hypothesis, and it is thus proved, that the condition $\delta A = 0$ is fulfilled for the case under consideration.

In the theory of elliptic functions certain linear differential equations prove to be of importance, the coefficients of which are doubly periodic, while the integrals can also be represented as doubly periodic functions. In particular, two equations have been investigated, Lamé's and Picard's differential equations of the second order. In previous papers Professor Krause established a system of differential equations as an analogue of Lamé's equation. In his present paper he exhibits as the simplest analogue of Picard's equation a system of two linear differential equations, the coefficients and also the integrals of which are quadruply periodic functions.

Professor Shaw's paper is a complete generalization of quaternions to an algebra of n units. This is rendered possible by considering what has heretofore been called a *product* of two multiple numbers as simply a *function* of two such numbers, or of two sets of n variables. By the introduction of the symbol D defined by

$$D\bar{a}\beta = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

where $$a = \sum_1^n a_i e_i, \quad \beta = \sum_1^n b_i e_i,$$

the e's being defining units; and by the introduction of a set of A's so defined that

$$A . a_1 a_2 \cdots a_r A . \beta_1 \cdots \beta_{r+1} = \begin{vmatrix} \beta_1 \cdots & \beta_{r+1} \\ D\bar{a}_1\beta_1 \cdots & D\bar{a}_1\beta_{r+1} \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ D\bar{a}_r\beta_1 \cdots & D\bar{a}_r\beta_{r+1} \end{vmatrix},$$

every quaternion formula finds its extension. The paper includes the theory of matrices of order n as the linear vector operators of such algebras. It is shown that every such matrix may be written in the form $\varphi = \sum_{1,1}^{n,n} a_{rs} v_{01}{}^s v_{10}{}^r$, where v_{01} and v_{10} are matrices of order n, $v_{01}{}^n = 1$, $v_{10}{}^n = 1$, $v_{01}{}^s v_{10}{}^r = i^{-rs} v_{10}{}^r v_{10}{}^s$, i being a primitive nth root of unity. Just as in quaternions $(1 + \sqrt{-1}\, k)i = i + \sqrt{-1}\, j$ is a nullity, so in matrices in general $\theta_{r-1} = (1 - v_{10})(1 - iv_{10})(1 - i^2 v_{10}) \cdots (1 - i^{r-2} v_{10})(1 - i^r v_{10}) \cdots (1 - i^{n-1} v_{10}) v_{01}$ is a nullity, *i. e.*, $\theta_{r-1}^2 = 0$. Any expression $\theta = \theta_a + \theta_b + \cdots + \theta_f$ is also a nullity, *i. e.*, $\theta^p = 0$, p being an integer less than n. A complete list of expressions for the moduli, etc., of matrices and of combined matrices is given.

Professor Davis presented the following graphic method for the solution of a system of linear equations. Let (1), (2), (3), (4) stand for four equations in x, y, z, w; let (12) stand for the result of eliminating z, and $(12)'$ for the result of eliminating w from (1) and (2); while (123) is the result of eliminating both z and w from (1), (2), and (3). Further, let $(1)_{ab}$ be the graph in the xy-plane of the result of putting $z = a$ and $w = b$ in (1), while $(12)_a$ is the graph of the result of putting $w = a$ in (12). To solve the equations draw $(1)_{00}$ and $(2)_{00}$, $(1)_{10}$ and $(2)_{10}$. Each pair meets upon $(12)_0$, which draw. Similarly determine and draw $(13)_0$ together with $(12)_1$ and $(13)_1$. Each pair meets upon the graph of (123). Draw this graph and likewise determine and draw the graph of (124). These two graphs meet in P whose coördinates are the proper values of x and y to satisfy the original equations. Finally w equals the distance from $(12)_0$ to P divided by the distance from $(12)_0$ to $(12)_1$, while the value of z is obtained by accenting the parentheses in the value of w. The method can be extended to any number of equations in as many unknowns.

The groups referred to in the title of Professor Dickson's paper are included in the author's list of simple groups published in the BULLETIN for July, 1899. The paper will appear in the *Transactions*.

Besides fundamental generalities (cf. BULLETIN, December, 1899, p. 99), the second paper of Professor Moore, which is to appear in the *Transactions*, develops certain examples of which the most important is the following: The group $F_{2, R, 1}$ of binary fractional matrices $\left(\frac{a,\ b}{c,\ d}\right)$, $(ad - bc = 1)$, of determinant 1 in any field or realm R of rationality, has the generators

$$T = \left(\frac{0,\ -1}{1,\quad 0}\right), \quad S_s = \left(\frac{1,\ s}{0,\ 1}\right),$$

s being any mark of the realm R; and the corresponding abstract group is given in the abstract generator symbols T, S_s, by the system of generational relations :*

---

* Another generational determination of this group was given at this meeting in the paper of Mr. J. A. Smith.

$$(1)\quad T^2 = I, \quad (2)\quad S_0 = I, \quad (3)\quad S_{s_1}S_{s_2} = S_{s_1+s_2},$$
$$(4)\quad S_{s_1}TS_{s_2}TS_{s_3}TS_{s_4}TS_{s_5}T = I,$$

where I is the identity element, and in (3) s_1, s_2 are any two marks of the realm R, and in (4) s_1, s_2 are any two non-reciprocal marks of the realm R, and

$$s_3 = \frac{s_1 - 1}{s_1 s_2 - 1}, \quad s_4 = -(s_1 s_2 - 1), \quad s_5 = \frac{s_2 - 1}{s_1 s_2 - 1}.$$

It should be noted that the marks $s_1, \cdots, s_5$ are the roots of a quintic equation whose coefficients are rational functions with integral coefficients of two arbitrary parameters and which in the realm of its coefficients is solvable by radicals.

Professor Newson investigates the geometrical properties of the group of 216 collineations in the plane and its subgroups with respect to a pencil of cubic curves remaining invariant under the group. In particular he discusses the effect of the transformations of the different subgroups on the harmonic and equianharmonic cubics and the four inflexional triangles contained in the pencil and their relation to the invariant triangles belonging to the transformations. The structure of the group is analogous to that of the tetrahedron group, the four equianharmonic cubics, the four degenerate cubics, and the six harmonic cubics of the pencil being analogous to the four vertices, four faces, and six edges of the tetrahedron.

Mr. Smith's paper exhibits in the first place certain novel generators of linear substitution groups which may be either homogeneous and integral in two letters or fractional in one. The determinant is taken as unity and the parameters are taken either 1° integers modulo q, 2° marks of a Galois field of order q^n together with integers modulo q, or 3° continuous variables either real or complex. For each of these three fields two types of generating substitutions S_α and V_α are defined, and relations are found which suffice for complete definition of the isomorphic abstract groups. These generators are so defined in each of the fields above mentioned that the formal relations are the same for all, the fractional linear group requiring three relations, the integral only two. In the second part of the paper normal representations of all elements of the groups are given in terms of generating elements, and their periodicity is determined by means of these normal forms.

In Professor Davis's second paper it is pointed out that the 48 operations which bring the cube into coincidence with itself are operations which bring any one of the eight triangles into which a face is divided by its lines of symmetry into coincidence with the 48 like triangles of the surface of the cube. Any of the subgroups of operations brings this same triangle into coincidence with merely a part of the 48 triangles. A cube with these triangles all of one color and the rest of the cube another color is an object which, taking account of color, comes into coincidence with itself only under the operations of the subgroup. Similar methods can be used for the representation of all regular groups and their subgroups.

THOMAS F. HOLGATE,
Secretary of the Section.

ON THE HISTORY OF THE EXTENSIONS OF THE CALCULUS.

BY J. G. HAGEN, S. J.

THE results attained by means of the infinitesimal calculus naturally evoked similar attempts in other directions. We may distinguish between two kinds of new theories, differing from each other mainly in their origin. The one is the natural outcome of an ever recurring need of solving practical problems or of giving existing theories a broader basis; the other owes its origin to purely abstract speculation. The former is not the property of any one mathematician; its nomenclature is not arbitrarily chosen, and its general introduction is only a question of time. This is clear from the history of all those branches of mathematics that have come into general use.

The case is somewhat different with the purely speculative theories. But though these may never succeed in demonstrating their practical utility, yet they are of value to mathematical science since they throw new light on existing methods and on their usefulness. For our purpose, however, a summary outline of these theories will suffice.

The present abstract is confined to those theories that are in close relation to the infinitesimal calculus and the theory of functions, and excludes, for instance, all geometrical methods and what are called "principles" or methods of demonstration.

In enumerating the following theories it cannot be the intention of the writer to classify them with the abstract theories that will never command general attention. They are mentioned merely because they have not yet been generally adopted. How long a new theory may remain a hidden treasure is amply illustrated by Grassmann's Ausdehnungslehre. Perhaps the reason why some of the theories considered have appeared of slight value is that they are formal in their nature and were invented before the full import and difficulty of Taylor's theorem was revealed. We expressly exclude from this remark Cauchy's calculus of residuals and the calculus of finite differences. How far the "operations" of this latter calculus may be promoted by the modern theory of functions is a question of great interest, which has probably not yet been fully answered.

While some of these new theories that have a set of formulas similar to the infinitesimal calculus are generally known, a synoptical view of all of them may be instructive. The references to the original sources would be be very numerous and cumbersome and are omitted here, as they will be given elsewhere.*

I. *Derivation with a General Index.*

The attempts to invent a calculus, of which the differential and integral calculuses should be particular branches, date back as far as Leibniz. Their history was outlined by Liouville, Tardy, and Cantor. That a negative index of differentiation is equivalent to an integration, and *vice versa*, was first stated by Leibniz. The first examples, in which the index n of an n^{th} differential coefficient is put equal to $\frac{1}{2}$, were given by Leibniz and Euler. Later, Fourier expressed a differential coefficient by a double integral, and added the passing remark that the index of the differential may be any number. While these few examples were intended as mere curiosities, the theory of the new calculus was established by Liouville and has since remained substantially in the shape that he gave it, although it was perfected later by Riemann, Gruenwald, Most, Letnikoff, and Krug.

Liouville gave three independent definitions of the derivation with a general index: first by development into a series, then by finite differences, and finally by definite integrals.

1. Liouville supposed the given function to be developed into an *exponential series*, of which he took the n^{th} differen-

---

* The writer's Synopsis of higher mathematics, vol. III., pp. 85–99.

tial, and then assumed the index n to be any number. His hypothesis is generally considered unsatisfactory. Riemann, while yet a student, chose for the same purpose a series progressing according to *powers* of the increment h in the development of $f(x+h)$, such that the exponents increase by integral numbers. The n^{th} derivation is then the coefficient of $k_n h^n$, where the factors k_n are subject to the condition, that the development becomes Taylor's theorem when n is an integer.

The definition by series has been abandoned by later writers on the subject.

2. A second definition was given by Liouville in the following manner: He expressed the *finite difference* $\Delta^n f$ by Newton's formula, then divided it by h^n, and finally took the limit for $h = 0$. If then the index n is taken arbitrarily, we have a definition for the general derivation. Gruenwald was probably the first to assume this index to be a complex number. He also expressly added a lower and upper limit to this derivation, as is required in a calculus which is to comprise the integral calculus as a special branch, and which may be valid only within definite limits in the plane of complex coördinates. Gruenwald's *notation* of the n^{th} derivation of $f(x)$ within the interval from a to x seems to be the most convenient: $D^n[f(x)]_a^x$.

3. A third definition is that by *definite integrals.* Liouville gave two integrals with the limits zero and infinity, and all later definitions are only generalizations of these. Gruenwald, as before mentioned, changed the limits into a and x. Most gave four different integrals, in each of which the upper limit is a critical point of the function $f(x)$. Letnikoff's two integrals resemble those of Liouville, except for the upper limit x. Krug's treatise on the same subject seems to surpass all others by a more precise statement of the conditions to which the formulas are subject. One of his integrals is a curve integral, and reminds us of Cauchy's fundamental theorem; the other has the limits a and z, and comprises all the cases discussed by Gruenwald and Most.

It will not be necessary for the purpose of this abstract to mention the various theorems which Liouville has established for the operation $D^n[f(x)]_a^x$.

II. *Cauchy's "Calcul des Résidues."*

This calculus is entirely due to Cauchy as to its origin and complete development. Unfortunately this calculus in its entirety has not yet gained general recognition. Bertrand's "Calcul différentiel" is probably the only textbook that gives a complete theory of it. Of course, as far as this calculus is connected with Cauchy's fundamental theorem of integration around contours, it will ever remain a corner stone in the theory of functions.

1. The "Calcul des résidues" is not a generalization of the infinitesimal calculus, like the derivation with general index, but rather a branch parallel to it. Cauchy was probably led to this new idea by Lagrange's definition of the derivative as coefficient of the increment e^{+1} in the development of the function $f(z+e)$. For he defined the residuum as the coefficient of e^{-1}. This applies of course only to functions which have points of infinity. Suppose the equation $f(z) = \infty$ has the root z_r m times; then the function $(z - z_r)^m f(z) = \varphi(z)$ is in the point $z = z_r$ neither zero nor infinite. Hence, putting $z = z_r + e$, we have the development

$$f(z_r + e) = \frac{\varphi(z_r + e)}{e^m} = \frac{1}{e^m}\varphi(z_r) + \cdots + \frac{1}{e}\,\frac{\varphi^{(m-1)}(z_r)}{1.2\cdots(m-1)} + \frac{\varphi^{(m)}(z_r + \theta e)}{1.2\cdots m}, \qquad (\theta < \pm 1).$$

Here the coefficient of $\frac{1}{e}$ is called the *residuum* of $f(z)$ in regard to the m-fold root z_r:

$$\frac{\varphi^{(m-1)}(z_r)}{(m-1)!} = \frac{1}{(m-1)!}\,\frac{d^{m-1}}{de^{m-1}}\Big[\, e^m f(z_r+e) \,\Big]_{e=0}.$$

2. The "*integral residuum*" of the same function $f(z)$ is defined as the sum of the residuals in regard to all the roots of $f(z) = \infty$ within certain limits, *i. e.*, within $z = a + iA$ and $z = b + iB$, and is designated by

$${}_a^b E_A^B((f)) \quad \text{or} \quad {}_a^b E_A^B[f],$$

where the letter E is the initial of the word "Extraction." It was changed by Cauchy into a character somewhat similar to a capital epsilon.

The "*partial residuum*" refers to a function composed of factors $f_1, f_2, \cdots$, and is designated by enclosing in brackets that factor of which the residuum is taken. Thus, $E\,[f_1]\,f_2 \cdots$ refers to the roots of $f_1 = \infty$ alone.

The sign E follows all the rules of the sign of integration, so far as the associative, the distributive, and the commutative laws are concerned.

Cauchy has established many theorems of this calculus which it is not necessary to mention here. By numerous applications to nearly all branches of mathematics and physics he has proved the great power of his calculus.

III. *Schell's "Quotial and Instaural."*

This interesting but little known calculus is similar to Cauchy's "Calcul des résidues," in as much as it branches off from the definition of the differential in a new direction. But while Cauchy started from Lagrange's idea of the derivative, Schell takes Leibniz's and Euler's point of view. His original idea is to replace the limit of a finite *difference* by the limit of a finite *quotient*. He endeavored to establish a kind of dualism in the infinitesimal calculus by interchanging differences and quotients, sums and products, somewhat similar to Gergonne's and Poncelet's dualism in geometry, where line and point coördinates are interchanged.

There are only two other publications relating to this subject; the one by Walton on the symbolic operator $\left(x\frac{d}{dx}\right)^r$, the other by Gaussin on the "Definition du calcul quotientiel." Gaussin considers this calculus mainly from the arithmetical and algebraic point of view.

1. The definition of the "quotial" consists in this: The finite difference $\Delta x = x_1 - x$ and the differential $\lim\limits_{x_1=x} \Delta x = dx$ are replaced by the quotient $\theta x = x_1 : x$ and the "quotial" $\lim\limits_{x_1=x} \theta x = \vartheta x$ (or $= qx$ in Gaussin's designation). Hence $x_1 = x + \Delta x$ has the dualistic formula $x_1 = x\theta x$. For the "quotial" of a function we have

$$u = F(x),\ u_1 = F(x\theta x),\ \vartheta u = \lim_{u_1=u} \frac{u_1}{u} = \frac{F(x\vartheta x)}{F(x)}.$$

Now, while in the differential calculus we pass from a differential to a differential quotient, we here pass from a "quotial" to a difference, by means of the logarithm. The peculiarity of this logarithm lies in the variable base θx, with the limit $\vartheta x = \omega \doteq 1$. It is denoted by λ, and we have the following definition:

$$\textit{Quotial Logarithm of } F(x) = \lambda\vartheta F(x) = \lambda\frac{F(x\vartheta x)}{F(x)}.$$

The connection between this and the differential calculus appears from the following transformation :

$$\lambda\vartheta F(x) = \frac{x}{F(x)}\frac{dF(x)}{dx} = x\frac{d}{dx}\operatorname{lg} F(x).$$

which shows that the "quotial derivative" is substantially identical with a logarithmic differentiation.

On the other hand, it appears that "quotial derivatives" of higher order with the symbol

$$\lambda\vartheta^r \equiv \left(x\frac{d}{dx}\operatorname{lg}\right)^r$$

are not quite identical with Walton's operator $\left(x\frac{d}{dx}\right)^r$.

2. The "instaural," as the limit of a product, is the dualistic counterpart of the integral, the limit of a sum. We first write the quotial derivative in this form

$$\lambda\vartheta F(x) = \lambda\frac{F(x\omega)}{F(x)} = f(x), \text{ or } \frac{F(x\omega)}{F(x)} = \omega^{f(x)} \quad (\lim \omega = 1).$$

Then we divide the quotient of two given limits x_0 and x into n arbitrary factors

$$\frac{x}{x_0} = \omega_1 \cdots \omega_n,$$

and substitute the n values

$$x_0, \quad x_1 = x_0\omega_1, \quad x_2 = x_1\omega_2, \quad \cdots, \quad x_{n-1} = x_{n-2}\omega_{n-1},$$

in the above formula (putting at the same time for ω successively $\omega_1, \cdots, \omega_n$). Finally, multiplying all these equations, and remembering that $\omega_{r+1} = \frac{x_{r+1}}{x_r} = \vartheta x_r$, we have the product

$$\frac{F(x)}{F(x_0)} = \omega_1^{f(x_0)} \cdots \omega_n^{f(x_{n-1})} = P\omega_{r+1}^{f(x_r)} = P\vartheta x_r^{f(x_r)}.$$

If this product has a definite finite limit, it is called the "instaural of $f(x)$ between x_0 and x ("produit" according to Gaussin), and designated by

$$\frac{F(x)}{F(x_0)} = \overset{x}{\underset{x_0}{P}}\vartheta\sigma^{f(\sigma)}, \quad \text{or} \quad P\vartheta\sigma^{f(\sigma)} = \text{const. } F(x).$$

The connection between this formula and the integral calculus appears from the following identities:

$$f(x) = x\frac{d}{dx}\lg F(x), \quad \text{or} \quad C\cdot\exp\int\frac{f(x)}{x}dx = F(x),$$

$$\overset{x}{\underset{x_0}{P}}\vartheta_\sigma{}^{f(\sigma)} = e^{\int_{x_0}^{x}\frac{f(\sigma)}{\sigma}d\sigma}.$$

IV. *The Exponential Function of Higher Order.*

In considering the product as a repeated sum, and the power as a repeated product, it was but natural to look for a new expression which would repeat the power, and thus increase the number of elementary operations from seven to eight. The attempt, however, failed, for the reason, that base and exponent are not commutative, as are terms and factors. Yet we meet in the older literature the expression $r^{r^{\cdot^{\cdot^{r^a}}}}$. It is not known where it occurs first. Condorcet equated this power of higher order to a certain series, without giving any proof. Euler, in supplying this proof, gave a full discussion of the function, and found an interesting relation between it and the base of natural logarithms.

1. He puts $\beta = r^\alpha$, $\gamma = r^\beta$, $\delta = r^\gamma$, $\cdots$ and investigates the condition that the series $\alpha, \beta, \gamma, \cdots$ converge toward a finite value ω. Evidently this condition is $r^\omega = \omega$.

Euler then finds that the maximum value of the root r which can satisfy this condition is $r_1 = e^{1/e} = 1.44 \cdots$ and the corresponding value $\omega = e = 2.7 \cdots$ For all values $r > r_1$ the limit $\omega = x + iy$ is imaginary, and for all values $r < r_1$ the condition $r^\omega = \omega$ furnishes generally two values ω and ω', towards which the series $\alpha, \beta, \gamma, \cdots$ can converge.

2. Assuming the root $r = e$ and writing x for a the function may be expressed by the symbol $e^{e^{\cdot^{\cdot^{x}}}} = \exp^n x$, where "exp" is Cayley's notation for the exponential, and where n indicates the number of times the letter e is written. In this case the differential of the function can be written in the following manner:

$$\frac{d}{dx}\exp^n x = \exp^n x \cdot \exp^{n-1}x \cdots \exp x.$$

3. There are later essays on this function by Eisenstein and Woepke, which give no reference to Euler's treatise. They do not seem to have substantially advanced our knowledge of the nature of this function.

V. *Extension of the Calculus of Finite Differences.*

There are two very interesting new theories in the line of finite differences: one by Oettinger and the other by Dr. McClintock.

1. Oettinger creates a calculus perfectly analogous to the calculus of finite differences, another imitation of the geometric dualism. He runs side by side the two formulas

$$\Delta f(x) = f(x+h) - f(x) \quad \text{and} \quad \zeta f(x) = f(x+h) + f(x),$$

and all their consecutive corollaries. The original idea is the change of the *minus* in the former formula into *plus* in the latter.

The letter ζ (instead of Δ) is the initial of the word "Zunahme" (increase). Oettinger then develops a full theory of $\zeta^{\pm m}$ parallel to that of $\Delta^{\pm m}$, with many formulas and applications. This calculus does not seem to occur in the later literature.

2. Dr. McClintock has laid down his new calculus in an elaborate essay, which appeared in an early volume of the *American Journal of Mathematics.* He starts from the following symbol, where D is used for $\dfrac{d}{dx}$ as usual,

$$E^h \varphi(x) \equiv e^{hD} \varphi(x) \equiv (1+\Delta)\varphi(x) \equiv \varphi(x+h),\ \Delta x = h,$$

and designates the operation "E" as "enlargement."

A number of symbolic formulas with the operator E is then established, of which the following is remarkable for its simplicity, and seems to be the most important of all:

$$D \equiv \lg E.$$

Dr. McClintock then develops the theory of the functions of E, and uses it to give the theory of differentiation a broader basis, as the logarithmic branch of the doctrine of these functions.

It would be too long in this abstract to show how the formulas of the differential calculus are derived from the symbol $D = \lg E$. Nor can we do more than mention the remarkable application of this new method for finding simultaneously all the roots of an equation. It is hoped that this calculus of enlargement will give a new impulse to the use of symbolic operations which have proved so powerful in the hands of French and English mathematicians.

VI. *The Logarithmic Methods of Bergbohm and Oltramare.*

The facility with which exponentials can be handled in differentiating and integrating seems to have given rise to the two methods to be mentioned here. Both have this in common, that they pass from a given function to a logarithm or exponential, and, after a series of suitable transformations, back to the given function. This process appears almost like a roundabout way, by which the direct differentiation and integration is partly covered. Both methods were represented by their authors in a rather obscure and unsatisfactory manner.

1. The "new methods" of Bergbohm are based upon the symbol dx^{dy} with vanishing base and exponent. This symbol is first developed into the converging series

$$dx^{dy} = e^{dy\lg dx} = 1 + \frac{dy\lg dx}{1} + \frac{(dy\lg dx)^2}{1.2} + \cdots,$$

from which the fundamental formula is obtained

$$dx^{dy} = 1 + dy\lg dx.$$

Of this formula the logarithm is taken, and then the antilogarithm or numerus, both in regard to a certain vanishing base $d\beta$. These operations are designated by λ and ν. In this manner the further formulas are derived

$$\frac{dy\lg dx}{\lg d\beta} = \lambda(1 + dy\lg dx), \quad \nu\frac{dy\lg dx}{\lg d\beta} = 1 + dy\lg dx,$$

since $\lambda dx . \lg d\beta = \lg dx$.

The application consists in this, that by the above mentioned transformations any expression df is changed into $f'dx$, and again any expression $f'dx$ into df. By the former process the derivative is obtained, by the latter the integral. The method is tested by some of the elementary formulas.

2. Oltramare's "Calcul de generalisation" rests on the symbol

$$Ge^{xu+yv+zw+\cdots} = \varphi(x, y, z, \cdots),$$

by which the operation with the "characteristic" G is said to be defined.

The *partial* generalization, which refers to u only, or to v only, etc., is denoted by G_u, G_v, $\cdots$; and a *repeated* generalization by G^2, G^3, etc. The *inverse* operation G^{-1} is called "degeneralisation."

The principal operations in this calculus consist in developing both members of the equation

$$Ge^{(x+a)u+(y+b)v+\cdots} = \varphi(x+a,\ y+b,\ \cdots)$$

according to powers of the increment a, and equating the coefficients of equal powers. By this process the following fundamental formula is found:

$$Ge^{xu+yv+\cdots}\psi(u,\ v,\ \cdots) = \psi(D_x,\ D_y,\ \cdots)\varphi(x,\ y,\ \cdots).$$

In the applications the function ψ has to be developed according to powers of $D_x = \dfrac{d}{dx}$, $D_y = \dfrac{d}{dy}$, $\cdots$, after which the differentiations are to be performed.

The idea of this calculus seems to be to replace a given function $\varphi(x, y, \cdots)$ by the exponential function $Ge^{xu+\cdots}$, and then, after some suitable transformations, to return from the exponential to the given function. This suggests the idea that G is an operator like Σ, although the author avoids saying so, perhaps owing to the objections which had been raised against Liouville's assumption, that every function can be developed into an exponential series. The work of Oltramare was recently reviewed by Professor Lovett in the BULLETIN.

GEORGETOWN UNIVERSITY,
May, 1900.

BURNSIDE'S THEORY OF GROUPS.

Theory of Groups of a Finite Order. By W. BURNSIDE,* M.A., F.R.S.; Professor of Mathematics at the Royal Naval College, Greenwich. Cambridge, The University Press, 1897. 8vo., xvi + 388 pp.

THIS work enjoys the distinction of being the first treatise on the theory of groups which does not consider the applications. As may be inferred from the title, the author has practically confined his attention to the groups of a finite order. The known regions which are bounded by these restrictions are, at the present time, not too extensive to be described in one volume. Happily, our author is so familiar with these regions that he does not confine himself entirely to leading the reader by known roads to the many interesting objective points. He has pointed out many short routes as well as a number of new objects of interest.

---

* The joint author of Burnside and Panton's "Theory of Equations" is W. S. Burnside, an Irishman, professor of mathematics and fellow of Trinity College, Dublin. He is not related to the author of the work here considered, who is of Scotch extraction and has not published any other book. He has, however, published a number of memoirs—most of which relate to the theory of groups—in the *Proceedings of the London Mathematical Society* and in the *Messenger of Mathematics*. It seemed desirable to mention these facts since the two authors—William Burnside and William Snow Burnside—are frequently confused. Cf. Netto's review, Schlömilch's *Zeitschrift*, vol. 44 (1899), p. 20.

The theory of groups of a finite order (under the form of substitution groups) was first developed as a branch of the theory of equations; and the works of Lagrange,* Abel, and Galois on the solution of equations have contributed most powerfully towards its early development. In recent years it has been pointed out by Jordan, Klein, Cayley, and others that this subject has extensive geometrical applications. This has led to the study of groups in the abstract; *i. e.*, independent of any particular mode of representation. The work before us aims to treat the subject in this modern spirit.

Nevertheless considerable space is devoted to substitution groups. This may possibly be partly due to the fact that the subject was first developed from the standpoint of substitution groups. The author states in the preface that it is done because, "in the present state of our knowledge, many results in pure theory are arrived at most readily by dealing with properties of substitution groups." The concepts substitutions and group are so closely related that the former seems a natural precursor of the latter; and the study of many of the group properties, such as the group of isomorphisms and the transformation of conjugate subgroups and operators, seems to lead naturally to substitution groups. A knowledge of substitution groups can thus be utilized very frequently in the study of the abstract group properties, even if it is not essential for the study of these properties.

A number of interesting theorems such as those which relate to primitivity, imprimitivity, transitivity, and intransitivity apply only to substitution groups. Many readers would regret to see these landmarks of early triumphs banished from the theory to which they led. To such readers the method followed by our author will be entirely satisfactory, while those who have confined their attention to the more abstract theories might have preferred to find fewer substitution group illustrations—especially in the early chapters of the work. The notation of substitution groups has, however, been excluded from the proofs and investigations of chapters II. to VII., in which most of the fundamental properties have been developed.

---

* Pierpont, "Lagrange's place in the theory of substitutions," BULLETIN, 2d series, vol. 1 (1895), p. 196.

The value of the book is increased by a number of carefully selected examples. These examples are generally chosen with a view towards continuing or completing the discussions in the text rather than with a view towards illustrating the theoretical developments. Consequently many of them seem too difficult to be taken up at the first reading. Some of them have not been stated with sufficient care: *e. g.*, number 2 on page 10 is evidently not true when $m = 1$ and $n > 2$; and the well known quaternion group is an exception to number 4, page 195, which is referred to Dyck. It may be remarked that the group of order 12 which contains six operators of order four is an exception to Dyck's theorem (*Mathematische Annalen*, vol. 22, p. 101) from which the given example seems to have been obtained. The second part of example 2, page 89, cannot be proved, since it is easy to construct groups of the specified type which are not generated by two operators that satisfy the given conditions.

As the book is written for the beginner, "to introduce to the reader the main outlines of the theory of groups of finite order without any applications," the first three chapters are devoted to the most elementary matters—the explanation of the notation of substitutions, the definitions and elementary illustrations of the terms group, subgroup, direct product,* etc., and a few fundamental theorems. The theorem stated on page 22 to the effect that every group of a finite order can be represented as a regular substitution group is not due to Dyck, as stated in the table of contents. It was given explicitly by Capelli at an earlier date (*Giornale di Matematiche*, 1878, p. 45) and somewhat less explicitly by Jordan at a still earlier date (Traité des substitutions, 1870, p. 60). The symbol G/Γ used to represent the quotient group of G with respect to its self-conjugate subgroup Γ seems to be due to Jordan (*Bulletin de la Société Mathématique*, vol. 1 (1873), p. 46). It is certainly not due to Hölder as is stated on page 38.†

---

* This definition of direct product does not appear to be as simple as possible; cf. *Transactions*, vol. 1 (1900), no. 1, p. 66. It may be remarked that Burkhardt gives an incomplete definition of direct product in his article on "Endliche discrete Gruppen," *Encyk. der math. Wissenschaften*, vol. 1, p. 219.

† Netto makes the same error in his new Algebra, vol. 2, 1900, p. 343.

A number of minor inaccuracies, which the reader will readily discover, occur in these as well as in the later chapters. Such inaccuracies tend to confuse the beginner, who is so prone to study the book, instead of using it as a tool to study the subject. For instance, on page 16, line 4, it is stated that the order of the product of two commutative operators is the least common multiple of the orders of these operators. That this is not always true may be seen from each of the following facts: the product of the two commutative substitutions $(abcd)(efgh)$ and $(afch)(bgde)$ is of order two, and the product of an operator and its inverse is identity. The statement is evidently always true when the groups generated by the two commutative operators have only identity in common. This condition is sufficient but it is not necessary. The interchange of the substitutions represented by D and E, on page 19, example VII., is an error of less importance.*

Chapter IV. is devoted to abelian groups. This subject is so important that a more extensive exposition would doubtless have proved agreeable to a large number of the readers. Several of the most interesting theorems are given as examples at the end of the chapter; but most beginners will probably be unable to prove them within a reasonable length of time. After a few preliminaries the author proves that any abelian group of order p^a (p being any prime number) contains a set of independent generators and that the orders of these generators are invariants of the group. This is followed by some very interesting investigations in regard to the subgroups of abelian groups of order p^a. Unfortunately the treatment of this subject is not always as complete as might be desired and there seem to be a few inaccuracies. For instance the notation of the latter part of theorem III., p. 58, seems to me to imply that a group of type (5, 2) contains a subgroup of type (3, 3), which is evidently impossible.

In chapter V. we meet the El Dorado of the theory of groups,—the groups of order p^a, p being any prime number. Cauchy proved that every group whose order is divisible by p must contain one or more subgroups of order p. More than twenty-five years later Sylow proved that a group contains one and only one system of conjugate subgroups of order p^a whenever its order is divisible by p^a but not by p^{a+1}. These results indicated the great importance of a careful study of the groups of order p^a. Fortunately they give rise to a large number of very interesting and unusually simple theorems. Many of these are old, but a considerable part of them are of recent origin being largely due to the investigations of Frobenius and Burnside. The general theorem published in the *Messenger of Mathematics*, vol. 27, 1897, appeared too late to be included in the present work.

---

* Mr. Kuhn called my attention to this interchange of substitutions.

The explicit theorems of this chapter together with those which are brought out by the illustrative examples give a comprehensive view of the groups of order p^a. The examples are drawn, to a large extent, from the publications of Hölder, Young, and Cole and Glover. On pages 68 and 69, Burnside undertakes the determination of every group G of order p^{r+s} in which the subgroup H_1 which is composed of the self-conjugate operators is the cyclical group of order r, while G/H_1 is an abelian group of type (1, 1, ⋯, with s units). He concludes that the number of these groups is $2^{\frac{1}{2}s}$, which is evidently incorrect.*

Chapter VI. is devoted to Sylow's theorem and the important extension due to Frobenius. For abelian groups this theorem was proved in Chapter IV. and the developments of the present chapter are somewhat simplified thereby. The fundamental importance of the theorem is clearly brought out by a number of carefully selected examples. In one of these examples all the possible groups of order 24 are determined. All these groups had been determined somewhat earlier,† but no reference is given to the earlier publications on this subject. The reference on p. 105 in regard to the published abstract groups is incomplete since all the groups whose order is less than 48 were known.‡ In giving the extension of Sylow's theorem the author follows Frobenius quite closely.

The composition series of a group, which is considered in Chapter VII., plays a very important rôle not only in the theory of groups but also in the Galois theory of equations. The foundation for the developments along this line was laid by Jordan in his article published in *Liouville*, vol. 14 (1869), p. 139. The reference on p. 119 should be to this article, instead of to Jordan's Traité des substitutions Here Jordan proved the very important theorem that the factors of composition of a given group are invariant. Hölder has extended these results somewhat by proving that the factor groups are also invariant. Burnside follows Hölder's developments quite closely in the first part of the chapter, and he terminates it with a few theorems on soluble groups followed by several examples. In his recent Algebra, vol. 2, 1900, Netto uses chief series (Hauptreihe) in a somewhat different sense from Burnside, hence his example, p. 338, does not substantiate the criticism which Netto makes in the foot-note on p. 337 in regard to Burnside's theorem given on p. 123 of the present work.

---

* Mr. Fite called my attention to this error.

† *Quar. Jour. of Math.*, vol. 28 (1896), p. 274.

‡ *Ibid.*, p. 32.

The next three chapters (VII.–X.) are devoted to the theory of substitution groups. Only a brief sketch of this theory could be given in the limited space, especially since a great part of it is devoted to the determination of all the possible substitution groups of given degrees. The errors which occur in these enumerations have been noted elsewhere.* On p. 189 an error which occurs in Jordan's Traité (p. 65) is pointed out without noting that Jordan had corrected the error himself soon after the appearance of his work. The beautiful theorems of Jordan in regard to the limit of transitivity, which were published in the same article† as the corrections mentioned, would probably have been more useful than the one given on p. 152.

The material of these chapters seems, in general, to have been wisely selected. The many illustrative examples enable the student to get a clear insight into the real meaning of the terms transitivity, intransitivity, primitivity, and imprimitivity. The construction of intransitive groups furnishes a good means to acquire a thorough working knowledge of quotient groups and isomorphisms. The references in these chapters are not as complete as those of most of the other chapters. In some of the illustrative examples only a brief outline is given, so that the reader has to supply a number of intermediate statements to complete the proofs. Corollaries II., on pp. 177 and 184, evidently do not apply to the primitive groups of a prime order.

The principle of isomorphisms is one of those thought saving developments upon which special stress has been laid in recent years. In chapter XI. the isomorphisms of a group with itself are considered. The beginning of this chapter is not auspicious, for in the second paragraph it is stated that a group of order 2 is the only one in which each operator corresponds to itself when we let the inverse of each operator of a group correspond to itself. This evident inaccuracy is, however, followed by one of the best and most interesting chapters of the entire volume. The developments would doubtless have been simplified by giving some other methods of making a group isomorphic with itself, but the entire subject matter is so new that a great degree of simplicity could scarcely be expected.

---

* BULLETIN, vol. 5 (1899), p. 249.

† Jordan, *Bull. de la Soc. Math. de France*, vol. 1 (1873), p. 41. Cf., BULLETIN, vol. 4 (1897), p. 144.

In this chapter Burnside introduces a very useful new term (holomorph) which enables him to employ an interesting theorem in regard to regular groups, due to Jordan,* in the study of the group of isomorphisms. It would perhaps have been better to define the holomorph of a regular group G as the substitution group which includes all the possible substitutions in the elements of G that are commutative with G instead of the abstract group which is simply isomorphic to this substitution group. The largest subgroup of the given substitution group that does not involve a given element is evidently simply isomorphic to the group of isomorphisms of G.† In article 171, on the group of isomorphisms of an abelian group of order p^n and type (1, 1, ⋯, to n units), reference should have been given to Moore's earlier article bearing on the same subject ‡ and the theorem that the symmetric group of n elements is a complete group, except when $n = 6$, should have been referred to the BULLETIN, vol. 1, p. 258, as well as to *Mathematische Annalen*, vol. 46, p. 345.

Chapters XII. and XIII. are devoted to the graphical representation of a group and to Cayley's color groups. Here we meet for the first time some groups of an infinite order. The first chapter is, to a great extent, a reproduction of Dyck's well known article which was published in the *Mathematische Annalen* in 1882, vol. 20. The first volume of the *American Journal of Mathematics* contains a brief note on color groups by Cayley. Recently Maschke published a long article on this subject in the same journal, vol. 18, 1896, pp. 156–188. Burnside gives only a brief sketch of these groups, devoting merely five pages to them. Chapter XIV. contains a very brief sketch of the linear group, which forms the subject matter of the greater part of Jordan's Traité des substitutions. In the analysis of the fractional linear groups Gierster's noted memoir, published in volume 15 of the *Mathematische Annalen*, is closely followed. Moore has called attention § to the fact that the proof given on page 339, to the effect that the group of isomorphisms of the group of order 16 which has four independent generators is simply isomorphic to the alternating group of degree 8, is incorrect; and that the proper references are lacking in regard to the developments on pp. 336–339 and p. 342.

---

* Traité des substitutions, 1870, p. 60.

† On page 221, vol. 1, of the *Encyclopädie der mathematischen Wissenschaften*, it is stated that the group of isomorphisms of G consists of those substitutions in the same elements that are commutative with G. This is evidently incorrect; cf. BULLETIN, vol. 5 (1899), p. 245.

‡ Moore, BULLETIN, vol. 2 (1895), p. 33

§ Moore, *Math. Annalen*, vol. 51, p. 419.

In the final chapter (XV.) of the book the soluble and composite groups are studied. In speaking of this chapter the author says (p. 334): "if the results appear fragmentary, it must be remembered that this branch of the subject has only recently received attention; it should be regarded rather as a promising field of investigation than as one which is thoroughly explored." Sylow's proof that every group whose order is p^a is soluble was one of the first general results in this direction. Recently Frobenius succeeded in establishing an equally general and interesting theorem by proving that all the groups whose order is not divisible by the square of a prime number are soluble. A large number of theorems of less extensive application have recently been published. These are clearly set forth in the present chapter, which seems more replete with recent results and problems awaiting solution than any other.

In conclusion we would say that the book seems to us to be a strong one notwithstanding the fact that the inaccuracies are somewhat numerous.* We have endeavored to call attention to a sufficient number of these to put the reader on his guard not to accept the results without a careful examination of the proofs. It is to be hoped that the demand for the book will warrant a new and carefully revised edition. A work containing such a rich store of facts, and nothing but facts, would be exceedingly valuable. It should be remembered that some of the mathematical works which have had a great influence on the developments in certain directions have been greatly marred by errors, so that we should not allow the minor defects of the present work to blind us to its many merits.

The student of the theory of groups will find here a rich storehouse of material, and the investigator will find numerous suggestions in regard to problems which await solution and methods of attacking them. The very errors will doubtless serve as a starting point for many investigations, since there are few things that can furnish a stronger impetus to many beginners than the thought of correcting a noted mathematician. It is therefore to be hoped that the book will be a means to arouse "interest among English (speaking) mathematicians in a branch of pure mathematics which becomes the more fascinating the more it is studied,'' and thus accomplish the author's object.*

---

* The book contains no references to Italian sources, although Italy has contributed very materially towards the development of this subject.

G. A. MILLER.

CORNELL UNIVERSITY,
March, 1900.

SHORTER NOTICES.

Traité de Nomographie. Par MAURICE D'OCAGNE. Paris, Gauthier-Villars, 1899. ix + 480 pp.

GIVEN a function of several variables, to read off its approximate values for all values of the variables, by simple inspection of a diagram drawn once for all—such is the general problem the investigation of which has led to this attractive volume. The diagram is often provided with travelling parts. The whole apparatus is called an *abacus*.

The problem is primarily a practical one. The technical arts force upon us relations or laws. So soon as a law is of frequent occurrence, its abacus is desirable. Even where great accuracy is required, an abacus is useful for getting a first approximation: for instance in the calculation of annuities (Abaque de M. Prévot), or in the solution of Kepler's equation

$$a - e \sin a = \mu$$

(abaque de M. d'Ocagne).

The scientific question as to what laws are capable of such exhibition is reserved for the final chapter. The rest of the book consists of classified instances. These instances are of quite surprising range and power. Practically a new subject is sprung upon us, claiming to be useful in so many directions that it would strain the faculty of an institute of technology to review the book in full detail. Thus we find an abacus (again by M. Prévot) used in the construction of the chromatic harp, an abacus of the penetration of light from a lighthouse (M. Allard), of the march of troops (M. Goedseels), of the deviation of the compass for an assigned ship (M. Lallemand), of the gauging of yachts (M. Chancel). These few random instances will serve to indicate the possibilities of the abacus in abolishing needless arithmetic. But the applications to the classic problem of cuttings and embankments must be especially mentioned, as M. d'Ocagne implies that the subject really grew out of this problem.

---

* P. vi.

In the construction of an abacus, free use is made of the elements of analytic geometry, and in particular of both point and line coördinates. But the curve of the geometer is replaced at the outset by the scale (échelle). That is, the relation $y=f(x)$ is shown not by a curve but by the Y-axis itself, the point of this axis whose distance from the origin is $f(x)$ being marked simply x. Thus the common slide rule is built of two parts, either of which is a scale of logarithms, representing in the manner stated $y=\log_{10}x$.

So a scale of cosines might bear the numbers 0, 10, 20, ..., 180, the number x being distant from the origin $\cos x^0$, or $l\cos x^0$ if $2l$ is the length of the scale.

Such scales play a great part in the construction of abaci. But referring to the book itself for the other means employed, we shall do better to give a specific instance of an abacus.

Consider two points of a sphere, whose latitudes are λ and λ' and the difference of whose longitudes is L. The angle at the center is φ where

$$2\cos\varphi = (1+\cos L)\cos(\lambda-\lambda') - (1-\cos L)\cos(\lambda+\lambda'). \tag{1}$$

Write $u=-l\cos(\lambda+\lambda'),\ v=l\cos(\lambda-\lambda')$.

Then $u(1-\cos L)+v(1+\cos L)=2\,l\cos\varphi. \qquad (2)$

Take a square of side $2l$, whose center is the origin. Mark a scale of cosines along the right side upwards, along the left side downwards, and along the top to the right. From the right scale read the number marked $\lambda-\lambda'$; its ordinate is u. From the left scale read the number $\lambda+\lambda'$; its ordinate is v. These numbers u and v are parallel coördinates of a line (as in Salmon, Conics, p. 386 of the sixth edition). And whatever values we assign to u and v, subject to (1), the line passes through a point; that is, (2) is the equation of a point. By an easy calculation this point is

$$x=l\cos L, \quad y=l\cos\varphi.$$

We lay then a line through the points $\lambda\pm\lambda'$, and observe where it cuts the vertical through the point of the top scale marked L; the ordinate of this point is $l\cos\varphi$, and φ itself is read off from the scale at the right.

The square with its scales, and the moveable line, constitute then the abacus of the equation (1). Naturally, the square is marked with vertical and horizontal lines to assist the reading.

The questions raised in the book should appeal not only to the technical man, but also to the teacher of elementary analytic geometry, at least to those teachers who care to heed that class of students whose cry is "of what use is this?"

FRANK MORLEY.

An Elementary Treatise on the Theory of Equations. By S. M. BARTON, Ph.D. D. C. Heath & Co., Boston, 1899. 8vo, xii + 198 pp.

THE title of Professor Barton's book suggests at once to the English-speaking student of mathematics the well-known treatises of Todhunter and of Burnside and Panton, and calls to mind how consistent English practice has been in assigning the theory of equations to distinctive works on the subject. Among the comparatively few books thus styled may be mentioned Chapman's "An Elementary Course in Theory of Equations," published in this country a few years ago.

The present treatise is much more limited in scope than the English works above mentioned. There is no attempt to deal with the formal side of the higher algebra. The book is intended expressly for undergraduate instruction in our colleges and technical institutions, and the contents and treatment are accordingly quite narrowly prescribed by the requirements of the usual college course.

The work falls into two parts: I., an elementary exposition of determinants; II., the theory of equations proper.

Part I.—The first two chapters give the principal theorems of determinants, and the third consists of applications to linear equations and a consideration of special determinant forms. The subject is introduced by considering the permutations of a group of elements, after which the development follows the usual course. The author here undoubtedly exposes himself to criticism for devoting so much space, some seventy-five pages in all, to a theory of which he is able to make very little use in the remaining chapters.

Part II.—In this part Burnside and Panton have been laid under heavy contribution. An inspection of the table of contents and a review of the pages show, as the author states, that the development has followed very closely in the lines of the first ten chapters of the treatise cited. It will perhaps be fair to characterize this part as an attempt to adapt a portion of the Burnside and Panton to the needs of the general student, and at the same time to provide an introduction to the more extensive treatises on the subject. To secure the requisite condensation, an occasional change in arrangement has been made, statements and proofs abridged, and certain parts omitted. In all essential respects, however, the proofs, method, and arrangement of the English book have been retained.

In so far as the excellent features of the large treatise are found reproduced in the smaller work, to that extent is this grade of text-book enriched. A decided gain is the retention of the purely algebraical methods of proof. There is undoubtedly a tendency, quite marked in some of our college text-books on algebra, to replace the algebraical proofs of certain theorems by the methods of the calculus. The gain in an elementary text-book by a substitution of this character is neither apparent nor real, and does not warrant so radical a departure from algebraical methods; on the contrary, the change is attended by a loss in simplicity and uniformity of development.

It is to be regretted that Professor Barton has not availed himself of the opportunity to improve upon Burnside and Panton in the matter of drawing a sharp distinction between the variable x and its absolute value. The need of a careful discrimination in this direction is particularly felt in the chapter on the properties of entire polynomials.

The condensation and abridgements previously referred to have not always been effected without loss. Occasionally indeed there are statements which, unqualified, are not true. Again, there is a failure at times to express with sufficient directness and simplicity the nature and purposes of the propositions. Attention may be here directed to several cases in which changes seem desirable. In the introduction to Part II., page 77, the double assumption that a and β are distinct roots and that a is not zero, is a rather unhappy conjunction of hypotheses. So in chapter VII., page 125, following the statement of the general theorem, the corollary: "Corollary I. Every root of an equation is a divisor, *whole or fractional*, of the absolute term of the equation" should be eliminated. As it stands, it is incorrect; and if restored to the form in which it appears in Burnside and Panton, it is void of significance. On page 162, the reference to the sign of the intermediary Sturm function $f_r(x)$ is superfluous, and, moreover, the function does not necessarily change sign when x passes through a.

For completeness of statement, we note that the introduction to the second part is followed by a chapter of sixteen pages on complex numbers, containing De Moivre's theorem and the trigonometric solution of the binomial equation. There is also a chapter of six pages on elimination, with illustrations of Euler's and Sylvester's methods. The volume concludes with the methods of approximating to the real roots of numerical equations. Each theorem of the text, as far as possible, is immediately supplemented by problems showing its applications and place in the theory.

In conclusion, there are many admirable features in the work which would make it in some respects a valuable adjunct to instruction, and it is to be hoped that in a second edition a careful revision and the elimination of the rather numerous typographical errors will bring these qualities more clearly to light and place it in a position of considerable usefulness.

JAMES MACLAY.

The Theory and Practice of Interpolation: Including mechanical quadrature and other important problems concerned with the tabular values of functions. With the requisite tables. By HERBERT L. RICE, M.S. The Nichols Press, Lynn, Mass. 1899. Cr. 8vo., 234 + ix pp.

THE theory and practice of interpolation is, in its essentials, based on the fact that in nearly all functions which arise in physical problems, a small change of the variable produces a small change of the function. The simplest case is the one in which we may consider the ratio of the two small changes as constant: in the language of the subject, the first differences are constant. When this assumption will not give sufficiently accurate results, we have to consider the difference of the first differences or the *second* differences, and even differences of higher order. Ultimately we neglect differences of some definite order and thus implicitly reduce the problem to the consideration of the values of a function which, between certain limits and to a given degree of accuracy, may be considered rational, integral and algebraic. Round this problem a mass of literature has grown up. The values of a function are required for certain values of the variable. To obtain these every time from a formula may be troublesome, or even impossible if no such formula is known; tables are therefore made giving the values of the function for certain values of the variable, and the subject teaches how we can thence obtain an approximate value of the function for the given value of the variable, and conversely.

Mr. Rice, in this volume, appears to aim at placing before the reader all the formulæ which are likely to be required for computing purposes, rather than to give an account of the principles which underlie the methods used. He has developed many formulæ very fully so as to show how they may be applied to any particular case, and has illustrated their use by many numerical examples. The book is essentially a practical one and as such will doubtless be of great service to a student wishing to learn the methods in use, as well as to one who knows them but desires to have the results gathered together in one volume; and the tables at the end will greatly increase its value. To one interested in the subject from the mathematical point of view only, it will perhaps be found to be lacking in interest. But this separation of the theoretical from the practical may be necessary here if either is to be of service. From the practical side, the processes used carry their own justification; that is, the results appear to justify the means employed. The theorist would probably prefer to investigate their validity and determine their limitations. One of the chief problems for the latter is the consideration of the limits between which any of the functions which arise for consideration can be represented, either theoretically or usefully for computing purposes, by a rational integral function, and the degree of error committed in so doing.

There are three principal formulæ used—those of Newton, Stirling, and Bessel—which really amount to a development by Taylor's theorem, with the assumption that we can always stop at some particular term and neglect the remainder. The differences between the three formulæ simply consist in the manner in which the various orders of differences are grouped; the best formula for any particular purpose is generally that which gives the required accuracy with the least amount of calculation. All these details are fully set forth by the author, and the reader will have no difficulty in immediately finding for practical use anything he may require.

One or two details call for slight criticism from the point of view of accurate statement. On p. 40 a remark is made which would leave the reader to infer that interpolation may give more accurate results than direct calculations even when both methods are available; this, if true, at least requires explanation. Again the author uses the elliptic integrals (with a real variable) as examples of non-integrable expressions when the modulus is near unity; this is misleading, as a transformation enables us in such a case to approximate as close as we wish to the numerical value of the integral. The writer of this notice has not tested any of the formulæ but they appear to be carefully worked out and the author seems to be familiar with the best methods of computation. ERNEST W. BROWN.

Vorlesungen über Geschichte der Trigonometrie. Von DR. A. VON BRAUNMÜHL. Erster Theil. Leipzig, B. G. Teubner, 1900. 260 pp.

VON BRAUNMÜHL's history of trigonometry is to be in two parts. The volume before us is the first part, and carries the narrative down to the invention of logarithms. If the second half of the work rises to the level of the first, the book will at once take its place as the most complete and authoritative work on the history of trigonometry.

In the perusal of this volume we realize the rapid progress which has been made in the study of mathematical history. The "restitution of decayed intelligence" has been carried on with great diligence. Formerly the origin of trigonometry was ascribed to Hipparchus and Ptolemaeus; now the earliest attempts to establish relations between the sides and the angles of a right triangle can be traced back to a period as long before the time of Hipparchus, as Hipparchus was before our time. The Ahmes papyrus of about 2000 B. C. contains five problems involving such relations.

In Whewell's History of the inductive sciences we read of the "sterility of Arabian genius," but in recent years a closer acquaintance with Arabic research in mathematics forces us to the admission that the Arabs displayed considerable originality. Von Braunmühl makes it plain that in trigonometry much of what has been ascribed to Europeans of the fifteenth century was worked out at an earlier period by the Arabs. The author suggests that even our present knowledge may require considerable revision after Arabic manuscripts have been read; which at present, covered with the dust of centuries, lie untouched on the shelves of Spanish libraries.

The author has embodied in this volume a great deal of original research. He has brought to light the fact that the Greeks possessed a graphic method of solving spherical triangles which was extended by the Hindus and Arabs so as to become a fruitful aid in trigonometric computation. This remained in use as late as the seventeenth century. Von Braunmühl traces the beginning of the graphic treatment of spherical triangles back to Anaximander, but suspects that these graphic processes are much older, that they were known to the Egyptian and possibly also to the Chaldean astronomers.

The graphic method of solving spherical triangles is the oldest trigonometric *method* known to us. Ptolemaeus gives graphic processes in which *sine* is used, but curiously enough, in all trigonometric computation he employs instead *the chord of double the arc.* This anomaly finds its explanation in the fact that the Greeks treated the graphic solutions and the numerical solutions apart from each other; it remained for the Hindus and Arabs to unite the two methods and to recognize the advantage, in all cases, of using half the chord in place of the whole chord.

We have noticed no errors of importance. On page 88 the date of Alcuin's birth is given as 736. The same date occurs in Felix Müller's *Zeittafeln*, but the correct date is uncertain; it is probably 735. Snellius's baptismal name is spelled on page 70 *Willebrod*, in other places it is given correctly as *Willebrord*. FLORIAN CAJORI.

Histoire des Mathématiques. Par JACQUES BOYER. Illustrée de fac-similés de manuscrits et de portraits. Georges Carré et C. Naud, Éditeurs. Paris, 1900. 250 pp.

ON opening this book the reader is attracted by several facsimile reproductions from old mathematical books or documents and by a number of portraits of mathematicians. Thus he has before him a facsimile of part of the Egyptian Akhmim papyrus, of the title page of the Acta Eruditorum, of a page of Euclid's Elements, from a manuscript preserved in the Bibliothèque Nationale of Paris. There are nineteen portraits of mathematicians. The list comprises eleven Frenchmen, three Englishmen, two Germans, two Russians, and one Swiss. Two likenesses are of women, namely, of Mme. Du Chatelet and Mme. Kovalevski.

To write a general history of mathematics and confine it, as M. Boyer does, to the small compass of 247 pages is no easy task. Anything like completeness cannot be looked for. Perhaps all one can expect is that the information offered be accurate, that the broad movements in mathematical thought be brought before the reader and that the narrative be made attractive so as to invite more thorough study in larger treatises. In this last respect we think that

M. Boyer has done well; but in the other two his success is not always so evident.

As a rule the information is up to date and the facts are accurately presented. In the case of Mme. Kovalevski he puts us on our guard against the usual 1853 as the date of her birth, and gives it correctly as 1850. In some places the author fails to embody recent results of historical research. He mentions Argand on the graphic representation of the imaginary, but says nothing of Wessel; he does not mention Taurinus in connection with Gauss, Lobachevski and the Bolyais; the native place of Serenus is given as Antissa, but according to Heiberg it should be Antinoeia. M. Boyer refers to the question whether or not "the physical space of our experience" is that of Lobachevski and then says: "Peirce conclut par l'affirmative en se basant sur de délicates observations des parallaxes d'étoiles fixes." By the reference given in the alphabetical index this remark is attributed to Benjamin Peirce, but this is probably an error, for it was Mr. Charles S. Peirce who once maintained that he had demonstrated the above.

M. Boyer has failed to emphasize that, according to recent study of Arabic manuscripts, Nasir Eddin and other Arabs had really developed trigonometry to a much fuller extent than was formerly supposed. We were rather surprised to find no reference whatever to such original workers in algebra as D. F. Gregory, George Peacock, and George Boole, while an eighteenth century English algebraist of lesser note is not only mentioned but also honored with a portrait. The author has signally failed in setting before the reader a view of the development of theories. For instance, nothing worthy of notice is given on the growth of the theory of limits, or the gradual extension of the number concept. No matter how inadequate the space may be for recent history, it would seem as if the broad fact should be presented that at the beginning of the nineteenth century mathematics had no adequate logical foundation, and that Gauss, Abel, and Cauchy were compelled to begin to build anew on firmer ground. The alphabetical index is inaccurate and incomplete.

Notwithstanding these defects the book will be useful in attracting students to the study of the history of mathematics and making them acquainted with the names and some of the achievements of great mathematicians.

FLORIAN CAJORI.

Vorlesungen über Kreis- und Kugel-Functionen-Reihen. Von DR. JOHANNES FRISCHAUF. Leipzig, B. G. Teubner, 1897, pp. 60.

THIS little treatise will be welcomed by students of pure analysis and also by students of mathematical physics. The subject treated has already an extensive literature of its own; but the important results in this field are scattered through different memoirs of Dirichlet, Du Bois Reymond, Bonnet, Lipschitz, Dini, and others, while there has appeared no distinctly elementary treatment of the subject as a whole. Thorough and comprehensive treatments of Fourier's series and other allied developments have indeed been given by C. Neumann in his "Über die nach Kreis- Kugel- und Cylinder-Functionen fortschreitenden Entwickelungen," and by Dini in his "Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale," but these treatments could hardly be called elementary—certainly not in comparison with the above work. The author has followed a uniform and easy method throughout and has shown the convergence of Fourier's series, of Fourier's integral, and of the developments in terms of zonal and spherical harmonics in case the function to be developed has but a limited number of maxima and minima within a certain interval and is either continuous within this interval or else has but a finite number of so called "finite discontinuities." Some of the simpler cases are also considered in which the maxima and minima become infinite in number in the neighborhood of certain points, or in which the function at certain points becomes infinite in value. In brief, the author has given us an elementary, systematic treatment of these series for all cases which might ordinarily arise in connection with nature.

Chapter I. deals especially with Fourier's series. Chapters II. and III. are devoted to the discussion of certain general properties of zonal and spherical harmonics, and Chapter IV. relates to the developments in terms of the latter functions. Illustrative examples are given throughout.

W. B. FORD.

NOTES.

THE sessions of the coming summer meeting of the Society will be held on the afternoons of Wednesday, Thursday, and Friday, June 27th–29th, alternating with those of Section A of the American association for the advancement of science. A fourth session will be added if necessary. The place of meeting is room 506, Fayerweather Hall, Columbia University. The Council will meet on Thursday morning.

THE ninth conference of the German association for promoting instruction in the mathematical and natural sciences was held at Hamburg, June 4th–7th, 1900.

A NEW scientific journal, *Il Bolletino di matematiche e di scienze fisiche e naturali*, devoted to the interests of the teachers and students of the normal schools of Italy, has been lately founded at Bologna, under the editorship of Professor ALBERTO CONTÉ. The initial number appeared December 1, 1899, and each annual volume will consist of twenty-four numbers.

THE *Revue Philosophique* for April, 1900, contains two articles of interest to mathematicians: "La première antinomie mathématique de Kant," by C. DUNAN, and "l'Antinomie du transfini," by É. BOREL.

THE first and second numbers of the first volume of the third series of *Bibliotheca Mathematica* under the editorship of Gustaf Eneström appeared from the press of B. G. Teubner April 30th, 1900. The enlarged sphere of this journal is amply indicated by the table of contents of this double number of two hundred and ninety-six pages, containing a portrait of Sophus Lie as frontispiece, some thirty papers, obituary notices of C. I. Gerhard, F. Rosenberger, H. E. Wappler, and L. G. Gascó, a chronological list of the writings of Sophus Lie by F. Engel, accounts of the annual meetings held by mathematicians in Germany, France, England, and America last summer, reviews, new publications, and current notes of scientific interest.

THE last three parts of GINO LORIA's Le scienze esatte nell' antica Grecia are announced for early publication.

The first two parts appeared in 1893 and 1895. The concluding books are entitled respectively: Il substrato matematico della filosofia naturale dei Greci; Il periodo argenteo della geometria Greca; L'aritmetica dei Greci.

THE first part of FELIX MÜLLER's Mathematisches Vocabularium in deutscher und französischer Sprache is in the press. The completed work will contain ten thousand terms of pure and applied mathematics; it has been prepared with the assistance of J. NEUBERG and A. WANGERIN, and represents years of labor. The authors hope that this vocabulary will prove to be 1° a useful supplement to all French and German dictionaries, 2° the forerunner of an exhaustive mathematical dictionary, 3° a lexicographical guide to the terminology of the mathematical sciences.

THE third volume of the second edition of M. CANTOR's Vorlesungen über Geschichte der Mathematik is in the press; it will appear in three parts.

FELIX L. DAMES, of Berlin, announces the appearance of the first part of the third and final volume of Professor J. G. HAGEN's Synopsis der höheren Mathematik. The three volumes are devoted respectively to algebraic analysis, geometry, and differential and integral calculus.

THE UNIVERSITY OF CHICAGO. The announcement for the summer quarter of 1900 appeared in the May number of the BULLETIN. During the three quarters (autumn, winter, spring), October, 1900, to June, 1901, the following mathematical courses will be offered, four or five hours weekly:—By Head Professor E. H. MOORE: Theory of groups with applications to algebra (introductory course followed by a seminar), autumn and winter quarters; Functions of a real variable, spring quarter; Advanced integral calculus I, autumn quarter; Advanced algebra II, winter quarter.—By Professor OSKAR BOLZA: Theory of elliptic functions (Weierstrass) with applications, winter and spring quarters; Calculus of variations, winter quarter; Theory of functions of a complex variable, spring quarter.—By Associate Professor H. MASCHKE: Selected chapters of algebra, autumn quarter; Projective geometry, autumn quarter; Modern analytic geometry (continuation), winter quarter; Advanced integral calculus II, winter quarter.—By Assistant Professor J. W. A. YOUNG: Theory of equations I, autumn quarter.—By Assistant Professor L. E. DICKSON: Lie's theory of differential equations, winter quarter; Continuous groups, spring quarter.—By Dr. H. E. SLAUGHT: Solid analytical geometry, spring quarter.—By Dr. J. H. BOYD: Theoretical mechanics, autumn quarter; Differential equations, spring quarter.—By Dr. F. R. MOULTON: Analytical mechanics I, II, autumn and winter quarters.

YALE UNIVERSITY. The following courses in mathematics are announced for the year 1900–1901:—By Professor J. WILLARD GIBBS: Vector analysis, three hours; Multiple algebra, two hours; Electromagnetic theory of light, two hours.—By Professor W. BEEBE: Celestial mechanics, three hours.—By Professor J. PIERPONT: Differential equations and theory of functions, three hours; Galois's theory, three hours; † Theory of numbers, three hours; † Partial differential equations and in particular those of dynamics, three hours.†—By Professor P. F. SMITH: Continuous groups, two hours; Transformations of space, two hours; † Differential geometry, two hours.†—By Professor G. P. STARKWEATHER: Theoretical mechanics (advanced), two hours.—By Dr. M. B. PORTER: Linear differential equations, three hours; Calculus (advanced), three hours.—By Dr. H. E. HAWKES: Modern geometry, three hours; Advanced algebra, three hours.†—By Mr. E. B. WILSON: Algebraic plane curves, three hours; † Theoretical mechanics (elementary), three hours.†

Courses marked † extend through only one-half of the year.

A MATHEMATICAL seminary, appointed as are those of the universities of Göttingen and Leipzig, has been equipped at the University of Jena.

THE mathematical prize of the Istituto Lombardo di scienze e lettere in Milano, will be awarded for the year 1901 for a study of the differential equations of electrotechnics that will indicate the most practicable methods for their approximate integration and illustrate the exposition with examples.

THE Istituto Veneto delle scienze, lettere, ed arti offers its prize for the year 1902 for a memoir on the projective character of two dimensional surfaces in space of n-dimensions.

PROFESSOR THOMAS CRAIG, of Johns Hopkins University, died at Baltimore, May 8th, 1900, aged 44 years. Professor

Craig was one of the pioneers in the great movement of the last quarter of a century which has brought American mathematicians in touch with continental ideas and methods. He graduated at Lafayette College in 1875, with the degree of civil engineer; but with the advent of Sylvester in America his natural interest in pure mathematics immediately asserted itself. At the opening of Johns Hopkins University in 1876, he was appointed to a fellowship and at the same time was entrusted with a part of the mathematical instruction. In 1878, he received the degree of doctor of philosophy from Johns Hopkins, his dissertation being entitled: "Representation of one surface upon another, and on some points in the theory of the curvature of surfaces." During the years 1879–1881, he was connected with the United States Coast Survey, continuing, however, to give certain lectures at the University. His connection with the University covered the entire twenty-four years of its existence. Since 1892 he held a full professorship of mathematics. A considerable part of his activity for twenty years was devoted to the *American Journal of Mathematics*, of which he was an editor from 1883, and editor-in-chief from 1893 to 1898. Besides numerous articles in the *American Journal of Mathematics* and elsewhere, Professor Craig published a treatise on the mathematical theory of projection (1879), a treatise on linear differential equations (1889), and two briefer works on fluid motion (1879).

PROFESSOR L. BIANCHI has been made a member of the editorial board of the *Annali di Matematica* in the place of the late E. Beltrami. The board now consists of L. Cremona, U. Dini, G. Jung, and L. Bianchi.

PROFESSORS W. FÖRSTER, of Berlin, and E. WEISS, of Vienna, recently celebrated twenty-five year jubilees as professors at the respective universities.

PROFESSOR L. BOLTZMANN, of Vienna, has accepted the call to a professorship of physics recently tendered him by the University of Leipzig.

MR. J. F. HUDSON, of Oxford University, has been appointed mathematical lecturer at University College, Bristol, in the place of Mr. J. F. MCKEAN, who resigned to become mathematical lecturer at the Royal Naval Engineering College, of Devonport.

PROFESSOR F. MORLEY, of Haverford College, has been appointed head of the department of mathematics at Johns Hopkins University.

At Yale University the following changes have taken place in the mathematical staff: In the Sheffield Scientific School, assistant professor Percey F. Smith has been promoted to a full professorship, and Dr. G. P. Starkweather to an assistant professorship of applied mechanics. Mr. E. B. Wilson has been appointed instructor in the academic department.

At Princeton University Mr. A. H. Wilson resumes his duties as instructor in mathematics after spending a year on leave at the University of Göttingen, and Dr. E. O. Lovett has been promoted to a full professorship in mathematics and granted a year's leave of absence.

A separate department of astronomy has been organized at the University of California and placed under the charge of associate professor A. O. Leuschner. Dr. E. M. Blake and Mr. D. N. Lehmer have been appointed instructors in the department of mathematics.

At Columbia University, Mr. C. J. Keyser, has been promoted to an instructorship in mathematics, and Mr. D. H. Pollard has been appointed assistant in mathematics.

Dr. G. H. Ling, of Wesleyan University, has been appointed professor of mathematics at the University of Cincinnati for the summer session of 1900.

Mr. A. H. Lybyer, of Princeton University, has been made professor of mathematics at Roberts College, Constantinople.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Brunel (G.). See Encyklopädie.

Burali-Forti (C.). Les propriétés formales des opérations algébriques. Turin, 1900. 12mo. 40 pp. Fr. 1.00

Burkhardt (H.). See Encyklopädie.

Carnoy. Cours de géométrie analytique. Vol. I: Géométrie plane. 6e édition. Paris, Gauthier-Villars, 1899. 8vo. Fr. 11.00

Cox (H.). Rudimentary treatise on integral calculus. New edition. London, Lockwood, 1900. 12mo. 136 pp. (Weale's series.) 1s. 6d.

Dehn (M.). Die Legendre'schen Sätze über die Winkelsumme im Dreieck. [Diss.] Leipzig, Teubner, 1900. 8vo. 38 pp. M. 1.20

Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. II., redigiert von H. Burkhardt. Heft 2 und 3: G. Brunel, Bestimmte Integrale [Schluss]; P. Painlevé, Gewöhnliche Differentialgleichungen, Existenz der Lösungen; E. Vessiot, Gewöhnliche Differentialgleichungen, elementare Integrationsmethoden; E. von Weber, Partielle Differentialgleichungen. Leipzig, Teubner, 1900. 8vo. Pp. 161–399. M. 7.50

Geck (E.). Ueber die singulären Punkte algebraischer Flächen. [Diss.] Tübingen, Pietzcker. 1900. 8vo. 40pp. M. 1.00

Gutzmer (A.). See Jahresbericht.

Hauck (G.). See Jahresbericht.

Hentschel (O.). Ausführung einiger konformen Abbildungen. I. (Progr.) Salzwedel, 1899, 4to. 8 pp., 4 plates.

Hermite (C.). See Serret (J. A.).

Herff (E.). Die Maxima und Minima einer verwandelbaren Funktion. (Progr.) Sigmaringen, 1899. 4to. 18 pp.

Jahresbericht der deutschen Mathematiker-Vereinigung, enthaltend die Chronik der Vereinigung für das Jahr 1899 und die auf der Versammlung zu München gehaltenen Vorträge. Herausgegeben von G. Hauck und A. Gutzmer. Leipzig, Teubner, 1900. 8vo. 231 pp.

Kinn (G. A.). Die Anwendung unendlicher Produkte in der Funktionentheorie. (Progr.) Sächsisch-Regen, 1899. 4to. 32 pp.

Painlevé (P.). See Encyklopädie.

Pescione (A.). Le coniche in coordinate cartesiane; appunti di geometria analitica. 1899. 8vo. Fr. 2.50

Picard (E.) et Simart (G.). Théorie des fonctions algébriques de deux variables indépendantes. Vol. II, fascicule 1. Paris, Gauthier-Villars, 1899. 8vo. 4 + 206 pp. Prix du volume complet Fr. 14.00

Principales formules de la théorie des fonctions elliptiques; tableau résumé publié par les *Nouvelles annales de mathématiques*.* Paris, Gauthier-Villars, 1900. 8vo. (Concours pour l'agrégation de mathématiques.) Fr. 0.75

Radelfinger (F. G.). Linear differential equations. (*Philosophical Society of Washington*, *Bulletin*, Vol. 14, pp. 21–35.) Washington, 1900. 8vo.

Serret (J. A.). Cours de calcul différentiel et intégral. 5e édition, augmentée d'une note sur les fonctions elliptiques, par C. Hermite. (En 2 volumes.) Vol. I: Calcul différentiel; 13 + 617 pp. Vol. II: Calcul intégral; 13 + 904 pp. Paris, Gauthier-Villars, 1900. 8vo. Fr. 25.00

Simart (G.). See Picard (E.).

Sporer (B.). Zur Ableitung allgemeiner Eigenschaften algebraischer Kurven. (Progr.) Ehingen, 1899. 4to. 24 pp.

Torroja (E.). Tratado de geometria de la posicion y sus aplicaciones á la geometria de la medida. Madrid, 1899. 4to. 813 pp. Fr. 25.00

Vessiot (E.). See Encyklopädie.

Villafrañe y Viñals (J. M.). Tratado de analisis matematica (algebra superior). 2a edicion, corregida y aumentada. Barcelona, 1899. 4to. 996 pp. Fr. 25.00

Weber (E. von). See Encyklopädie.

Weiner (F.). Eine Anwendung der Hermite'schen U-Funktionen. (Progr.) Wien, 1899. 8vo. 16 pp.

II. ELEMENTARY MATHEMATICS.

Aleu (M. L.). Elementos de matematica. Geometria. 2a edicion. Madrid, 1899. 4to. 8 + 268 pp. Fr. 6.50

Arndt (E.). Hauptsätze der ebenen Geometrie, nebst Uebungsaufgaben zum Gebrauche an Volks- und Mittelschulen. 7te Auflage. Berlin, Oehmigke, 1900. 8vo. 4 + 56 pp. M. 0.50

Chrystal (G.). Algebra. Elementary text-book for higher classes of secondary schools, etc. Part 2. 2nd edition. Edinburgh, Black, 1900. 8vo. 632 pp. 12s. 6d.

Comberousse (C de). See Rouché (E.).

Constable (W. G.) and Mills (J.). Elementary algebra, including quadratic equations. London, Longmans, 1900. 12mo. 272 pp. Cloth. 2s.

F. (F.). Manuel d'algèbre et de trigonométrie. Paris, Poussielgue [1900]. 16mo. 8 + 216 pp.

Focke (M.) und Krass (M.). Leitfaden zur Einführung in die Stereometrie und Trigonometrie. Sonderausgabe für die Untersekunda aus den gleichnamigen Lehrbüchern. 3te Auflage. Münster, Coppenrath, 1900. 8vo. 32pp. Boards. M. 0.60

Harmuth (T.). Textgleichungen geometrischen Inhalts. Für den Gebrauch beim Unterricht entworfen. 2te Auflage. Berlin, Springer, 1900. 8vo. 4 + 75 pp. Boards. M. 1.20

Krass (M.). See Focke (M.).

Mills (J.). See Constable (W. G.).

Röllner (F.). (*a*) Beweis eines Gesetzes der gleichzeitig gleichschnellen Rotationen. (*b*) Ueber Aehnlichkeit und Symmetrie als grundlegende Principien der Geometrie. (Progr.) Römerstadt, 1899. 8vo. 17 pp.

ROUCHÉ (E.) et COMBEROUSSE (C. DE). Traité de géométrie. 7e édition, revue et augmentée, par E. Rouché. Première partie: Géométrie plane. Paris, Gauthier-Villars, 1900. 8vo. 42 + 548 pp. Fr. 7.50

TESTI (G. M.). Nozioni di geometria ad uso più specialmente delle allieve dei corsi complementari (già scuole preparatorie normali). 4a edizione migliorata. Livorno, Giusti, 1900. 16mo. 8 + 100 pp. Fr. 1.00

WEISBACH (J.). Tafel der vielfachen Sinus und Cosinus, sowie der vielfachen Sinus versus von kleinen Winkeln, nebst Tafel der einfachen Tangenten. 6te Auflage. Berlin, Weidmann, 1900. 8vo. 28 pp. M. 1.00

III. APPLIED MATHEMATICS.

ALBRECHT (T.). Bericht über den Stand der Erforschung der Breitenvariation am Schlusse des Jahres 1899. Herausgegeben vom Centralbureau der internationalen Erdmessung. Berlin, Reimer, 1900. 4to. 26 pp., 1 plate. M. 3.00

BLAHA (E.). See LEIST (C.).

CHARBONNEL (J.). La loi du trapèze. Paris, Dunod, 1900. 8vo. 75 pp.

EGGERS (W.). Lehrbuch der darstellenden Geometrie. Teil I: Die Elemente der darstellenden Geometrie. 2te Auflage. Leipzig, Seemann, 1900. 8vo. 7 + 54 pp., 5 plates. Boards. M. 1.50

GUBATZ, LÜDEKE, und WEIGEL. 301 Aufgaben aus der darstellenden Geometrie für Maschinenbauer, Kesselschmiede und verwandte Gewerbe. Mit kurzen praktischen Lösungen und 333 Zeichnungen. Leipzig, Seemann, 1900. 8vo. 34 pp. Boards. M. 2.25

HALL (E. H.). Elementary lessons in physics; mechanics (including hydrostatics) and light. New York, Holt, 1900. 12mo. 10 + 182 pp. Cloth. $0.65

HELLER (J. F.). See SMOLIK (F.).

HOFF (J. H. VAN'T). Vorlesungen über theoretische und physikalische Chemie. Heft 3: Beziehungen zwischen Eigenschaften und Zusammensetzung. Braunschweig, Vieweg, 1900. 8vo. 10 + 36 pp. M. 4.00

JOHNSON (J. B.). Theory and practice of surveying; for surveyors and engineers generally, but especially for students of engineering. 15th edition, revised and enlarged. New York, Wiley, 1900. 8vo. 900 pp. Cloth. $4.00

LACHMANN (M.). See RODET (J.).

LEIST (C.). Die Steuerungen der Dampfmaschinen. Zugleich als vierte Auflage des gleichnamigen Werkes von E. Blaha. Berlin, Springer, 1900. 8vo. 15 + 770 pp. Cloth. M. 20.00

LÜDEKE. See GUBATZ.

MERCHICH (M.). Isagoge in principia mathematica oeconomices politicae, eruditis orbis catholici congressum quintum mense augusto anni 1900 Monaci celebraturis tradita. Vienna, Horvát-Kimle, Merchich, 1900. 8vo. 26 pp. M. 0.40

MONTCHOISY (DE). Cours pratique et théorique de machines à vapeur, professé à l'Ecole supérieure de maistrance de la marine. 3e édition, (En deux volumes.) Vol. I : 15 + 623 pp. ; Vol. II : 671 pp. Paris. Challamel, 1900. 8vo.

NOVAT (J.). Cours pratique de résistance des matériaux, professé à la Société d'enseignement professionel du Rhône. Paris, Béranger, 1900. 18mo. 2 + 440 pp.

RITTER (W.). Anwendungen der graphischen Statik. Nach C. Culmann bearbeitet. Teil 3 : Der kontinuierliche Balken. Zürich, Raustein, 1900. 8vo. 12 + 270 pp., 4 plates. M. 9.60

RODET (J.). Berechnung der Leitungen von Mehrphasenströmen. Uebersetzt von M. Lachmann. Leipzig, Leiner, 1900. 8vo. 7 + 55 pp. M. 2.75

SCHMIDT (J.). Anleitung zur Konstruktion von Sonnenuhren. (Progr.) Plan, 1899. 8vo. 21 pp., 3 plates.

SMOLIK (F.). Elemente der darstellenden Geometrie. Neu bearbeitet von J. F. Heller. 2te, vermehrte Auflage. Leipzig, 1900. 8vo. Cloth. M. 4.00

WEIGEL. See GUBATZ.

SOME REMARKS ON TETRAHEDRAL GEOMETRY.

BY DR. H. E. TIMERDING.

(Read before the American Mathematical Society, June 27, 1900.)

1. Any two quadric surfaces have in general a common self-conjugate tetrahedron. If we refer to it a system of homogeneous point coördinates x_1, x_2, x_3, x_4, the equations of the two quadrics take the forms:

$$
\begin{aligned}
&g_1x_1^2 + g_2x_2^2 + g_3x_3^2 + g_4x_4^2 = 0, \\
&h_1x_1^2 + h_2x_2^2 + h_3x_3^2 + h_4x_4^2 = 0.
\end{aligned}
\tag{1}
$$

Now the polar planes of a point with respect to these two quadrics cut each other in a straight line, which we may also find as the polar line of the given point in the following way, starting from the quartic curve of intersection of the two quadrics: We draw the two chords of the quartic curve which pass through the point, and determine on each of them the fourth harmonic to its two curve points and the given point. Joining these two fourth harmonic points we have at once the required polar line.

We fix a straight line in space by Plückerian line coördinates, viz., if x_i, y_i be the coördinates of any two points upon the line, by the six expressions

$$p_{ik} = x_iy_k - x_ky_i. \tag{2}$$

Between them the identical relation exists

$$p_{23}p_{14} + p_{31}p_{24} + p_{12}p_{34} = 0. \tag{3}$$

The polar lines of all points in space, with respect to a quartic space curve of the first species, form a complex of lines. This complex has been called by Reye a tetrahedral complex. If z_i be the coördinates of the point, we immediately find for the coördinates of its polar line

$$p_{ik} = g_{ik}z_iz_k, \tag{4}$$

by making

$$g_{ik} = g_ih_k - g_kh_i. \tag{5}$$

These magnitudes g_{ik}, which also satisfy the condition

$$g_{23}g_{14} + g_{31}g_{24} + g_{12}g_{34} = 0 \tag{6}$$

may be regarded as the coördinates of the quartic curve referred to its fundamental tetrahedron, whose vertices are the vertices of the quadric cones passing through the curve.

The equations (4) give a representation of the lines of the tetrahedral complex by means of the coördinates of their "poles." From them we find the other representation

$$(7)\qquad \frac{p_{23}p_{14}}{g_{23}g_{14}} = \frac{p_{31}p_{24}}{g_{31}g_{24}} = \frac{p_{12}p_{34}}{g_{12}g_{34}},$$

whence the necessary condition (3) is, by (6), at once derived.

The fundamental tetrahedron has the singular property that every straight line lying in one of its faces, and every line passing through one of its vertices belongs to the tetrahedral complex. In fact, one coördinate of each of the three pairs p_{23}, p_{14}; p_{31}, p_{24}; p_{12}, p_{34} then vanishes, and the double equation (7) is satisfied.

The tetrahedral complex is of the second degree. Those of its lines which pass through any point constitute a quadric cone, which is circumscribed about the fundamental tetrahedron; and the lines lying in any plane envelop a conic section of this plane, which is inscribed in the fundamental tetrahedron, as will be shown hereafter.

Since the tetrahedral complex is given by the ratios of the three binary products

$$p_{23}\,p_{14},\quad p_{31}\,p_{24},\quad p_{12}\,p_{34},$$

and their sum is always 0, any equation

$$(8)\qquad \lambda_1 p_{23}p_{14} + \lambda_2 p_{31}p_{24} + \lambda_3 p_{12}p_{34} = 0$$

may be regarded as defining a tetrahedral complex.

We may enquire for the quartic space curves, by aid of which this complex is to be obtained. Their fundamental tetrahedron must, of course, be that of the tetrahedral complex, and the condition their coördinates g_{ik} must satisfy is at once found, by supposing

$$(6)\qquad g_{23}g_{14} + g_{31}g_{24} + g_{12}g_{34} = 0,$$

to be

$$(9)\qquad \lambda_1 g_{23}g_{14} + \lambda_2 g_{31}g_{24} + \lambda_3 g_{12}g_{34} = 0.$$

This form is the same as that of the proposed equation of the complex of lines, and we can say that this triply infinite set of quartic curves is also a tetrahedral complex.

2. Instead of taking the polar planes of points with respect to two quadric surfaces, we may join the poles of any plane with respect to the same surfaces, and we shall thus obtain the same tetrahedral complex of lines. For, if u_i, v_i be the coördinates of any two planes passing through a given straight line, we may also denote the magnitudes

$$q_{ik} = u_i v_k - u_k v_i \tag{10}$$

as coördinates of the line, and we then have

$$\begin{aligned} q_{23} &= p_{14}, & q_{31} &= p_{24}, & q_{12} &= p_{34}, \\ q_{14} &= p_{23}, & q_{24} &= p_{31}, & q_{34} &= p_{12}. \end{aligned} \tag{11}$$

But the line, which joins the poles, with respect to the surfaces (1), of a plane with coördinates w_i has itself in this sense the following coördinates :

$$q'_{ik} = g_{ik} \frac{w_i}{g_i h_i} \frac{w_k}{g_k h_k}, \tag{12}$$

whence, by virtue of (11), the relation (7) is again derived.

Now let us consider, instead of only two quadrics, all quadrics having the same common curve. Then for any point we obtain a pencil of polar planes with respect to these surfaces, and the pencils thus found for all points of space are in a projective relation. This relation is such that the planes passing through the same vertex of the fundamental tetrahedron are always corresponding planes; and the planes joining the lines of the tetrahedral complex, as axes of these pencils of polar planes, to the vertices of the fundamental tetrahedron have all therefore the same anharmonic ratio. Hence reciprocally : *If we determine the straight lines such that the planes joining them with four given points have the same anharmonic ratio, then these lines form a tetrahedral complex.*

If we take the quadrics inscribed in the same quartic developable, then the poles of any plane with respect to these quadrics lie upon a straight line, and the straight lines thus obtained are in a projective relation. This relation is such, that the points where the straight lines meet the planes of the fundamental tetrahedron are corresponding points of the lines. Hence, as above : *If we determine the straight lines whose points of intersection with four given planes have the same anharmonic ratio, these lines form a tetrahedral complex.*

As a corollary we find that, the fundamental tetrahedron being known, a tetrahedral complex is wholly given by *one* of its lines. With regard to every tetrahedron the straight lines in space are divided, in an entirely determinate manner, into a set of tetrahedral complexes.

If we draw in two faces of the fundamental tetrahedron, through their points of intersection with a line of the tetrahedral complex, the straight lines to those vertices of the tetrahedron which are not common to both planes, then we get, varying the line of the complex, in the two planes two projective pencils of rays, and every line meeting two corresponding rays of these pencils belongs to the tetrahedral complex. Hence it is evident how all lines of the complex may be found, if the fundamental tetrahedron and one line are given.

The analytical proof is very easy. For, the points of intersection of the line whose coördinates are given by (4) with the first and second plane of the fundamental tetrahedron have coördinates x_i and y_{ii} determined thus:

$$x_1 = 0, \quad x_2 : x_3 : x_4 = p_{12} : p_{13} : p_{14} = g_{12}z_2 : g_{13}z_3 : g_{14}z_4;$$
$$y_2 = 0, \quad y_1 : y_3 : y_4 = p_{21} : p_{23} : p_{24} = g_{21}z_1 : g_{23}z_3 : g_{24}z_4.$$

Hence

$$\mu = \frac{x_3}{x_4} = \frac{g_{13}}{g_{14}} \cdot \frac{z_3}{z_4}, \quad \nu = \frac{y_3}{y_4} = \frac{g_{23}}{g_{24}} \cdot \frac{z_3}{z_4},$$
$$\frac{\mu}{\nu} = -\frac{g_{31}\,g_{24}}{g_{23}\,g_{14}} = \mathit{const.},$$

which proves the theorem.

3. From the poles of any planes, with respect to one quadric, we obtain the poles of the same planes, with respect to another quadric, by a collinear transformation. Hence: *If we transform the points of space by a collinear transformation and join every point with its corresponding point, these lines will form a tetrahedral complex, whose fundamental tetrahedron is determined by the self-conjugate points.* Reciprocally: *If we transform the planes of space by a collinear transformation and cut every plane by its corresponding plane, the lines of intersection will form a tetrahedral complex, whose fundamental tetrahedron is determined by the self-conjugate planes.*

For an analytical proof suppose

$$h_1 = h_2 = h_3 = h_4;$$

then [compare (4)] a line of the complex has coördinates of the following form:

(13) $$p_{ik} = (g_i - g_k) z_i z_k$$
or
(13′) $$p_{ik} = y_i z_k - y_k z_i,$$
by making
(14) $$y_i = g_i z_i \qquad (i = 1, 2, 3, 4).$$
But these equations represent a general collinear transformation, whose fundamental tetrahedron is that of the tetrahedral complex.

Instead of the one transformation just obtained, we may employ any transformation of the form

(15) $$y_i = (g_i - \lambda) z_i,$$

where λ is an arbitrary parameter, and arrive at the same tetrahedral complex. All these transformations are said to form a *pencil*. They convert every line of the complex into itself, and we obtain by them successively all points of the line from any one point upon it.

Starting from planes, we get our tetrahedral complex by any transformation of the form

(16) $$v_i = (\gamma_i - \lambda) w_i, \qquad (i = 1, 2, 3, 4),$$

where v_i and w_i are plane coördinates and

(17) $$\gamma_i = \frac{1}{g_i}.$$

In fact, these equations give, for the line of intersection of two corresponding planes, the following coördinates:

(18) $$g_{ik} = v_i w_k - v_k w_i = (\gamma_i - \gamma_k) w_i w_k,$$

and this line belongs to the same tetrahedral complex as that given by (13).

The transformation (16) is expressed, in point coördinates, as follows:

(19) $$y_i = \frac{1}{\gamma_i - \lambda} z_i, \qquad (i = 1, 2, 3, 4).$$

All these transformations, which we obtain by varying λ, are said to constitute a *series*. By them the planes through any line of the complex pass one into another, and are all to be found from any one among them.

4. If, in the expressions

(13) $$p_{ik} = (g_i - g_k) z_i z_k$$

we put

$$(20) \qquad z_i = f_i u_i\,, \quad \text{whence} \quad u_i = \frac{z_i}{f_i},$$

and, regarding the u_i as plane coördinates, write q_{ik} for p_{ik}, we do not change the tetrahedral complex. But

$$(21) \qquad \frac{z_1}{f_1}x_1 + \frac{z_2}{f_2}x_2 + \frac{z_3}{f_3}x_3 + \frac{z_4}{f_4}x_4 = 0$$

represents the polar plane of the point with coördinates z_i with respect to the surface

$$(22) \qquad \frac{x_1^2}{f_1} + \frac{x_2^2}{f_2} + \frac{x_3^2}{f_3} + \frac{x_4^2}{f_4} = 0\,,$$

and this is a general quadric, for which the fundamental tetrahedron is a self-conjugate one. Thus we see, that *a tetrahedral complex remains unaltered, if in place of each of its lines we take the reciprocal polar of this line with respect to any quadric surface having the fundamental tetrahedron for a self-conjugate tetrahedron.*

On the other hand, let

$$g_{ik} = y_i z_k - y_k z_i$$

be the coördinates of any line; then its reciprocal polar for the surface (22) has the coördinates

$$p_{ik} = \frac{g_{ik}}{f_i f_k}\,,$$

hence, whatever be the values of the f_i, we have

$$\frac{p_{23}p_{14}}{g_{23}g_{14}} = \frac{p_{31}p_{24}}{g_{31}g_{24}} = \frac{p_{12}p_{34}}{g_{12}g_{34}}.$$

If we take the reciprocal polars of a straight line for all quadric surfaces having a certain common self-conjugate tetrahedron, these polars will form a tetrahedral complex which contains the given line and whose fundamental tetrahedon is the common self-conjugate tetrahedron.

5. By regarding, in the equations (19), z_i as fixed and λ as variable, we see that every point is transformed by the transformations of the series into the points of a cubic space curve. We must suppose the right members of the equations to be multiplied by the common factor

$$(\gamma_1 - \lambda)\;(\gamma_2 - \lambda)\;(\gamma_3 - \lambda)\;(\gamma_4 - \lambda)\;;$$

then they all are cubic functions of λ and vanish for three of the values γ_1, γ_2, γ_3, γ_4, whence for $\lambda = \gamma_i$ three of the coördinates y are $= 0$, and only y_i is $\neq 0$. Thus the cubic curve is circumscribed about the fundamental tetrahedron. I call this curve a pole curve of the tetrahedral complex, and the point with coördinates z_i its pole.

This curve has the singular property, that *every one of its chords is a line of the tetrahedral complex*, as it is easily shown. For, let λ, λ' be the parameters of the points where the pole curve is met by a chord; then the coördinates of this chord are

$$p_{ik} = \frac{z_i}{\gamma_i - \lambda} \cdot \frac{z_k}{\gamma_k - \lambda'} - \frac{z_k}{\gamma_k - \lambda} \cdot \frac{z_i}{\gamma_i - \lambda'}.$$

Hence, if we put

$$w_i = \frac{z_i}{(\gamma_i - \lambda)\,(\gamma_i - \lambda')},$$

write g_{ik} for p_{ik}, and suppress the common factor $\lambda - \lambda'$, we find

$$g_{ik} = (\gamma_i - \gamma_k) w_i w_k \qquad \text{[see (18)]}.$$

Every cubic space curve is contained upon a doubly infinite set of quadric surfaces. They all are ruled surfaces, and one of the two sets of straight lines upon them is formed by chords of the cubic curve. In our case the surfaces passing through the pole curve are covered by an infinity of complex lines, and may be generated by a line moving in the tetrahedral complex. We shall call them simply quadrics of the complex. Any two pole curves lie upon *one* quadric, and this is always a quadric of the complex.

6. Before inquiring into the general equation of these quadrics, let us find first the equations of the four cones projecting a pole curve from the vertices of the fundamental tetrahedron. This is immediately done if we omit one of the four equations (19), for instance the last, and eliminate λ from the proportion

$$y_1 : y_2 : y_3 = \frac{z_1}{\gamma_1 - \lambda} : \frac{z_2}{\gamma_2 - \lambda} : \frac{z_3}{\gamma_3 - \lambda}$$

or

$$\frac{z_1}{y_1} : \frac{z_2}{y_2} : \frac{z_3}{y_3} = (\gamma_1 - \lambda) : (\gamma_2 - \lambda) : (\gamma_3 - \lambda).$$

Multiplying the several terms of this proportion by $\gamma_2 - \gamma_3$, $\gamma_3 - \gamma_1$, $\gamma_1 - \gamma_2$ and adding, we get

$$(\gamma_2 - \gamma_3)\frac{z_1}{y_1} + (\gamma_3 - \gamma_1)\frac{z_2}{y_2} + (\gamma_1 - \gamma_2)\frac{z_3}{y_3} = 0.$$

Finally we make

$$r_{ik} = \frac{\gamma_i - \gamma_k}{z_i z_k}, \tag{23}$$

and then obtain the required equations of the four cones in question

$$\left\{\begin{aligned} r_{34}y_3y_4 + r_{42}y_4y_2 + r_{23}y_2y_3 &= 0, \\ r_{43}y_4y_3 + r_{14}y_1y_4 + r_{31}y_3y_1 &= 0, \\ r_{24}y_2y_4 + r_{41}y_4y_1 + r_{12}y_1y_2 &= 0, \\ r_{32}y_3y_2 + r_{13}y_1y_3 + r_{21}y_2y_1 &= 0. \end{aligned}\right. \tag{24}$$

The magnitudes r_{ik}, again subject to the condition

$$r_{23}r_{14} + r_{31}r_{24} + r_{12}r_{34} = 0, \tag{25}$$

fix the pole curve among the cubic curves circumscribed about the fundamental tetrahedron, and may be regarded as its coördinates in the system of these curves. Then the equations (23) show that one of these curves is a pole curve of the tetrahedral complex of lines given by the equation

$$\lambda_1 p_{23}p_{14} + \lambda_2 p_{31}p_{24} + \lambda_3 p_{12}p_{34} = 0$$

if between its coördinates the relation exists

$$\lambda_1 r_{23}r_{14} + \lambda_2 r_{31}r_{24} + \lambda_3 r_{12}r_{34} = 0.$$

The pole curves of a tetrahedral complex of lines form again a tetrahedral complex among the cubic space curves circumscribed about the fundamental tetrahedron.

7. If we multiply the equations (24) respectively by η_1, η_2, η_3, η_4 and add them all, we immediately find the following expression for a quadric surface containing the pole curve:

$$\begin{aligned} &(\eta_1 - \eta_4)r_{23}y_2y_3 + (\eta_2 - \eta_3)r_{14}y_1y_4 \\ &+ (\eta_2 - \eta_4)r_{31}y_3y_1 + (\eta_3 - \eta_1)r_{24}y_2y_4 \\ &+ (\eta_3 - \eta_4)r_{12}y_1y_2 + (\eta_1 - \eta_2)r_{34}y_3y_4 = 0. \end{aligned} \tag{26}$$

These surfaces constitute a double infinity, since the equation of one of them is not altered, if we add any arbitrary constant to the η's, or multiply them by a common factor.

If we put the foregoing equation identical with the following

$$a_{23}y_2y_3 + a_{14}y_1y_4 + a_{31}y_3y_1 + a_{24}y_2y_4 + a_{12}y_1y_2 + a_{34}y_3y_4 = 0, \tag{27}$$

supposing always

$$(28) \qquad a_{ik} = a_{ki}$$

and denoting generally the a_{ik} as coördinates of a quadric circumscribed about the fundamental tetrahedron, we find for these magnitudes such expressions as

$$a_{23} = (\eta_1 - \eta_4) r_{23}, \quad a_{14} = (\eta_2 - \eta_3) r_{14}, \quad \text{etc.},$$

or if we substitute the values of r_{ik} from (23), we get

$$(29) \quad a_{23} = \frac{(\eta_1 - \eta_4)(\gamma_2 - \gamma_3)}{z_2 z_3}, \quad a_{14} = \frac{(\eta_2 - \eta_3)(\gamma_1 - \gamma_4)}{z_1 z_4}, \text{ etc.},$$

and hence

$$(30) \qquad \frac{a_{23} a_{14}}{G_1} + \frac{a_{31} a_{24}}{G_2} + \frac{a_{12} a_{34}}{G_3} = 0,$$

where

$$G_1 : G_2 : G_3$$

$$= (\gamma_2 - \gamma_3)(\gamma_1 - \gamma_4) : (\gamma_3 - \gamma_1)(\gamma_2 - \gamma_4) : (\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)$$

(31)

$$= (g_2 - g_3)(g_1 - g_4) : (g_3 - g_1)(g_2 - g_4) : (g_1 - g_2)(g_3 - g_4)$$

so that

$$(32) \qquad G_1 + G_2 + G_3 = 0;$$

and

$$(33) \qquad \frac{p_{23} p_{14}}{G_1} = \frac{p_{31} p_{24}}{G_2} = \frac{p_{12} p_{34}}{G_3}$$

represents the tetrahedral complex of lines. The equation (30), which defines a set of *four* dimensions among the *five*-dimensional multitude of quadrics circumscribed about the fundamental tetrahedron, gives the necessary and sufficient condition that one of these quadrics is a quadric of the tetrahedral complex.

Especially we may consider the cones among these quadrics of the complex. Every point in space is the vertex of one of them, and the cone itself is generated by the complex lines passing through this point. The equation of such a complex cone is at once derived from (26), if we make

$$(\eta_1, \eta_2, \eta_3, \eta_4) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4),$$

and substitute for the r_{ik} their values (23). We then get

$$(34) \qquad \begin{cases} a_{23} = \dfrac{G_1}{z_2 z_3}, & a_{31} = \dfrac{G_2}{z_3 z_1}, & a_{12} = \dfrac{G_3}{z_1 z_2}, \\[2ex] a_{14} = \dfrac{G_1}{z_1 z_4}, & a_{24} = \dfrac{G_2}{z_2 z_4}, & a_{34} = \dfrac{G_3}{z_3 z_4}, \end{cases}$$

whence

$$(35)\qquad \sqrt{a_{23}a_{14}} + \sqrt{a_{31}a_{24}} + \sqrt{a_{12}a_{34}} = 0.$$

Rationalizing, we may write this same relation

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix} = 0,$$

and this is in fact the condition, that the quadric (27) should become a cone. Thus, the complex cone belonging to a point with coördinates z_i has the equation

$$(36)\; G_1\left(\frac{y_2y_3}{z_2z_3} + \frac{y_1y_4}{z_1z_4}\right) + G_2\left(\frac{y_3y_1}{z_3z_1} + \frac{y_2y_4}{z_2z_4}\right) + G_3\left(\frac{y_1y_2}{z_1z_2} + \frac{y_3y_4}{z_3z_4}\right) = 0.$$

8. The foregoing considerations may be repeated in the reciprocal form, without undergoing any other change or restriction, except that in place of the γ_i we must take their reciprocal values g_i. Instead of the pole curves *circumscribed* about the fundamental tetrahedron we then get cubic space curves *inscribed* in it, whose osculating planes intersect in pairs in lines of the complex. The coördinates of the points of these curves are represented by a variable parameter as follows:

$$(37)\qquad x_i = m_i(g_i - \lambda)^3, \qquad (i = 1, 2, 3, 4),$$

where the m_i are certain magnitudes. Indeed, it is at once clear that the points in which this curve intersects any plane of the fundamental tetrahedron coincide. These four planes are, therefore, osculating planes of the curve and their points of contact correspond respectively to the values g_1, g_2, g_3, g_4 of the parameter λ.

There exists not only a system of quadrics *circumscribed* about the fundamental tetrahedron, which are covered by complex lines, but also a system of *inscribed* quadrics enjoying the same property. Any quadric inscribed to the fundamental tetrahedron has an equation of the form

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & x_1 \\ a_{21} & 0 & a_{23} & a_{24} & x_2 \\ a_{31} & a_{32} & 0 & a_{34} & x_3 \\ a_{41} & a_{42} & a_{43} & 0 & x_4 \\ x_1 & x_2 & x_3 & x_4 & 0 \end{vmatrix} = 0,$$

and the same equation (30) expresses the condition that the surface may be generated by lines of the complex.

The lines of the tetrahedral complex which lie in a given plane envelop a conic section of this plane touching the four planes of the fundamental tetrahedron. We call it a a complex curve. Then from (36) we see at once that, if

$$w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 = 0$$

be the equation of the plane, the equation of its complex curve in plane coördinates v_i is

$$(38)\quad G_1\left(\frac{v_2v_3}{w_2w_3}+\frac{v_1v_4}{w_1w_4}\right)+G_2\left(\frac{v_3v_1}{w_3w_1}+\frac{v_2v_4}{w_2w_4}\right)+G_3\left(\frac{v_1v_2}{w_1w_2}+\frac{v_3v_4}{w_3w_4}\right)=0.$$

Hence the four cones projecting the conic from the vertices of the fundamental tetrahedron have, in point coördinates, the following equations :

$$(39)\quad \begin{cases} \qquad\qquad s_{12}\sqrt{x_2} + s_{13}\sqrt{x_3} + s_{14}\sqrt{x_4} = 0, \\ s_{21}\sqrt{x_1} \qquad\qquad + s_{23}\sqrt{x_3} + s_{24}\sqrt{x_4} = 0, \\ s_{31}\sqrt{x_1} + s_{32}\sqrt{x_2} \qquad\qquad + s_{34}\sqrt{x_4} = 0, \\ s_{41}\sqrt{x_1} + s_{42}\sqrt{x_2} + s_{43}\sqrt{x_3} \qquad\qquad = 0, \end{cases}$$

if we make

$$(40)\qquad s_{23} = \sqrt{G_1w_2w_3}, \quad s_{14} = \sqrt{G_1w_1w_4}, \quad \text{etc.}$$

The s_{ik}, bound to the identical relation

$$(41)\qquad s_{23}s_{14} + s_{31}s_{24} + s_{12}s_{34} = 0,$$

are the coördinates of the complex-curve among the totality of conics inscribed to the fundamental tetrahedron. From (40) we derive

$$(42)\qquad \frac{s_{23}s_{14}}{G_1} = \frac{s_{31}s_{24}}{G_2} = \frac{s_{12}s_{34}}{G_3}.$$

The complex-curves of a tetrahedral complex of lines form again a tetrahedral complex.

9. We add, in conclusion, a few remarks on the pole curves of our complex, the coördinates of whose points are represented by a variable parameter, as follows :

$$(19)\qquad y_i = \frac{z_i}{\gamma_i - \lambda} \qquad (i = 1, 2, 3, 4).$$

The plane joining three points of the curve corresponding to parameters λ, λ', λ'' has the equation :

$$
\begin{aligned}
(43)\qquad & \frac{(\gamma_1-\lambda)(\gamma_1-\lambda')(\gamma_1-\lambda'')}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)(\gamma_1-\gamma_4)}\frac{y_1}{z_1} \\
& +\frac{(\gamma_2-\lambda)(\gamma_2-\lambda')(\gamma_2-\lambda'')}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)(\gamma_2-\gamma_4)}\frac{y_2}{z_2} \\
& +\frac{(\gamma_3-\lambda)(\gamma_3-\lambda')(\gamma_3-\lambda'')}{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)(\gamma_3-\gamma_4)}\frac{y_3}{z_3} \\
& +\frac{(\gamma_4-\lambda)(\gamma_4-\lambda')(\gamma_4-\lambda'')}{(\gamma_4-\gamma_1)(\gamma_4-\gamma_2)(\gamma_4-\gamma_3)}\frac{y_4}{z_4}=0.
\end{aligned}
$$

Hence we find the equation of the osculating plane at the point whose parameter is λ

$$
(44)\quad \frac{(\gamma_1-\lambda)^3}{f'(\gamma_1)}\frac{y_1}{z_1}+\frac{(\gamma_2-\lambda)^3}{f'(\gamma_2)}\frac{y_2}{z_2}+\frac{(\gamma_3-\lambda)^3}{f'(\gamma_3)}\frac{y_3}{z_3}+\frac{(\gamma_4-\lambda)^3}{f'(\gamma_4)}\frac{y_4}{z_4}=0.
$$

where $f'(\lambda)$ denotes the derivate function of

$$
f(\lambda)=(\gamma_1-\lambda)(\gamma_2-\lambda)(\gamma_3-\lambda)(\gamma_4-\lambda).
$$

Now the osculating planes of a skew cubic in the three points where a plane meets the curve intersect in a point of this plane, and thus by means of the cubic curve there is associated with every plane a point lying in it, and reciprocally to every point a plane passing through it. This correspondence between the planes and points in space has been called by *Moebius* a null system (Nullsystem);* every point is the null point of its corresponding plane, and every plane the null plane of its corresponding point. Denoting by u_i plane coördinates and by x_i the coördinates of the conjugate point, we have, for any such transformation, a representation of the following form :

$$
(45)\qquad
\begin{aligned}
u_1 &= \phantom{p_{21}x_1+{}} p_{12}x_2+p_{13}x_3+p_{14}x_4, \\
u_2 &= p_{21}x_1 \phantom{{}+p_{32}x_2} + p_{23}x_3+p_{24}x_4, \\
u_3 &= p_{31}x_1+p_{32}x_2 \phantom{{}+p_{43}x_3} + p_{34}x_4, \\
u_4 &= p_{41}x_1+p_{42}x_2+p_{43}x_3 \phantom{{}+p_{34}x_4} ,
\end{aligned}
$$

---

* This name has been chosen, because for the lines in a plane passing through its null point, or for the lines through a point lying in its null plane, the moment of a certain system of forces is zero.

where always

$$p_{ik} + p_{ki} = 0.$$

If we ask what the values of these magnitudes p_{ik} are for the pole curve (19) just considered, we find the following expressions:

$$p_{23} = \frac{(\gamma_1 - \gamma_4)(\gamma_2 - \gamma_3)^2}{z_2 z_3}, \quad p_{14} = \frac{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_4)^2}{z_1 z_4},$$

$$(46) \quad p_{31} = \frac{(\gamma_2 - \gamma_4)(\gamma_3 - \gamma_1)^2}{z_3 z_1}, \quad p_{24} = \frac{(\gamma_3 - \gamma_1)(\gamma_2 - \gamma_4)^2}{z_2 z_4},$$

$$p_{12} = \frac{(\gamma_3 - \gamma_4)(\gamma_1 - \gamma_2)^2}{z_1 z_2}, \quad p_{34} = \frac{(\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)^2}{z_3 z_4},$$

or, if we like,

$$(46') \quad \begin{array}{lll} p_{23} = G_1 r_{23}, & p_{31} = G_2 r_{31}, & p_{12} = G_3 r_{12}, \\ p_{14} = G_1 r_{14}, & p_{24} = G_2 r_{24}, & p_{34} = G_3 r_{34}, \end{array}$$

[compare (23) and (31)].

To verify this result, it suffices to show that the plane corresponding to any point of the pole curve is the osculating plane at this point. Let λ be the parameter of the point and

$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0$$

the equation of the corresponding plane. Then, considering first the coördinate u_4, we find for it

$$\begin{aligned} (\gamma_1 - \lambda)(\gamma_2 - \lambda)(\gamma_3 - \lambda) u_4 z_4 \\ &= (\gamma_2 - \gamma_3)(\gamma_1 - \gamma_4)^2 (\gamma_2 - \lambda)(\gamma_3 - \lambda) \\ &+ (\gamma_3 - \gamma_1)(\gamma_2 - \gamma_4)^2 (\gamma_3 - \lambda)(\gamma_1 - \lambda) \\ &+ (\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)^2 (\gamma_1 - \lambda)(\gamma_2 - \lambda). \end{aligned}$$

But the right side of this equation is easily shown to be

$$(\gamma_2 - \gamma_3)(\gamma_3 - \gamma_1)(\gamma_1 - \gamma_2)(\gamma_4 - \lambda)^2.$$

This equation is to be multiplied by $(\gamma_4 - \lambda)$ on both sides; and if we add the corresponding equations for u_1, u_2, u_3, which are at once obtained by a cyclic permutation, and omit the common factor $(\gamma_1 - \lambda)(\gamma_2 - \lambda)(\gamma_3 - \lambda)(\gamma_4 - \lambda)$ at the left, we finally get

$$u_1 = (\gamma_3 - \gamma_4)\,(\gamma_4 - \gamma_2)\,(\gamma_2 - \gamma_3)\,\frac{(\gamma_1 - \lambda)^3}{z_1},$$

$$u_2 = (\gamma_4 - \gamma_1)\,(\gamma_1 - \gamma_3)\,(\gamma_3 - \gamma_4)\,\frac{(\gamma_2 - \lambda)^3}{z_2},$$

$$u_3 = (\gamma_1 - \gamma_2)\,(\gamma_2 - \gamma_4)\,(\gamma_4 - \gamma_1)\,\frac{(\gamma_3 - \lambda)^3}{z_3},$$

$$u_4 = (\gamma_2 - \gamma_3)\,(\gamma_3 - \gamma_1)\,(\gamma_1 - \gamma_2)\,\frac{(\gamma_4 - \lambda)^3}{z_4},$$

as it should be.

From (46) we still derive the following relation :

$$(47) \qquad \sqrt[3]{p_{23}p_{14}} + \sqrt[3]{p_{31}p_{24}} + \sqrt[3]{p_{12}p_{34}} = 0.$$

This is the necessary condition for the determining magnitudes of a zero system, if it is to be found from a skew cubic *circumscribed* about the fundamental tetrahedron. But we should have found the same equation by starting from a skew cubic *inscribed* in the fundamental tetrahedron. In fact, by applying the polar transformation, which is simply represented as follows :

$$(y_1,\ y_2,\ y_3,\ y_4) = (u_1,\ u_2,\ u_3,\ u_4),$$

we get instead of the circumscribed pole curve an inscribed cubic space curve, which furnishes the same zero system, and hence also the equation (47) remains unaltered.

STRASSBURG I. E.,
March 15, 1900.

ON SINGULAR TRANSFORMATIONS IN REAL PROJECTIVE GROUPS.

BY PROFESSOR H. B. NEWSON.

(Read before the American Mathematical Society, April 28, 1900.)

A SINGULAR transformation in a continuous group has been defined as one that can not be generated from an infinitesimal transformation of the group. There have recently appeared in this country several papers* dealing with the question of singular transformations in continuous groups, and in particular in the subgroups of the projective group. The authors of all these papers use the same method, viz., Lie's theory of continuous groups, and consider the variables and parameters to be complex numbers.

In this paper it is proposed to treat those transformations in real projective groups which can not be generated from the real infinitesimal transformations of these groups. I shall make use of a radically new method of treatment, in fact an independent theory of these groups. In what follows the discussion will be limited to real projective transformations in one and two dimensions. The method admits of ready application to three and higher dimensions.

§ 1. *Real Transformations in one Dimension.*

Every projective transformation in one dimension

$$T: \quad x_1 = \frac{ax+b}{cx+d}, \tag{1}$$

can be reduced to one or other of the normal forms

$$(a) \quad \frac{x_1 - A'}{x_1 - A} = k\frac{x - A'}{x - A}, \quad (b) \quad \frac{1}{x_1 - A} = \frac{1}{x - A} + a. \tag{2}$$

In the first form k is the cross ratio of the two invariant points A, A' and the pair of corresponding points x, x_1. For any transformation of this form the cross ratio k is the same for all pairs of corresponding points. The invariant points A, A' and the cross ratio k are the geometric constants of the transformation.

---

* Rettger, *Amer. Jour. of Math.*, vol. 22, p. 60; *Proc. Amer. Acad.*, vol. 33, p. 493; Williams, *Proc. Amer. Acad.*, vol. 35, p. 97; Taber, BULLETIN, vol. 6, p. 199, and several other papers by the same author. For references to these, see footnote on page 61 of Rettger's paper in *Amer. Jour. of Math.*, vol. 22.

There are two cases under (a) to be considered: (1) When the points A, A' are both real, k is also real; this is the hyperbolic case. (2) When the points A, A' are conjugate imaginary, k is a complex number of the form $e^{i\theta}$, *i. e.*, $|k| = 1$. These cases must be discussed separately. In the parabolic case (b) we have a single real invariant point A and a real constant a. Every real projective transformation of one dimension is either hyperbolic, elliptic, or parabolic.

In the hyperbolic case all transformations which have the same real invariant points A, A' but different values of k form a one parameter group, and k is the variable parameter of the group. Two transformations T and T_1 whose cross ratios are respectively k and k_1, combine to form T_2 whose cross ratio k_2 is equal to the product of k and k_1; thus $k_2 = kk_1$. The identical transformation in the group $hG(A, A')$ is given by $k=1$; and two distinct infinitesimal transformations in the group are given by $k = 1 \pm \delta$. Every transformation of the group for which k is positive and $k > 1$ can be generated by repeating the infinitesimal transformation $k = 1 + \delta$ a sufficient number of times; every transformation for which k is positive and $k < 1$ can be generated from the other infinitesimal transformation $1 - \delta$; the transformations of the group for which k is negative are all singular transformations in the sense that they can not be generated from either infinitesimal transformation of the group.* All transformations of (1) for which the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} < 0$$

are hyperbolic with negative k, and are therefore singular transformations.

In the one parameter elliptic group eG_1 two transformations also combine by multiplying their constants; thus $k_2 = kk_1$, or $e^{i\theta_2} = e^{i\theta}. e^{i\theta_1}$, whence $\theta_2 = \theta + \theta_1$. The identical transformation is given by $e^{i\theta} = 1$, *i. e.*, by $\theta = 0$. Two infinitesimal transformations are given by $k = e^{\pm i\delta\theta}$. Evidently all transformations in the elliptic group eG_1 can be generated from either infinitesimal transformation of the group.

In the parabolic case we have a single invariant point A and a constant a, both of which are real. Two transformations of a one parameter group combine by adding the constants, thus $a'_2 = a + a_1$. The identical transformation is given by $a = 0$, and two infinitesimal transformations have as constants $\pm \delta a$. Every transformation with positive a in the group pG_1 can be generated from the infinitesimal transformations $+ \delta a$, and every transformation with negative a can be generated in like manner from $- \delta a$. Hence there are no singular transformations in the real parabolic group pG_1.

---

* The pair of inverse transformations for which $k = 0$ and ∞ respectively need not be excluded from the group, but their exceptional character as limiting cases should be noted.

THEOREM 1. *In one dimension all real hyperbolic transformations with positive k, all real elliptic and parabolic transformations are non-singular; all hyperbolic transformations with negative k are singular transformations.*

The one parameter hyperbolic group is composed of three subdivisions as follows: all transformations with $1 < k < \infty$ form subdivision I.; all with $0 < k < 1$ form subdivision II.; all with negative k form subdivision III. A one parameter parabolic group is composed of two subdivisions; all transformations with positive a form subdivision I., and all with negative a form subdivision II. The one parameter elliptic group shows no such subdivisions.

One of the foundation stones of Lie's theory of continuous groups is that every one parameter continuous group contains one infinitesimal transformation, and that all finite transformations of the group can be generated from the infinitesimal transformation of the group. We have seen that the real one parameter groups hG_1, eG_1, and pG_1 each contain two distinct infinitesimal transformations, one positive and the other negative. There is no vital contradiction here, for in Lie's theory no distinction is made between positive and negative infinitesimal values of the variable parameter of a group: in the most general form of his theory the parameter is a complex variable, so that positive and negative distinctions are out of place. However, for real groups the distinction between the two infinitesimal transformations of a one parameter group requires recognition.

We have seen that all elliptic and parabolic one parameter groups can be generated from infinitesimal transformations belonging to those groups, but all the transformations of a hyperbolic group can not be so generated. Yet the hyperbolic one parameter group hG_1 is clearly continuous. It follows at once without further elaboration that the continuity of a real one parameter projective group is not coextensive with its generation from an infinitesimal transformation of the group.

§ 2. *Real Projective Transformations in the Plane.*

We come now to the consideration of real projective transformations or collineations in the plane. Every projective transformation of the plane belongs to one of the five types enumerated by Lie in his "Continuierliche Gruppen," page 66. The normal forms of the equations of these five types were given by the writer in the *Kansas University Quarterly*, Series A, vol. 7, pages 43–66. Making use of these normal forms we can determine the structure and properties of the continuous groups of projective transformations in the plane in a manner similar to that used above for one dimension.

Type I. What I have called the implicit normal form for a transformation T of type I is given by the equations

$$\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} = k \frac{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}}, \quad \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} = k' \frac{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}}. \quad (3)$$

Here A, B; A', B'; and A'', B'' are the coördinates of the three vertices of the invariant triangle of T, and k and k' are two independent cross ratios; k is the cross ratio of the one dimensional transformation along the invariant line joining (A, B) and (A', B'); k' is the cross ratio of the one dimensional transformation along the line joining (A, B) and (A'', B'').

There are two cases to be considered. The invariant triangle is either real in all its parts or has one real and two conjugate imaginary vertices. These are respectively the hyperbolic and elliptic cases and must be treated separately. In the hyperbolic case k and k' are both real quantities, while in the elliptic case they are both complex quantities.

Let $T(k, k')$ and $T(k_1, k_1')$ be two hyberbolic transformations having the same invariant triangle. It may readily be verified from equation (3) that their resultant is $T(k_2, k_2')$ where $k_2 = kk_1$ and $k_2' = k'k_1'$. All transformations having the same invariant triangle form a group of two parameters, k and k' being the independent parameters of the group. k and k' may assume in turn all possible real values; thus our group $hG_2(ABC)$ is *continuous*.

The two parameter group hG_2 contains an infinite number of one parameter subgroups, and we proceed to determine these. All transformations in hG_2 for which the two parameters satisfy a relation of the form $k' = k^{1-r}$, where r is a constant, form a one parameter subgroup ; and conversely, in all one parameter subgroups, k and k' satisfy a relation of this form. There are different subgroups for different values of r. Geometrically,* r is interpreted as the constant cross ratio of certain four points on the tangent to a path curve of $hG_1(r)$, viz., the point of tangency T and the points of intersection of the tangent with the sides of the invariant triangle. These four points are all real for a real hyperbolic group and hence r is also real.

In order to study the distribution of the ∞^2 transformations of hG_2 into one parameter subgroups we resort to a geometrical device as follows : Let k and k' be the rectangular coördinates of a point in a plane (not to be confused with the plane of our transformation). It is evident, since k and k' are independent parameters, that there is a point in the plane corresponding to every transformation of the group $hG_2(ABC)$. Since all transformations, whose parameters k and k' satisfy the relation $k' = k^{1-r}$, form a one parameter subgroup of hG_2, we see that the curve $y = x^{1-r}$ represents this subgroup and the individual points of the curve represent the individual transformations of the group. If we give to r all real values, we have a family of curves which represent the system of subgroups of $hG_2(ABC)$.

From the properties of this system of curves we deduce the following results : If r is an irrational number, the curve $y = x^{1-r}$ contains no real point for which either coördinate is negative ; the curve lies entirely in the first quadrant. If r is a rational fraction with odd numerator and odd denominator, y cannot be negative, and the curve lies above the axis of x in the first and second quadrants. If r is rational with odd numerator and even denominator, the curve lies in the first and fourth quadrants. If r is rational with even numerator and odd denominator, the curve lies in the first and third quadrants.

Every curve passes through the point (1, 1), which shows that the identical transformation belongs to every subgroup. The curves of our family contain every point in the first quadrant, but not every point in the second, third, and fourth quadrants. Consequently our two parameter group $hG_2(ABC)$ contains transformations *which do not belong to any of its subgroups.* Such a transformation has one or both of its cross ratio constants negative, and their values are such that they do not satisfy an algebraic equation of the form $k'^m = k^n$, where m and n are integers.

---

* Lie, "Contin. Gruppen," p. 78.

The variable parameter of every one parameter group in $hG_2\,(ABC)$ is k; and every one parameter group contains two real infinitesimal transformations, viz., when $k = 1 \pm \delta$. Each infinitesimal transformation generates its corresponding portion of the group. Every transformation in the group $hG_2(ABC)$ for which both k and k' are positive can be generated from an infinitesimal transformation of the group, while no transformation for which either k or k' is negative can be so generated. Thus the group hG_2 contains three singular transformations for every non-singular * one.

The failure of Lie's theory to express fully the relations here set forth for the real group $hG_2(ABC)$ is due to the fact that he assumes k and k' to be of the form e^t and e^{at}, thus silently excluding from consideration negative values of k and k'.

We turn now to the consideration of the real elliptic group $eG_2(ABC)$ in which the invariant triangle has one real vertex A and two conjugate imaginary vertices B and C. The invariant line BC is a real line, and the one dimensional transformation along that line is elliptic, for the two invariant points on it are conjugate imaginary. The quantities k and k' in equation (3) are both complex numbers; but the necessary and sufficient condition that the transformation T shall be real is that k and k' shall be conjugate imaginary numbers. Thus the real elliptic group $eG_2(ABC)$, instead of having two independent variable parameters k and k', has only one, viz, k; but this is a complex number and may assume in turn all possible complex values. Consequently the group $eG_2(ABC)$ contains a transformation corresponding to every point in the complex plane.

The question of the distribution of the transformations of $eG_2(ABC)$ into one parameter subgroups and their generation from infinitesimal transformations of the group need not detain us long; for this group is holoedrically isomorphic with the group $G_2(mn)$ of circular transformations of the complex plane. This latter group was fully discussed by the writer in the BULLETIN, Vol. 4, pp. 112–115. We learn, therefore, that every transformation of $eG_2(ABC)$ belongs to at least two distinct subgroups and may be gen-

---

* The inappropriateness of the term singular transformation is hereby shown. The term would be more appropriate if applied to those transformations in $hG_2(ABC)$ which do not belong to one parameter subgroups.

erated from at least two infinitesimal transformations of the group. We sum up the discussion of type I. as follows:

THEOREM 2. *Every real elliptic projective transformation of the plane belongs to some one parameter group and may be generated from the infinitesimal transformation of the group; the same is true for every hyperbolic transformation for which k and k' are both positive. All other hyperbolic transformations are singular; and many of these singular transformations do not belong to one parameter groups.*

Type II.—The implicit normal form of a transformation of type II. is as follows:

$$\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ c & s & 0 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}} = k \frac{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ c & s & 0 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}, \qquad \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ c & s & 0 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}} = \frac{\begin{vmatrix} x & y & 1 \\ A' & B' & 1 \\ c & s & 0 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}} + \alpha.$$

$$(c^2 + s^2 = 1.)$$

The invariant figure consists of two lines, their point of intersection, and a second point on one of these lines. We designate a transformation of this type by T', and its invariant figure by (ABl). Along the invariant line AB there is a one dimensional hyperbolic transformation whose cross ratio is k; while along the invariant line Al is a one dimensional parabolic transformation whose constant is α. For a real transformation k and α are both real quantities. Two transformations $T'(k, \alpha)$ and $T_1'(k_1, \alpha_1)$ having the same invariant figure combine to form $T_2'(k_2, \alpha_2)$ where $k_2 = kk_1$ and $\alpha_2 = \alpha + \alpha_1$; T_2' has the same invariant figure as T' and T_1'. Thus all transformations of type II. leaving (ABl) invariant form a two parameter group $G_2'(ABl)$, the two parameters being k and α.

The group $G_2'(ABl)$ contains ∞^1 one parameter subgroups; to show these subgroups we assume a relation between these quantities k and α, and consider only those transformations of the group G_2' for which k and α satisfy the assumed relation. Let $k = a^{\alpha}$, where a is a constant; all transformations in G_2' satisfying this relation form a one parameter subgroup of G_2', α being the parameter.

Within the group G_2' there are ∞^1 such one parameter subgroups, one for each positive value of a.

In order to study the distribution of the transformations in G_2' into subgroups and their generation from infinitesimal transformations, we resort to the same device as in type I. and make k and a the rectangular coördinates of a point in a plane. The family of curves $y = a^x$ represents the system of one parameter subgroups of G_2'. For positive values of a these curves lie in the first and second quadrants and completely fill the upper half of the plane. There are no continuous curves for negative values of a, and hence continuous subgroups of G_2' exist only for positive values of a.

Two particular curves of the family deserve special attention. For a very large value of a, the curve $y = a^x$ differs but little from the axis $x = 0$; hence in the limit when $a = \infty$ the line $x = 0$ is a curve of the family; on the other hand when $a = 1$ the curve reduces to the line $y = 1$. In the first case $x = 0$ is the only curve of the family that penetrates into the lower half of the plane, and consequently the corresponding group is the only continuous subgroup of G_2' containing transformations with negative values of k. The transformations of the group corresponding to $a = \infty$ are in reality not of type II. but of type IV.; they are perspective transformations, the line Al being the axis and the point B the vertex of the transformations of the group. The transformations of the group corresponding to $a = 1$ are of type V.; they are elations, the line AB being the axis and A the vertex of the transformations of the group.

Each one parameter subgroup of G_2' contains two infinitesmal transformations, one positive and the other negative. Every transformation in a subgroup of G_2' may be generated from one of its infinitesimal transformations, except the transformations with negative k in the perspective subgroup $a = \infty$. The transformations properly of type II. in G_2' for which k is negative *do not belong to continuous subgroups of* G_2' and cannot be generated from infinitesimal transformations of the group.

THEOREM 3. *All real projective transformations of type II. for which k is negative are singular transformations and do not belong to one parameter groups.*

Types III., IV., V.—The remaining types of projective transformations in the plane viz., types III., IV., V. may be passed over rapidly. The invariant figure of a transformation of type III. is a lineal element Al. There are ∞^3

transformations of type III. leaving invariant the same lineal element and these form a three parameter group G_3'' [Lie, (13)].* This three parameter group breaks up into ∞^1 two parameter subgroups G_2'' [Lie, (24)], and into ∞^2 one parameter subgroups G_1'' [Lie, (37)]. Every transformation in G_3'' belongs to a one parameter subgroup and may be generated from a real infinitesimal transformation of the group. Hence there are no singular transformations of this type. I give the facts only without the proof.

Every real projective transformation of type IV. leaves invariant a real point and a real line, the vertex and axis respectively of the transformation. Every line through the vertex is an invariant line and has on it two real invariant points. Along each invariant line is a one dimensional hyperbolic transformation, and all are characterized by the same cross ratio k. The latter may assume in turn all real values and we thus have a one parameter group of perspective transformations. The structure of this group is identical with that of the hyperbolic group in one dimension. Hence all transformations of this group for which k is negative are singular perspective transformations.

Along each of the invariant lines of a projective transformation of type V. is a one dimensional parabolic transformation with a real constant a. a may assume in turn all real values and thus we have a one parameter group of type V. which is identical in structure with a one dimensional parabolic group. Hence there are no singular tranformations of type V.

THEOREM 4. *All real projective transformations of type IV. for which k is negative are singular transformations.*

It is unnecessary to discuss in detail all groups of real projective transformations in the plane. The groups containing singular transformations discussed above enter as subgroups into many of the groups of higher orders; and consequently these latter groups also contain such singular transformations. Any one who is acquainted with the structure of the projective groups of the plane can readily make the application for himself.

KANSAS STATE UNIVERSITY,
April, 1900.

---

* These numbers refer to Lie's list of perspective groups given on pages 288–291 of "Continuierliche Gruppen."

ON GROUPS OF ORDER 8!/2.

BY IDA MAY SCHOTTENFELS.

(Read before the American Mathematical Society, June 29, 1900.)

§ 1. In the BULLETIN, vol. IV. (1898), pp. 495–510, Dr. L. E. Dickson discusses the structure of the hypoabelian groups. Among the simple groups of the system J, occurs one of order $8!/2$ ($p^n = 2^1$, $m = 3$); this 2.3 or senary linear group is defined as the totality of linear substitutions on 2.3 indices, as follows:

$$(1)\qquad \begin{aligned} \xi_i' &= \sum_{j=1}^{3} (\alpha_j^{(i)} \xi_j + \gamma_j^{(i)} \eta_j), \\ \eta_i' &= \sum_{j=1}^{3} (\beta_j^{(i)} \xi_j + \delta_j^{(i)} \eta_j), \end{aligned} \qquad (i = 1, 2, 3),$$

satisfying the relations

$$(2)\qquad \begin{aligned} &\sum_{i=1}^{3} \begin{vmatrix} \alpha_j^{(i)} & \gamma_j^{(i)} \\ \beta_j^{(i)} & \delta_j^{(i)} \end{vmatrix} = 1, \qquad \sum_{i=1}^{3} \begin{vmatrix} \alpha_j^{(i)} & \gamma_k^{(i)} \\ \beta_j^{(i)} & \delta_k^{(i)} \end{vmatrix} = 0, \\ &\sum_{i=1}^{3} \begin{vmatrix} \alpha_j^{(i)} & \alpha_k^{(i)} \\ \beta_j^{(i)} & \beta_k^{(i)} \end{vmatrix} = 0, \qquad \sum_{i=1}^{3} \begin{vmatrix} \delta_j^{(i)} & \delta_k^{(i)} \\ \delta_j^{(i)} & \delta_k^{(i)} \end{vmatrix} = 0, \end{aligned}$$

$$(j \neq k\,;\ j,\ k = 1,\ 2,\ 3)\,;$$

$$(3)\qquad \sum_{j=1}^{3} \beta_j^{(i)} \delta_j^{(i)} = 0, \quad \sum_{j=1}^{3} \alpha_j^{(i)} \gamma_j^{(i)} = 0, \quad \sum_{i,j=1}^{3} \alpha_j^{(i)} \delta_j^{(i)} = m,$$

$$(i = 1,\ 2,\ 3\,;\ m = 1,\ 2,\ 3).$$

The present paper determines that the above group is abstractly the alternating group $G^{8}_{8!/2}$, *and thus establishes a new proof of its simplicity.**

Writing the substitutions (1) in square array, and considering the elements of the group as the matrices of these coefficients, we have

---

*Dr. L. E. Dickson in the *Proc. of the Lond. Math. Soc.*, vol. 30, "The structure of certain linear groups with quadratic invariants," pp. 81 et seq., has proved the correspondence of these groups.

$$\begin{pmatrix} \alpha_1^{(1)} & \gamma_1^{(1)} & \alpha_2^{(1)} & \gamma_2^{(1)} & \alpha_3^{(1)} & \gamma_3^{(1)} \\ \beta_1^{(1)} & \delta_1^{(1)} & \beta_2^{(1)} & \delta_2^{(1)} & \beta_3^{(1)} & \delta_3^{(1)} \\ \alpha_1^{(2)} & \gamma_1^{(2)} & \alpha_2^{(2)} & \gamma_2^{(2)} & \alpha_3^{(2)} & \gamma_3^{(2)} \\ \beta_1^{(2)} & \delta_1^{(2)} & \beta_2^{(2)} & \delta_2^{(2)} & \beta_3^{(2)} & \delta_3^{(2)} \\ \alpha_1^{(3)} & \gamma_1^{(3)} & \alpha_2^{(3)} & \gamma_2^{(3)} & \alpha_3^{(3)} & \gamma_3^{(3)} \\ \beta_1^{(3)} & \delta_1^{(3)} & \beta_2^{(3)} & \delta_2^{(3)} & \beta_3^{(3)} & \delta_3^{(3)} \end{pmatrix} \quad \text{with the identity matrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\alpha_j^{(i)}, \beta_j^{(i)}, \gamma_j^{(i)}, \delta_j^{(i)}$ are the marks 0 and 1 of the $G.F. [2^1]$.

These matrices compound according to the general law of composition of matrices

$$A = (a_{ij}), \quad B = (b_{jk}), \quad C = (c_{ik}), \quad AB = C, \quad \sum_{j=1}^{6} a_{ij} b_{jk} = c_{ik}.$$

§ 2. Access to the above described group is obtained through the Dickson generators $M_i M_j$, $N_{ij\lambda}$ ($\lambda = 1$; $i, j = 1, 2, 3$),* of the table p. 442, and elements $E_1', E_2', \cdots, E_6'$ are determined having the following properties:

$$\Sigma(E_1', \cdots, E_6') \; \{E_1'^3 = E_{i+1}'^2 = (E_i' E_{i+1}')^3 = (E_i' E_j')^2 = I\}$$

$$(i, i+1, j = 1, 2, \cdots, 6 ; \; i+1 < j),$$

where I is the identity element.

§ 3. Making use of Theorems B and C of Professor Moore's paper "Concerning the abstract groups of order $k!$ and $\frac{1}{2}k!$ holoedrically isomorphic with the symmetric and alternating groups on k letters," *Proceedings of the London Mathematical Society*, Dec. 10, 1896, vol. 28, pp. 358–359, the definite conclusions are arrived at that the generators $E_1', \cdots, E_6'$ satisfying the relations $\Sigma(E_1', \cdots, E_6')$, with order greater than the limit prescribed in the above theorem C, and therefore, by theorem C, satisfying no further relations, generate a group $G(E_1', \cdots, E_6')$, and this group $G(E_1', \cdots, E_6')$ is holoedrically isomorphic to the alternating group of degree eight $G^8_{8!/2}$, and is therefore simple; but the original senary group, as Dr. Dickson † has proved, is of order 8!/2 ; it is therefore identical with its, as we may say, alternating subgroup $G(E_1', \cdots, E_6')$, *and is therefore simple.*

§ 4. The elements $M_i M_j$, P_{ij} ‡ ($i, j = 1, 2, 3$), taken in pairs, thus $M_1 M_2$, P_{12}; $M_1 M_3$, P_{13}; $M_2 M_3$, P_{23}, play a most important rôle; the discovery of their simple combinational properties forms the basis of this paper, namely the determination of S, an element of period seven,

---

* BULLETIN, loc. cit., p. 496, § 8; $N_{ij\lambda} (\lambda = 0)$ is the identity element in the $G.F. [2^n]$.

† loc. cit., § 5.

‡ BULLETIN, loc. cit., § 3 ; $M_i M_k$ transformed by $N_{ij1} = Q_{ij1}$, Q_{ij1} transformed by $Q_{ji1} = P_{ij}$.

$$S = (M_1M_2 . N_{121} . N_{131} . P_{12} . P_{23});$$

the element T of period three

$$T = (N_{121} . M_1M_2)^2;$$

and the elements W and R defined in the next paragraph.

§5. The elements $E_1', \cdots, E_6'$ were derived from suitable combinations of the two elements, W of period four and R of period fifteen,

$$W = (TS^4 . P_{13}), \quad R = (TS^4 . M_1M_3);$$

in fact $E_1', \cdots, E_6'$ are identically the functions of W and R that $e_1, \cdots, e_6$ are of c_0 and c_1 in Prof. Moore's article: "Concerning the general equations of the seventh and eighth degrees," *Mathematische Annalen*, vol. 51, p. 437, in which the identity is established of the alternating group of degree eight $G^8_{8!/2}$, and a certain quaternary group. R and W enjoy the following properties: *

$$\Sigma(R, W); \{W^4 = R^{15} = (WR^5)^6 = (WR^{10})^4 = (WR^9)^6 = (WR^3)^4 = (WR^6)^4 = \mathrm{I}\}.$$

§ 6. *Table exhibiting the matrices corresponding to the notation* M_iM_j, N_{ij}, etc.†

$$\begin{array}{cccc}
M_1M_2 & M_1M_3 & N_{121} & N_{131} \\
\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} &
\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix} &
\begin{pmatrix} 1&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&1&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} &
\begin{pmatrix} 1&0&0&0&0&1\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&1&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}
\end{array}$$

$$\begin{array}{ccc}
P_{12} & P_{13} & P_{23} \\
\begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix} &
\begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix} &
\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}
\end{array}$$

---

* The relations $\Sigma(R, W)$ are the same as the relations $\Sigma(C_0, C_1)$ of Prof. Moore's "Congruence group," loc. cit., p. 436, marked (*).

† The Dickson generators $M_2M_3 = M_1M_2 \cdot M_1M_3$ and N_{231} are not used in the present paper.

$$\begin{array}{cccc} S & T & R & W \\ \begin{array}{c}\begin{pmatrix}1&0&0&1&0&0\end{pmatrix}\\ \begin{vmatrix}0&0&1&0&0&0\\1&0&0&0&1&0\\0&0&0&0&0&1\\0&1&1&0&0&1\\1&0&0&0&0&0\end{vmatrix}\end{array} & \begin{array}{c}\begin{pmatrix}1&0&0&1&0&0\end{pmatrix}\\ \begin{vmatrix}0&0&1&0&0&0\\0&1&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{vmatrix}\end{array} & \begin{array}{c}\begin{pmatrix}0&1&1&1&1&1\end{pmatrix}\\ \begin{vmatrix}1&0&0&1&0&1\\0&1&0&1&1&0\\0&0&0&1&0&1\\1&0&0&0&0&1\\0&0&1&1&1&1\end{vmatrix}\end{array} & \begin{array}{c}\begin{pmatrix}0&0&1&1&1&1\end{pmatrix}\\ \begin{vmatrix}1&0&0&0&0&1\\0&1&0&1&1&0\\0&0&0&1&0&1\\1&0&0&1&0&1\\0&1&1&1&1&1\end{vmatrix}\end{array} \end{array}$$

$$\begin{array}{cccc} E_1' & E_2' & E_3' & E_4' \\ \begin{array}{c}\begin{pmatrix}1&0&0&0&0&0\end{pmatrix}\\ \begin{vmatrix}1&1&1&1&0&1\\1&0&0&1&0&1\\1&0&1&0&0&1\\0&0&0&0&0&1\\1&0&1&1&1&1\end{vmatrix}\end{array} & \begin{array}{c}\begin{pmatrix}0&1&0&1&0&1\end{pmatrix}\\ \begin{vmatrix}0&0&1&0&0&0\\0&1&0&0&0&0\\1&0&1&0&0&1\\0&1&1&0&1&0\\0&0&0&0&0&1\end{vmatrix}\end{array} & \begin{array}{c}\begin{pmatrix}1&0&0&0&0&0\end{pmatrix}\\ \begin{vmatrix}1&1&1&1&0&1\\1&0&0&0&0&0\\1&0&1&0&0&1\\1&0&1&0&1&0\\0&0&0&1&0&0\end{vmatrix}\end{array} & \begin{array}{c}\begin{pmatrix}0&1&1&1&1&1\end{pmatrix}\\ \begin{vmatrix}0&0&0&0&1&0\\0&1&1&0&1&0\\0&1&0&1&1&0\\0&1&0&0&0&0\\1&1&1&1&1&0\end{vmatrix}\end{array} \end{array}$$

$$\begin{array}{cc} E_5' & E_6' \\ \begin{array}{c}\begin{pmatrix}0&1&0&1&0&0\end{pmatrix}\\ \begin{vmatrix}0&0&1&0&0&1\\0&1&0&0&0&1\\1&0&1&0&0&1\\1&0&1&1&1&1\\0&0&0&0&0&1\end{vmatrix}\end{array} & \begin{array}{c}\begin{pmatrix}1&0&0&0&0&0\end{pmatrix}\\ \begin{vmatrix}0&1&1&0&1&0\\0&0&0&0&1&0\\1&0&1&0&0&1\\0&0&1&0&0&0\\1&0&0&1&1&0\end{vmatrix}\end{array} \end{array}$$

CHICAGO,
August, 1899.

LOBACHEVSKY'S GEOMETRY.

(SECOND PAPER.)

LOBACHEVSKY'S earliest published work on geometry, translated by Engel under the title "Ueber die Anfangsgründe der Geometrie," * contains the elements of a system of analytic geometry under the hypothesis of a variable angle of parallelism, together with numerous applications to the determination of the lengths of arcs, areas, and volumes. Some of this matter appears also in Lobachevsky's article on "Géométrie imaginaire" (Crelle, vol. 17) and more in his "Pangéométrie" (Kasan, 1856); but it is probably safe to say that the knowledge of this part of his work is not so general as that of the more elementary side of his theory, partly because of the difficulties involved in reading the last mentioned articles, and partly because of the fact that the widely known "Geometrische Untersuchungen" does not contain this analytic treatment. It is not the least of Professor Engel's services in publishing his volume on Lobachevsky that he accompanies this part of the work with lucid notes, more copious than the text itself, which smooth away many difficulties in the path of the reader. It is the purpose of the present article to present a few of the chief points in this treatment, and in particular to emphasize the analytic connection pointed out by Engel between Lobachevsky's equations and those of the projective measurement of Cayley.

---

* See the BULLETIN for May, 1900, p. 339.

Lobachevsky's angle of parallelism, $\Pi(x)$, where x is the perpendicular distance from a point on one parallel to the other, is analytically defined by the equations

$$\tan \tfrac{1}{2}\Pi(x) = e^{-x}, \qquad (1)$$

$$\Pi(0) = \frac{\pi}{2}, \quad \Pi(+\infty) = 0, \quad \Pi(-x) = \pi - \Pi(x).$$

From these the functions $\sin \Pi(x)$, $\cos \Pi(x)$, $\tan \Pi(x)$ are easily written in terms of the exponential or the hyperbolic functions; but we will retain Lobachevsky's notation, the strangeness of which disappears very quickly in practice.* Angles and distances are measured as usual by the repeated application of a unit of measure to the thing measured, but the two units of angle and of length can not be assumed independently of each other. If we adopt the usual unit of angle such that the entire angular magnitude about a point in a plane is equal to 2π, then the unit of length is fully defined by the above equations, if the angle of parallelism for any value of x is assumed or experimentally determined. In the space in which we live, we can only know that the unit of length is extremely great compared with the greatest length which enters into experience.

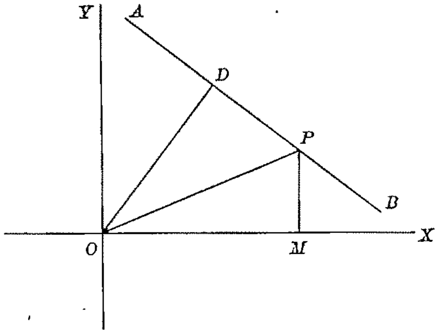
In a right triangle with sides a, b, c, and with angles A, B, $\frac{\pi}{2}$, the following relations hold:

$$\begin{array}{llr}
\sin \Pi(c) = \sin \Pi(a) \sin \Pi(b), & \sin \Pi(c) = \tan A \tan B, & \\
\tan \Pi(c) = \tan \Pi(a) \sin A, & \tan \Pi(c) = \tan \Pi(b) \sin B, & \\
\cos \Pi(a) = \cos \Pi(c) \cos B, & \cos \Pi(b) = \cos \Pi(c) \cos A, & (2) \\
\sin A = \sin \Pi(b) \cos B, & \sin B = \sin \Pi(a) \cos A, & \\
\tan A = \cos \Pi(a) \tan (b), & \tan B = \cos \Pi(b) \tan \Pi(a). &
\end{array}$$

---

*The interesting little book by J. Frischauf, "Absolute Geometrie nach Johann Bolyai," Leipzig, 1872, contains some of the equations of this article in the exponential notation.

Let OX and OY be two coördinate axis at right angles to each other. From any point P of the plane drop a perpendicular to OX meeting it at M. Then the distances OM (x) and $MP(y)$ are the coördinates of P, the usual conventions as to signs being maintained. It is essential that the coördinates be always drawn as just stated; a perpendicular from P to OY will not be equal to x, for no rectangle exists on Lobachevsky's plane. If OP, the distance of the point from the origin, be denoted by r, and the angle XOP be denoted by ϑ, equations (2) give



$$\begin{aligned} \sin \Pi(r) &= \sin \Pi(x) \sin \Pi(y), \\ \sin \vartheta &= \frac{\tan \Pi(r)}{\tan \Pi(y)}, \\ \cos \vartheta &= \frac{\cos \Pi(x)}{\cos \Pi(r)}. \end{aligned} \tag{3}$$

If r is constant, while x and y vary, the first of these equations represents a circle with the center at the origin.

Let AB be any straight line, p the length of the perpendicular OD from the origin, α the angle XOD. Then, in the right triangle ODP,

$$\cos(\alpha - \theta) \cos \Pi(r) = \cos \Pi(p),$$

whence, by (3),

$$\cos \alpha \cos \Pi(x) + \sin \alpha \sin \Pi(x) \cos \Pi(y) = \cos \Pi(p). \tag{4}$$

This equation, in which p may have any value from 0 to $+\infty$, and α any value from 0 to 2π, is Engel's form of the equation of the straight line. We may obtain the form given by Lobachevsky by assuming that AB cuts OY at a point $(0, l)$ under an angle A. Then in the right triangle ODA,

$$\cos \Pi(p) = \cos \Pi(l) \cos\left(\frac{\pi}{2} - a\right),$$

$$\tan\left(\frac{\pi}{2} - a\right) = \sin \Pi(l) \cot A,$$

whence by substitution in (4),

$$\cos \Pi(y) = \frac{\cos \Pi(l)}{\sin \Pi(x)} - \sin \Pi(l) \cot A \cot \Pi(x). \qquad (5)$$

This is called by Lobachevsky the general equation of the straight line. It is so in the sense that the equation of any straight line may be given this form by a proper choice of axes. If the axes are fixed, however, the equation fails for lines parallel to OY or divergent from it. The case of parallelism may be included by taking the limiting form of (5) as l increases indefinitely, and the case of divergence may be met by introducing the conceptions of imaginary lengths or angles, which we can only do after we have an analytic expression for each. Engel's equation has no such defect, and we will base our work upon it.

It is worth noticing that $x = c$ represents a straight line, while $y = c$ does not; the former being obtained from (4) by placing $a = 0$, $p = c$, while the latter is not contained in (4). Both of these facts are geometrically evident.

The derivation of expressions for distance and angle is facilitated by first studying the distance DP and the angle OPD of the figure. If we place DP equal to d, we have, by (2) and (3),

$$\sin \Pi(d) = \frac{\sin \Pi(r)}{\sin \Pi(p)} = \frac{\sin \Pi(x) \sin \Pi(y)}{\sin \Pi(p)},$$

whence

$$\cos^2 \Pi(d) = \frac{\sin^2 \Pi(p) - \sin^2 \Pi(x) \sin^2 \Pi(y)}{\sin^2 \Pi(p)}.$$

The latter equation gives, by use of (4),

$$\cos \Pi(d) = \pm \frac{\sin a \cos \Pi(x) - \cos a \sin \Pi(x) \cos \Pi(y)}{\sin \Pi(p)},$$

where the sign is determined by assuming a positive direction on DP.

Similarly, if the angle OPD is called β,

$$\sin \beta = \frac{\sin \Pi(x) \sin \Pi(y) \cos \Pi(p)}{\cos \Pi(r) \sin \Pi(p)},$$

$$\cos \beta = \pm \frac{\sin a \cos \Pi(x) - \cos a \sin \Pi(x) \cos \Pi(y)}{\cos \Pi(r) \sin \Pi(p)},$$

the sign of $\cos \beta$ corresponding to that of $\cos \Pi(d)$.

Passing to the general case, suppose two lines (a_1, p_1) and (a_2, p_2) intersect at P. The angle between them is the algebraic difference of the angles β_1 and β_2 made by each with OP. Placing $\varphi = \beta_1 - \beta_2$, we find

$$\cos \varphi = \frac{\cos (a_1 - a_2) - \cos \Pi(p_1) \cos \Pi(p_2)}{\sin \Pi(p_1) \sin \Pi(p_2)}, \tag{6}$$

the simplification being effected by means of (3) and (4). This equation affords a simple analytic criterion for the relative position of two lines, they being intersecting, parallel, or divergent according as $\cos \varphi$ is numerically less than, equal to, or greater than unity.

In a similar manner, the distance between two points (x_1, y_1) and (x_2, y_2) is the algebraic difference of the two distances d_1 and d_2 measured as above along the line joining the two points. If we call $d_1 - d_2 = D$, and make use of Lobachevsky's formula,

$$\sin \Pi (d_1 - d_2) = \frac{\sin \Pi(d_1) \sin \Pi(d_2)}{1 - \cos \Pi(d_1) \cos \Pi(d_2)},$$

we find

$$\sin \Pi(D) = \frac{\sin \Pi(x_1) \sin \Pi(x_2) \sin \Pi(y_1) \sin \Pi(y_2)}{1 - \cos \Pi(x_1) \cos \Pi(x_2) - \sin \Pi(x_1) \sin \Pi(x_2) \cos \Pi(y_1) \cos \Pi(y_2)}. \tag{7}$$

Neither (6) nor (7) is given by Lobachevsky, but his equation for the length of a line measured from its intercept on OY and expressed in the parameters of equation (5) may be derived from (7).

Engel calls attention to the fact that equation (4) is linear in two functions of x and y, and that a simple substitution leads to a system of projective measurement as developed by Cayley. Following this line of thought, let us place, with Engel,

$$\begin{aligned}\xi &= \cos \Pi(x),\\ \eta &= \sin \Pi(x) \cos \Pi(y),\end{aligned} \tag{8}$$

thus reducing (4) to

$$\xi \cos \alpha + \eta \sin \alpha = \cos \Pi\,(p).$$

Now from equations (3), together with (1), if a point is at a finite distance r from the origin,

$$0 < \sin\ \Pi(x)\ \sin\ \Pi(y) < 1,$$

while as r increases indefinitely, $\sin \Pi(x)\ \sin \Pi(y)$ approaches zero as a limit. Therefore, since

$$\sin \Pi(x)\ \sin \Pi(y) = \sqrt{1 - \xi^2 - \eta^2};$$

if (ξ, η) be interpreted as Cartesian coördinates on a plane, the entire plane of Lobachevsky corresponds to the interior of the conic

$$\xi^2 + \eta^2 - 1 = 0$$

and the infinitely distant portion of the plane corresponds to the conic itself. We will take this conic as the fundamental conic of a system of projective measurement, so choosing the arbitrary constants involved that, if M is the distance between two points (ξ_1, η_1) and (ξ_2, η_2), and if ψ is the angle between two lines (u_1, v_1, w_1) and (u_2, v_2, w_2),*

$$\cos iM = \frac{1 - \xi_1 \xi_2 - \eta_1 \eta_2}{\sqrt{(1 - \xi_1^2 - \eta_1^2)\,(1 - \xi_2^2 - \eta_2^2)}}, \tag{9}$$

$$\cos \psi = \frac{u_1 u_2 + v_1 v_2 - w_1 w_2}{\sqrt{(u_1^2 + v_1^2 - w_1^2)\ (u_2^2 + v_2^2 - w_2^2)}}. \tag{10}$$

Comparing these with D and φ, computed above for the corresponding points and lines, we have $D = M$, $\psi = \varphi$. The latter assertion follows at once from (6), by placing $\cos \alpha = u$, $\sin \alpha = v$, $-\cos \Pi(p) = w$. To establish the former, we substitute from (8) in (7), obtaining

$$\sin \Pi(D) = \frac{\sqrt{(1 - \xi_1^2 - \eta_1^2)(1 - \xi_2^2 - \eta_2^2)}}{1 - \xi_1\xi_2 - \eta_1\eta_2} = \frac{1}{\cos iM},$$

---

* For these and other references to projective geomety, see Klein's Autographirte Vorlesungen über Nicht-Euklidische Geometrie, vol. 1. (Göttingen, 1892).

or $$\frac{2}{e^D + e^{-D}} = \frac{2}{e^M + e^{-M}},$$

of which the only solution in positive real qualities is $D = M$.

It was first shown by Klein and is now well known that the projective geometry in which a real conic serves as the basis of measurement has all the characteristic features of Lobachevsky's geometry. The equations (8) establish a direct analytic relation between the two geometries, since they image the entire Lobachevsky plane upon the interior of the (ξ, η)-plane with preservation both of angle and of distance, with the understanding that these magnitudes are to be measured in the proper way on each plane. To any real point (ξ, η) lying within the fundamental conic, corresponds one and only one real point (x, y) and conversely. In order to extend this correspondence to imaginaries without destroying the one to one relation, we may agree to consider on the (x, y)-plane only complex quantities of the type $a + bi$, where $0 \leqq b < 2\pi$. With this convention any two straight lines intersect in one and only one point, since two equations of type (4) have always an analytic solution. Lobachevsky's distinction between intersecting, parallel, and divergent lines appears accordingly as one in real finite variables only. Divergent lines meet in an imaginary point, corresponding to a real point (ξ, η) outside of the conic.

The use of the (ξ, η)-plane offers a two-fold advantage: first, it often makes the peculiar properties of Lobachevsky's geometry directly visible; and secondly, it simplifies the analytic treatment. As an example of the first use, consider Lobachevsky's theorem that two straight lines either intersect, are parallel, or have a common perpendicular. On the (ξ, η)-plane two lines are always perpendicular to the polar of their point of intersection, and this polar appears in the interior of the conic only when the two lines intersect on the exterior; so that, Lobachevsky, working in reals, could only recognize the common perpendicular for divergent lines. In the same connection, the impossibility of the existence of a rectangle is evident from the properties of poles and polars.

The nature of the circle is also easily studied on the (ξ,η)-plane. If, in equation (7), we hold D constant and take (x_1, y_1) as a fixed point, while replacing (x_2, y_2) by the variable point (x, y), we have the equation of the circle. The corresponding equation (9) is

$$k(1-\xi_1\xi_2-\eta_1\eta_2)^2=1-\xi^2-\eta^2,$$

which is the equation of a conic tangent to the fundamental conic at the points where it is cut by the polar of (ξ_1, η_1).

There are then three cases, according as (ξ_1, η_1) is within, on, or outside the conic. These correspond on Lobachevsky's plane respectively to an ordinary circle with real center and finite radius, to a limit circle approached by any circle as the radius is indefinitely increased and the center becomes indefinitely remote, and to a curve all points of which are equidistant from a real straight line. The first two kinds are given by Lobachevsky, but it is doubtful if he knew that the last was also a circle with imaginary center and radius.

An expression for the element of arc may be obtained readily from (9) by placing $\xi_1=\xi$, $\eta_1=\eta$, $\xi_2=\xi+d\xi$, and $\eta_2=\eta+d\eta$ and neglecting infinitesimals of higher order. There results

$$ds=\frac{\sqrt{d\xi^2+d\eta^2-(\eta d\xi-\xi d\eta)^2}}{1-\xi^2-\eta^2}.$$

By means of (8) and the formula of differentiation

$$d\Pi(x)=-\sin\Pi(x)dx,$$

this becomes $$ds=\sqrt{dy^2+\frac{dx^2}{\sin^2\Pi(x)}},$$

a result which may be obtained with more difficulty from (7) and is derived by Lobachevsky directly from an infinitesimal triangle. Either of these expressions applied to the circle

$$\sin\Pi(x)\sin\Pi(y)=\sin\Pi(r)$$

or $$1-\xi^2-\eta^2=\sin^2\Pi(r)$$

gives as the entire circumference

$$C=2\pi\cot\Pi(r)=\pi(e^r-e^{-r}).$$

Similarly the expression given by Klein for an area on the (ξ, η)-plane

$$S=\int\int\frac{d\xi d\eta}{(1-\xi^2-\eta^2)^{3/2}}$$

becomes by change of variables

$$S=\int\int\frac{dxdy}{\sin \Pi(y)},$$

and either of these applied to the circle gives as the entire area

$$A=\pi(e^{\frac{r}{2}}-e^{-\frac{r}{2}})^2.$$

The integration of the last integral with respect to y gives as the area dS of a narrow strip on the plane

$$dS=\cot \Pi(y).\ dx.$$

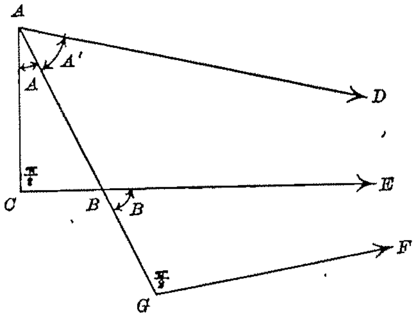
This expression is found by Lobachevsky as a result of the theorem that the area of a triangle is equal to π minus the sum of its angles. Conversely the latter theorem may be derived by integration as shown in the "Pangéométrie." To see this, consider first an infinite strip of the plane bounded by the positive part of OX, a portion of OY, and a line parallel to OX cutting OY at a distance a from O. The equation of this parallel, found by placing $l=a$, and $A=\Pi(a)$ in (5), is

$$\cos \Pi(y)=\cos \Pi(a).\ e^{-x}.$$

Hence

$$S=\int_0^\infty \cot \Pi(y)\ dx=-\int_a^0 \sin \Pi(y)\ dy=\frac{\pi}{2}-\Pi(a)=\frac{\pi}{2}-A.$$

Consider now a triangle right-angled at C, and draw through A a line parallel to CB. Prolong AB to G, so that GF perpendicular to AB shall be parallel to CB.



The triangle ABC equals the sum of the infinite strips $DACE$ and $EBFG$ diminished by $DAGF$. Substituting the areas of the strips, as found above,

$$ABC = \frac{\pi}{2} - (A + A') + \frac{\pi}{2} - B - \left(\frac{\pi}{2} - A'\right)$$

$$= \frac{\pi}{2} - A - B$$

$$= \pi - (A + B + C).$$

The theorem is thus proved for a right triangle, and is readily extended to an oblique triangle by dividing it into two right triangles by a perpendicular from any vertex.

In the foregoing pages no attempt is made to give an exhaustive statement of Lobachevsky's methods and results on the plane nor to indicate his extension of his methods to space.

FREDERICK S. WOODS.

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY, *May*, 1900.

BURKHARDT'S ELLIPTIC FUNCTIONS.

Functionentheoretische Vorlesungen. Von HEINRICH BURKHARDT. Zweiter Theil: *Elliptische Functionen.* Leipzig, Veit and Company, 1899. 8vo., x + 373 pp.

THE theory of elliptic functions has developed so rapidly and in so many different directions in recent years that an elementary treatise of moderate compass which would afford a rapid survey of its many and heterogeneous parts has been a long felt want. The admirable little treatise by Appell and Lacour is perfect in its way, but it addresses itself only to students who do not care to go very far into the theory of functions. It makes no pretentions to satisfy the needs of another large class of students, namely those who regard the theory of elliptic functions as merely one division of a greater theory and who thus study the elliptic functions not only on account of the interesting properties they offer *per se*, but also as a means of becoming more familiar with the principles and methods of the theory of functions, or as a stepping stone to the more abstruse theories of the abelian transcendents and automorphic functions.

The present volume meets the wants of this latter class most successfully. We are so impressed with its many merits that we do not hesitate to predict for it a rapid and widespread popularity.

The characteristic feature of the book is the predominance it gives to the ideas of Riemann. It is indeed remarkable, as Professor Burkhardt observes, that up to the present time no work on the elliptic functions has treated the theory from Riemann's standpoint. In several works on this subject we find reference to some of Riemann's ideas; but with the exception of Thomae's Abriss they are cursory and inadequate. We feel sure that this novel and valuable feature will be widely appreciated.

Another feature of the work is its comprehensiveness, accompanied by very moderate proportions. There is something so encouraging to the student in a text book of moderate size. The main divisions of the theory have received attention in accordance with their relative importance. By seeking everywhere the simplest form of treatment, Professor Burkhardt has succeeded in compressing a great deal into a very small compass. The student who reads this book with care will gain a very good idea of the modern theory of elliptic functions, in spite of the gigantic size this theory has assumed.

We indicate rapidly the contents. At the very outset an embarrassing question presents itself to the author of an elementary treatise on this subject: how are the elliptic functions to be introduced? Historically they arose in inverting the integral

$$u=\int_0^x \frac{dx}{\sqrt{f(x)}},$$

f being a polynomial of fourth degree.

The attempt to obtain analytic expressions of x considered as function of u led Abel and Jacobi to the theta functions, and these were used by the latter in his later university lectures as the fittest elements upon which to build up the theory. Already, then, in the infancy of this theory two lines of approach offered themselves, the one starting with an *implicit*, the other with an *explicit* definition. It was found that each had its advantages and disadvantages, and thus no traditional way of developing the theory has ever gained ground. In the last generation the two classic works were without doubt those by Briot and Bouquet and by Königsberger. The former starts with the thetas, the latter with the integral. Today we find the same lack of uniformity. Halphen's great treatise introduces the elliptic function by means of the integral definition; Weber and Krause, on the other hand, employ the thetas; finally the classic treatise of Tannery and Molk begin with Weierstrass's equivalent for the thetas, viz., the sigmas.

The advantages of starting with the integral definition seems to be chiefly these. First, it permits us to use Riemann's theories for which the elliptic integrals are merely a special, though extremely interesting, case. Secondly, in the physical applications it is as integrals that these transcendents appear. The objections are two-fold. For the student of pure mathematics who does not care to go beyond the elliptic functions, as well as for the student of physics, Riemann's theories form an unnecessary baggage. The second objection lies in the difficulty of establishing the one-valuedness of the inverse function at any early part of the course.

The advantage of beginning the elliptic functions with the thetas or sigmas is again two-fold. First, it defines them as explicit analytic expressions, which the student can see and from which he can deduce readily their principal elementary properties. Secondly, the existence theorem just mentioned falls away of itself. The disadvantage of this procedure is the unsatisfactory position it assigns the integrals.

Professor Burkhardt has followed a middle course. In chapter I. he has used the integral definition in connection with Riemann's surface. A simple proof of the uniformity of $x(u)$ for the case of real branch points is given by employing conformal representation. The general case is reserved for a later chapter.

In chapter II. an entirely new start is taken and we see no more of the elliptic functions as inverse of an integral until four chapters later, barring one or two fleeting references. The four chapters II., III., IV., V., occupying a little less than one hundred pages, give an account of double periodic functions in general, the $\wp$-function in particular, the functions σ, ϑ, ζ, and the functions of Hermite. The treatment follows the path opened up by Liouville and developed later by Hermite and Weber. There is not much chance here for an author to develop anything very novel when such masters have passed over the same route, but an attentive examination shows many minor merits.

The reviewer regrets that the historic functions of Jacobi, *sn, cn, dn,* are given an altogether inferior position. We touch here one of the serious difficulties which students encounter as soon as they begin to consult the literature of elliptic functions. They find to their dismay that there are two theories, running side by side, which, while but two aspects of the same thing, are yet so different as to make it impossible to pass from one to the other without considerable study. Now if students, when they first take up the elliptic functions, are taught almost altogether one theory, they are sure to experience a serious hindrance in their work later, since some mathematicians employ habitually the one, and some the other theory. It seems to us exclusive to maintain that either theory is the better; as well maintain that one system of coördinates always lends itself most simply to all problems. The ideal way is to have both theories equally in one's control. To this end it seems advisable, in an introductory course, to teach both simultaneously, pointing out as one goes along their interrelations. In this way the student does not acquire such a superior dexterity in one theory as to make it distasteful to him to employ the other. Such a course has not been followed here.

In chapter VI., which takes up the problem of inversion left unfinished in chapter I., we enter again the circle of ideas proper to Riemann. From now on they are our constant companions. This chapter treats, besides the problem of inversion, which is here brought to a close, various properties of integrals of the first, second, and third species; *e. g.*, Abel's theorem, the theory of Riemann-Roch, Legendre's relation, the interchange of argument and parameter in integrals of the third species, etc.

Chapters VII. to X. treat the following topics; reduction of elliptic integrals of the first species to canonical forms, the linear transformation, degeneration of the elliptic functions, and reality. In treating the linear transformation, an eighth root of unity enters as function of the coefficients of transformation. This was first determined by Hermite by making use of the sums of Gauss. Enneper and Thomae have shown how this root may be determined very simply apart from a plus or minus sign. Professor Burkhardt has been content to leave the problem at this point. For the more advanced parts of the theory this sign is simply indispensable. We take this occasion to remark that it may be determined à priori* by using no more of the theory of numbers than the simple law of reciprocity of quadratic residues.

True to his purpose to give Riemann's theory full sweep, Professor Burkhardt has not failed to show the relation between linear transformation of the periods and alterations in the system of cuts which make Riemann's surface simply connected. Unless we are very much mistaken, this subject is treated here for the first time in any treatise on the elliptic functions. The effect of varying the branch points along certain curves, closed or not, so that the surface returns to its original shape, is also studied and the notion of the monodromy of the branch points introduced.

---

* Weber gives an à posteriori demonstration; but such verifications are always unsatisfactory.

In chapter XI. we reach another grand division of the modern theory of elliptic functions, viz, the elliptic modular functions. The 51 pages which make up the chapter form one of the most interesting and instructive parts of the book. In the older theory the elliptic functions were studied almost uniquely as functions of a single variable, namely the argument u. But besides the argument, they involve certain parameters, the modulus $\varkappa$, the periods ω_1, ω_2, and the invariants g_2, g_3. With these, many important quantities are formed which do not contain u at all. As soon as the theory of functions became somewhat developed it must have occurred to many to investigate these quantities from a function-theoretical standpoint.

For example, what is the relation between $\varkappa$, which defines $sn(u, \varkappa)$ from the integral standpoint, and $\tau = \frac{iK'}{K}$, which we use to build the corresponding thetas. This question forces itself on one almost imperatively even in an elementary treatment.

The function $\varkappa$ of τ was found to possess the most remarkable properties. The most striking at first sight was that it had a natural boundary. Weierstrass's theory of functions, by starting with the notion of analytical continuation, had made the existence of such functions possible; and indeed isolated examples were discovered quite early. Thus it was known that the function

$$1 + 2q + 2q^4 + 2q^9 + \cdots,$$

which occurs already in the Fundamenta Nova,* possessed the unit circle as natural boundary. But here was a whole class which had this peculiarity. Another characteristic property is this; it is a one valued function remaining invariant for a subgroup of the general modular group

$$L = \left(\tau, \frac{a + b\tau}{c + d\tau}\right),$$

---

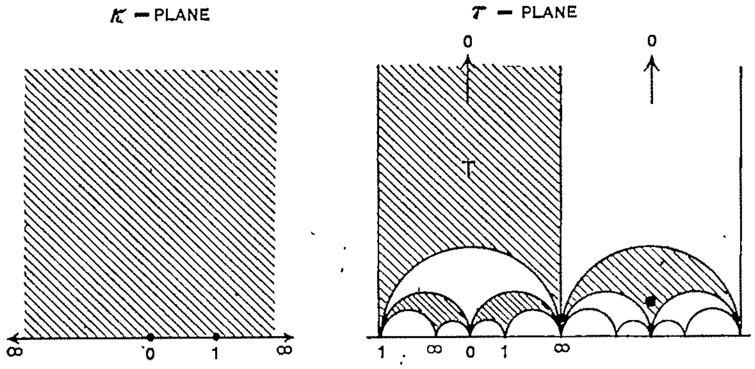
* Jacobi's Works, Vol. I., p. 235, § 65 (6).

a, b, c, d being integers and $ad - bc = 1$. The group in question, call it L_2, is, as Hermite first showed in his celebrated memoir "Sur la théorie des équations modulaires" (1859), defined by the congruences

$$a \equiv 0, \quad b \equiv 1, \quad c \equiv 1, \quad d \equiv 0 \qquad (\text{mod. } 2).$$

Such groups are called congruence groups; their order is given by the modulus, which is here 2. The conformal representation of κ by τ brings us to the important notion of fundamental domain. The points 0 and 1 divide the real κ axis into three segments $+\infty, 1$; $1, 0$; $0, -\infty$. In the τ plane these are represented respectively by the upper half of the circumference of a unit circle and the right lines yi, $1 + iy$, $(y > 0)$, which may be considered as the three sides of a circular triangle T whose vertices lie at $+1$, 0, ∞. If κ move now so as to cover every point once and only once in the upper half of the κ plane, τ passes over every point of this triangle once and only once.

Suppose now that κ passes into the negative half of its plane; it must cross over one of the three segments just mentioned. Then τ passes out of T crossing that side of the triangle which corresponds to the segment κ crossed. Let κ now cover all points of the negative half plane without going back into the positive half plane; we find that τ sweeps over a region which is got from T by reflection with respect to that side of T which τ crossed. This can therefore be nothing but a circular triangle, say T_1. If κ crosses back to the positive half plane and then covers again all its points, the same reasoning shows that τ passes out of T_1 and generates a new triangle T_2 also got from T_1 by reflection. As this process can go on indefinitely, we gradually cover the upper half of the τ plane by circular triangles as in the adjoined figure.



Any two successive triangles, as T, T_1, form a region within which $\varkappa$ (τ) takes on every value once and once only. For this reason it is called a fundamental region. The triangles $T, T_1, \cdots$ we observe are of two kinds, those like the one we first considered spreading out from left to right and having all the same size, and secondly those which approach nearer and nearer the real axis, constantly diminishing in size. This shows at once that the real τ axis is a natural boundary. Suppose $\varkappa$ describes a circuit about $\varkappa = 0$ in the positive direction; we find is τ subjected to the transformation $S = (\tau, \tau + 2)$. If τ describe a circuit about $\varkappa = 1$, τ undergoes the transformation $\Sigma = \left(\tau, \dfrac{\tau}{1 - 2\tau}\right)$. As every circuit $\varkappa$ can make is made up of these two, we see that the group L_2 mentioned above is identical with the group generated by the repeated combinations of $S^{\pm 1}$, $\Sigma^{\pm 1}$.

Now the elliptic function presents an unlimited number of such functions and we are thus led to ask what are their properties and mutual relations. The attempt to answer this question has given birth to the theory of elliptic modular functions which today has become an independent branch of mathematical science. The present chapter will prove a very valuable introduction to the classic treatise by Klein and Fricke on this subject.

Closely related with the theory of modular functions are the problems of transformation and division; these occupy chapter XII. The linear transformation was discussed in an earlier chapter. Here a rapid orientation of the general transformation problem is given. The theory of this problem might alone easily fill a book. Historically it arose in trying to find commodious methods of computing the elliptic integrals of the 1° species

$$u = \int_0^x \frac{dx}{\sqrt{1 - x^2 \cdot 1 - \varkappa^2 x^2}} = \int_0^\phi \frac{d\varphi}{\sqrt{1 - \varkappa^2 \sin^2 \varphi}} \qquad (1)$$

where $x = \sin \varphi$, and $0 < \varkappa < 1$. The transformation of Landen and Gauss showed that it was possible to transform (1) into an integral of the same form, aside from a factor, viz.,

$$u = M \int_0^y \frac{dy}{\sqrt{1 - y^2 \cdot 1 - \lambda^2 y^2}} = M \int_0^\psi \frac{d\psi}{\sqrt{1 - \lambda^2 \sin^2 \psi}} \qquad (2)$$

on making a simple change of the variable x. By repeating these transformations the moduli λ, λ_1, $\lambda_2 \cdots$ may be made to approach 0 or 1 as a limit. That is, the approximate value of u is made to depend on a degenerate elliptic integral, viz., either

$$\int_0^{\psi_\kappa} d\psi = \psi_\kappa \quad \text{or} \quad \int_0^{\psi_\kappa} \frac{d\psi}{\cos\psi} = \log\tan\left(\frac{\pi}{4} + \frac{\psi_\kappa}{2}\right).$$

When κ is near $\frac{1}{2}$ it is necessary to repeat the algorithm several times before a fair degree of approximation can be obtained. In attempting to get transformations of greater rapidity Jacobi was led to his celebrated theory of transformation which he later made the basis of his Fundamenta Nova. He proposed and solved the problem: Determine the coefficients a, b, the new modulus λ, and the multiplier M so that

$$y = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_nx^n}$$

converts (1) into (2).

From the standpoint of today the problem presents itself more naturally thus: Given two elliptic functions

$$f(u, \omega_1, \omega_2), \quad g(u, \bar{\omega}_1, \bar{\omega}_2),$$

constructed respectively on the networks

$$N(\omega_1, \bar{\omega}_2), \quad \bar{N}(\bar{\omega}_1, \omega_2);$$

what is their relation to one another? The most general relation we can assume between N and $\bar{N}$ is that they have a network M in common. Jacobi's problem is a special case of this, namely the case when one of the networks N, $\bar{N}$ becomes identical with M. In this case we have

$$\begin{aligned} \omega_1 &= a\bar{\omega}_1 + b\bar{\omega}_2, \\ \omega_2 &= c\bar{\omega}_1 + d\bar{\omega}_2, \end{aligned} \qquad (ad - bc = n),$$

and the transformation is said to be order n. For such a transformation the Jacobian formula (3) gives g as rational function of degree n in f. The case where $N = \bar{N}$ gives the linear transformation $n = 1$.

To treat the transformation in full requires us to pass to the thetas or their equivalents the sigmas. The most extensive and modern treatments are found in the works of Weber, Krause, and Tannery and Molk. In the elements, only the linear and quadratic transformations are useful and these are given with sufficient detail. The short sketch of the general theory which Professor Burkhardt gives will help the student to place these special transformations in their proper relation to the general theory.

The problem of division is this: The addition theorem shows that, for integral n, $\varphi(nu)$ is rational in $\varphi(u)$ and $\varphi'(u)$. This relation may however be regarded from another standpoint, viz., as defining $\varphi\left(\frac{u}{n}\right)$ as an algebraic function of $\varphi(u)$, $\varphi'(u)$. The problem is: what is the nature of this algebraic function; in particular express explicitly $\varphi\left(\frac{u}{n}\right)$ in $\varphi(u)$, $\varphi'(u)$. To solve completely either the problem of division or transformation we are led to certain algebraic equations called equations of transformation between modular functions which are of extraordinary interest. Professor Burkhardt has given an excellent account of these from a function-theoretical standpoint. They are also of equal interest from an algebraic and arithmetical point of view. For example, in algebra they define new algebraic irrationalities in terms of which the roots of a large class of algebraic equations can be expressed. Thus the equation between the old and new modulus $\varkappa$, $\bar{\varkappa}$ corresponding to a transformation of order 5 was used by Hermite (1858) to solve the general equation of fifth degree.

From the point of view of the higher arithmetic, they are chiefly interesting when complex multiplication takes place. This brings us to another great division of the modern theory of elliptic functions. What is complex multiplication? The answer lies in the following considerations: Up to the present point the periods of our elliptic functions have been left entirely free, with the sole restriction that

$$R\left(\frac{\tau}{i}\right) > 0. \qquad (4)$$

But an immense field is opened up when we require τ to satisfy a quadratic equation of a certain kind. Suppose, in fact, we ask with Abel: Is it possible to determine μ other than an integer, so that $\varphi(\mu u)$ is rational in $\varphi(u)$ and $\varphi'(u)$ as in ordinary multiplication? The key to this question lies in the elementary theorem: In order that an elliptic function $\psi(u)$ may be rational in $\varphi(u, \omega_1, \omega_2)$ and its derivative, it is necessary and sufficient that ψ admits ω_1, ω_2 as periods. Applying this condition to $\psi = \varphi(\mu u)$ we have at once

$$\begin{aligned}\mu\omega_1 &= a\omega_1 + b\omega_2,\\ \mu\omega_2 &= c\omega_1 + d\omega_2,\end{aligned} \tag{5}$$

which gives a condition for τ, viz.,

$$b\tau^2 + \tau(a - d) - c = 0. \tag{6}$$

If τ is an independent variable, (6) requires that $b = c = 0$, $a = d$, which put in (5) makes μ an integer. If, however, we suppose τ is not variable, but on the contrary satisfies an equation of the type (6); equations (5) show that μ satisfies the equation

$$\mu^2 - \mu(a + d) + n = 0,$$

setting $n = ad - bc$ as usual. This gives

$$\mu = \frac{a + d \pm \sqrt{\Delta}}{2} \tag{7}$$

where $\Delta = (a - d)^2 + 4bc = (a + d)^2 - 4n$ is the discriminant of (6). The condition (4) shows that $\Delta < 0$; hence (7) shows that μ is never real, whence the name complex multiplication. We see then that, when $\varphi(u, \omega_1, \omega_2)$ admits complex multiplication, τ is root of a quadratic equation with integral coefficients and negative discriminant. The converse is obviously true. Modular functions built up on such τ's are called *singular*. The corresponding equations of transformation enjoy the most remarkable properties. The simplest case of complex multiplication arises in connection with the lemniscate. In fact, the length u of an arc is given by

$$u = \int_{\wp}^{\infty} \frac{d\wp}{\sqrt{4\wp^3 - \wp}}.$$

The value of τ is here given by $\tau^2 + 1 = 0$, hence $\mu = \pm i$. Already in the Disquisitiones Arithmeticæ (1801) Gauss calls attention to the fact that the equations of transformation for this case can be solved algebraically, a theorem which Abel generalized to all cases in the Recherches.

Let us indicate in as few words as possible the rôle complex multiplication plays in the higher arithmetic, namely, in the theory of binary quadratic forms, with negative determinant,

$$Ax^2 + 2Bxy + Cy^2 = (A, B, C).$$

Such forms Gauss always took with even middle coefficient. The modern theory shows that this was unfortunate, as most material simplifications arise when we allow the middle coefficient to be either odd or even. As no elementary account of the new theory has yet been given, we follow here the classical notation. We saw that the necessary and sufficient condition for complex multiplication was that τ be root of an equation

$$r\tau^2 + s\tau + t = 0 \qquad (8)$$

with negative discriminant $\varDelta = s^2 - 4rt$. The coefficients we can suppose relative prime. According, then, as s is odd or even, we associate with (8) either the form

$$(2r,\ s,\ 2t) \quad \text{or} \quad (r,\ s/2,\ t).$$

The determinant D of this form is also negative. Conversely to every primitive form $F = (R,\ S,\ T)$ with negative determinant will correspond an equation of the type (8) which will give a singular τ.

Consider now the absolute invariant $J(\tau)$ built on the singular τ defined by (8) or the corresponding form F. Then to one form F with negative determinant corresponds one singular $J(\tau)$. Apply to τ a linear transformation, giving $\bar{\tau}$. Then $J(\bar{\tau}) = J(\tau)$. But the form $\overline{F}$ which corresponds to $\bar{\tau}$ is got from $\overline{F}$ by the same transformation; *i. e.*, F and $\overline{F}$ belong to the same class of quadratic forms. Thus not only does F give rise to a particular singular invariant $J(\tau)$ but every form in the same class as F gives rise to the same invariant. Let now $F_1,\ F_2,\ \cdots,\ F_H$ be properly primitive forms one from each of the $H(D)$ classes belonging to a given determinant D, and

$$J(\tau_1),\ J(\tau_2)\ \cdots,\ J(\tau_H) \qquad (9)$$

the corresponding singular invariants. The equations of transformation between J and its transformed show that the quantities (9) are roots of an irreducible equation with integral coefficients. This equation defines thus a numerical algebraic body, which on account of this intimate relation with the classes of binary quadratic forms is called a class body. The theory of these bodies is interwoven in the most wondrous and fascinating manner with the theory of composition of quadratic forms of negative determinant and their division into genera.

The present work, being only one volume in a course on the theory of functions, quite rightly does not even touch these questions. Indeed only three pages are devoted to complex multiplication. We have however felt justified in going out of our way to speak of these questions partly because of their intrinsic interest and partly because the reviewer deplores how little the higher arithmetic is cultivated in America. The theory of complex multiplication with its intimate relation to binary quadratic forms and algebraic numerical bodies offers a promising field for young men who seek to gain distinction as original investigators.

Leaving this subject, continue with our review. The next chapter, the thirteenth, treats the question of numerical computation. This vexatious subject, so important in all practical applications, is very satisfactorily handled here. Care is taken to give limits of error, a point often neglected.

The last three chapters are devoted to applications, one being selected from each of the three broad fields of geometry, analysis, and mechanics. For geometry it is the theory of elliptic curves, *i. e.*, curves defined in homogeneous coördinates by the equations

$$\rho x_1 = T_1(u), \quad \rho x_2 = T_2(u), \cdots,$$

the T's being conjugate intermediate functions of Hermite, *i. e.*, functions of the form

$$e^{au}\sigma(u - b_1)\sigma(u - b_2) \cdots \sigma(u - b_n) .$$

Curves of the third and fourth orders receive especial attention. The application to analysis is the discussion of Picard's equation of the second order, in particular Lamé's equation

$$\frac{d^2y}{du^2} = \{n(n+1)\wp u + b\}y. \tag{10}$$

The application to mechanics is the spherical pendulum. In discussing the horizontal motion of the bob, the author passes to polar coördinates, thereby missing a pretty application of Lamé's equation discussed in the chapter just preceding. In fact, keeping rectangular coördinates x, y, it is easily shown that $x + iy$ and $x - iy$ are two solutions of (10) for $n = 2$.

The rapid survey we have here made shows most clearly that we have in the present volume a text book which is rich to an uncommon degree in the latest results and points of view in the subject it treats. It fills most timely a place unoccupied by any other work and will, we are sure, prove itself invaluable to the large class of readers for which it is intended.

JAMES PIERPONT.

YALE UNIVERSITY.

CORRECTION.

In my paper "On the geometry of the circle," pp. 319–322 of the May number of the current volume of the BULLETIN, a term in the transformations involved was wrong and the results are partly incorrect. The following corrections, which were kindly pointed out by Professor P. F. Smith, should be made in the paper:

P. 320, lines 7 and 10, *for* bicircular *read* binodal.

P. 321, the last paragraph is only projectively correct. There are 40 double tangents and no single foci.

P. 322, the corrections corresponding to the preceding are to be made in the case of the sphere.

The eliminant of $\lambda, \mu, \cdots$, pp. 320–321, contains irrelevant factors; its reduced order is 16. The order of the surface, p. 322, reduces to 24.

VIRGIL SNYDER.

NOTES.

THE forty-ninth annual meeting of the American association for the advancement of science was held at Columbia University, New York City, June 23–30, 1900. The presence of no less than sixteen affiliated scientific societies, among them the AMERICAN MATHEMATICAL SOCIETY, added to the interest and importance of the occasion. It was decided to hold the next meeting of the Association at Denver, Colorado, August 24th–31st, 1901. Professor CHARLES S. MINOT was elected president of the Association. The officers of Section A are: vice-president, Professor JAMES MCMAHON; secretary, Professor H. C. LORD.

AT the meeting of the Cambridge philosophical society on May 21st, the following mathematical papers were presented: "On a certain diophantine inequality," by Major P. A. MACMAHON; "On rational space curves of the fourth order," by Mr. H. W. RICHMOND; "On the reduction of quadrics," by Mr. T. J. I. BROMWICH.

UNIVERSITY OF CALIFORNIA.—The following courses are announced for the year 1900–1901:—By Professor I. STRING-

HAM: History of mathematics, two hours; Logic of mathematics, two hours; Absolute geometry, two hours; Mathematical seminary, two hours.—By Professor M. W. HASKELL: Analytic projective geometry, three hours; Higher geometry, three hours.—By Professor G. C. EDWARDS: Differential equations, three hours.—By Professor A. O. LEUSCHNER: Theoretical astronomy, two hours; Perturbations, four hours.—By Mr. A. B. PIERCE: Theory of numbers, three hours.—By Dr. E. J. WILCZYNSKI: Transformation groups and differential equations, two hours.—By Mr. A. W. WHITNEY: Theory of probabilities, Life insurance, two hours.—By Dr. S. D. TOWNLEY: Method of least squares, three hours.

JOHN HOPKINS UNIVERSITY. During the year 1900–1901 the following advanced courses in mathematics are offered:—By Professor F. MORLEY: Theory of functions (second course), two hours; Geometry (second course), three hours; Seminary, one hour.—By Dr. A. COHEN: Theory of functions (first course), two hours; Lie's theory of transformation groups, two hours; Theory of numbers, three hours, first semester; Differential equations (second course), two hours, second semester.

AT a recent meeting of the Royal Society of Edinburgh, Professor A. R. FORSYTH, of the University of Cambridge, was elected as one of the British honorary fellows of the society.

THE honorary degree of doctor of science was conferred on Professor HENRI POINCARÉ, of Paris, on June 12th, by the University of Cambridge.

PROFESSOR GASTON DARBOUX was elected on May 21st to succeed the late JOSEPH BERTRAND as perpetual secretary for the section of mathematical sciences in the Paris academy of sciences. At the next meeting of the same academy, May 28th, Professor LUDWIG BOLTZMANN, of Vienna, recently called to Leipzig, was elected correspondent for the section of mechanics, in the place of the late EUGENIO BELTRAMI, of Rome; the same honor had been bestowed at the previous meeting on Professor J. WILLARD GIBBS, of Yale University.

AMONG the recipients of honorary degrees at the recent celebration at the University of Cracow was Professor SIMON NEWCOMB.

DR. E. D. ROE, JR., has been appointed to an associate professorship of mathematics in Syracuse University. Dr.

Ruth Gentry has been promoted to a similar position at Vassar College.

The following academic appointments have also recently been made: Dr. W. G. Bullard, instructor in mathematics, Syracuse University; Dr. C. C. Engberg, instructor in mathematics, University of Nebraska; Mr. E. B. Escott and Mr. William Marshall, instructors in mathematics, University of Michigan; Dr. L. W. Reid, instructor in mathematics, Haverford College; Dr. Anne L. Wilkinson, instructor in mathematics, Vassar College.

Dr. Mary F. Winston, has resigned the professorship of mathematics at the Kansas State Agricultural College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Baker (H. F.). See Memoirs presented, etc.

Beman (W. W.). See Fink (K.).

Berry (A.), Boltzmann (L.), Brown (E. W.). See Memoirs presented, etc.

Brückner (M.). Vielecke und Vielflache; Theorie und Geschichte. Leipzig, Teubner, 1900. 4to. 8 + 227 pp., 12 plates. Boards. M. 16.00

Burnside (W.). See Memoirs presented, etc.

Cassirer (E.). Descartes' Kritik der mathematischen und naturwissenschaftlichen Erkenntnis. Marburg, 1900. 8vo. 102 pp. M. 3.00

Eratosthenis Catasterismorum fragmenta Vaticana, edidit A. Rehm. Praemissum est de Catasterismorum recensionibus commentariolum. Auerbach, 1900. 8vo. 26 + 18 pp.

Fink (K.). A brief history of mathematics; an authorized translation of Dr. Karl Fink's Geschichte der Elementar-Mathematik, by W. W. Beman and D. E. Smith. With biographical notes and full index. Chicago, The Open Court Publishing Co., 1900. 8vo. 12 + 333 pp. Cloth. $1.50

Forsyth (A. R.). See Memoirs presented, etc.

Grassmann (R.). Die Differential- und Integralrechnung bei Vermeidung der Trugschlüsse eine höchst leichte Wissenschaft. Stettin, Grassmann, 1900. 12mo. 44 pp. Boards. M. 0.40

——. Die Funktionenlehre, namentlich die Differential- und Integralrechnung in strenger Formelentwicklung. (Neue Titelausgabe von "Die Folgelehre," 1895.) Stettin, Grassmann, 1900. 8vo. 12 + 189 pp. M. 1.75

——. Die Zahlenlehre oder Arithmetik in strenger Formelentwicklung. (Neue Titelausgabe von "Die Arithmetik," 1891.) Stettin, Grassmann, 1900. 8vo. 12 + 242 pp. M. 2.00

HOBSON (E. W.). See MEMOIRS presented, etc.

HÖLDER (O.). Anschauung und Denken in der Geometrie. Akademische Antrittsvorlesung, mit Zusätzen, Anmerkungen und einem Register. Leipzig, Teubner, 1900. 8vo. 75 pp. M. 2.40

LAMB (H.), LARMOR (J.). See MEMOIRS presented, etc.

LAURENT (H.). L'élimination. Paris, Carré et Naud, 1900. 16mo. 75 pp. (*Scientia*, physique mathématique, No. 7.) Boards.

LEE (A.). Integral tables of $F(r,v)$ and $H(r,v)$ functions. London, 1900. 8vo. 50 pp. 2s. 6d.

LINEBARGER (C. E.). See YOUNG (J. W. A.).

LIVEING (G. D.), LODGE (O. J.), LOVE (A. E. H.), LOVETT (E. O.). See MEMOIRS presented, etc.

LUDWIG (B. VON). Ueber die Nothwendigkeit der Beschränkung des Jacobi'schen Umkehrproblems auf Abel'sche Integrale erster Gattung. (Diss.) Halle, 1900. 8vo. 94 pp.

MANCHESTER (J. E.). Ueber Singularitäten ebener Kurven. (Diss.) Tübingen, 1899. 8vo. 38 pp.

MEMOIRS presented to the Cambridge Philosophical Society on the occasion of the jubilee of Sir George Stokes, Bart. Cambridge University Press, 1900. 4to. 28 + 447 pp., 1 portrait, 25 plates. Cloth. 21s.

(Contents: Order of the proceedings at the formal celebration by the University of Cambridge of the Jubilee of Sir G. G. Stokes, Bart., Lucasian Professor, 1849–1899. The Rede Lecture by A. Cornu: La théorie des ondes lumineuses. H. F. Baker, Theory of functions of several complex variables; A. Berry, Quartic surfaces which admit of integrals of the first kind of total differentials; L. Boltzmann and Mache, Ueber die Bedeutung der Konstante b des van der Waals'schen Gesetzes; E. W. Brown, Solution of a pair of simultaneous linear differential equations in the lunar theory; W. Burnside, Class of groups of finite order; A. R. Forsyth, Integrals of systems of differential equations; E. W. Hobson, Green's function for a circular disc; H. Lamb, Electromagnetic illustration of the theory of selective absorption of light by a gas; J. Larmor, Dynamics of a system of electrons or ions; G. D. Liveing, Effects of dilution, temperature and other circumstances on the absorption spectra of solutions of didymium and erbium salts; O. J. Lodge, Experiments on oscillatory discharge of an air condenser; A. E. H. Love, Propagation of waves of elastic displacement along a helical wire; E. O. Lovett, Contact transformations and Optics; H. M. Macdonald, Demonstration of Green's formula of electric density near the vertex of a right cone; P. A. MacMahon, Application of the partition analysis to the study of the properties of any system of consecutive integers; A. A. Michelson, Echelon spectroscope; G. Mittag-Leffler, Analytical representation of a uniform branch of a monogenic function; H. Poincaré, Groupes continus; H. W. Richmond, Minimal surfaces; A. Schuster, Periodogram of magnetic declination, 1871–95; C. Taylor, Geometry of Kepler and Newton; H. M. Taylor, Construction of a model showing the 27 lines on a cubic surface.

MACDONALD (H. M.), MACMAHON (P. A.), MICHELSON (A. A.). See MEMOIRS presented, etc.

MILLER (G. A.). On the transitive substitution groups which are isomorphic to a given group. (*Giornale di matematiche*, vol. 38.) 4to. 9 pp.

——. On the product of two substitutions. (*American Journal of Mathematics*, Vol. 22, pp. 185-190.) 4to.

MITTAG-LEFFLER (G.). See MEMOIRS presented, etc.

NERNST (W.). See YOUNG (J. W. A.).

OSGOOD (W. F.). Ueber einen Satz des Herrn Schönflies aus der Theorie der Funktionen zweier reeller Veränderlichen. (*Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, mathematisch-physikalische Klasse, 1900, Heft 1.) 8vo. 4 pp.

POINCARÉ (H.). See MEMOIRS presented, etc.

REHM (A.). See ERATOSTHENES.

RICHMOND (H. W.). See MEMOIRS presented, etc.

SCHÖNFLIES (A.). See YOUNG (J. W. A.).

SCHUSTER (A.). See MEMOIRS presented, etc.

SMITH (D. E.). See FINK (K.).

TAIT (P. G.). Scientific papers. Vol. II. Cambridge, University Press (New York, The Macmillan Company), 1900. 4to. 14 + 500 pp. Cloth. $6.50

TAYLOR (C.), TAYLOR (H. M.). See MEMOIRS presented, etc.

VECCHI (S.), Sulle figure complete, determinate da un numero qualunque di punti o da un numero qualunque di tangenti di una conica e sulle loro correlative nello spazio. Parma, Rossi-Ubaldi, 1899. 4to. 16 pp., 2 plates.

WEBER (E. VON). Vorlesungen über das Pfaff'sche Problem und die Theorie der partiellen Differentialgleichungen erster Ordnung. Leipzig, Teubner, 1900. 8vo. 11 + 622 pp. Cloth. (Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, No. 2.) M. 24.00

YOUNG (J. W. A.) and LINEBARGER (C. E.). The elements of the differential and integral calculus, based on Kurzgefasstes Lehrbuch der Differential- und Integralrechnung von W. Nernst und A. Schönflies. New York, Appleton, 1900. 8vo. 17 + 410 pp. Cloth. $2.20

II. ELEMENTARY MATHEMATICS.

BARDEY (E.). Arithmetische Aufgaben, nebst Lehrbuch der Arithmetik, vorzugsweise für Realschulen, höhere Bürgerschulen und verwandte Anstalten, neu bearbeitet und mit einer Logarithmentafel versehen von H. Hartenstein. 3te Auflage. Leipzig, Teubner, 1900. 8vo. 4 + 202 pp. Cloth. M. 2.00

——. Aufgabensammlung, methodisch geordnet, mehr als 8000 Aufgaben enthaltend, über alle Teile der Elementararithmetik, vorzugsweise für Gymnasien, Realgymnasien und Oberrealschulen. Neue

Ausgabe, nach der 24sten Auflage bearbeitet von F. Pietzker und O. Presler. Leipzig, Teubner, 1900. 8vo. 7 + 376 pp. Cloth. M. 3.20

BORTOLOTTI (E.). Nozioni pratiche di geometria per le scuole complementari. Roma, Società editrice Dante Alighieri, 1900. 8vo. 140 pp. Fr. 1.75

COMPLETE SOLUTIONS to papers in mathematics (second stage), 1887–99. Science and art examinations. London, Moffat & Paige, 1900. 12mo. Boards. 2s. 6d.

CONTI (A.). Elementi di aritmetica razionale. Parte I. Bologna, Zanichelli, 1900. 12mo. 304 pp. Fr. 2.40

——. Problemi di terzo grado: duplicazione del cubo; trisezione dell'angolo. Bologna, Zanichelli, 1900. 8vo. 56 pp., 3 plates.

DESCLAUX (G.). Cours primaire de trigonométrie pratique, à l'usage des écoles normales primaires et des écoles primaires supérieures. Paris, Hachette, 1900. 16mo. 79 pp. Fr. 1.00

FISHER (G. E.) and SCHWATT (I. J.). Rudiments of algebra. Philadelphia, Fisher and Schwatt, 1900. 12mo. 7 + 183 pp. Half-leather. $0.60

——. Secondary algebra. Philadelphia, Fisher and Schwatt, 1900. 12mo. Half-leather. $1.05

HAENTZSCHEL (E.). Ueber die verschiedenen Grundlegungen in der Trigonometrie. (Progr.) Berlin, Gaertner, 1900. 8vo. 31 pp.

HALSTED (G. B.). Noneuclidean geometry for teachers. 8vo. 14 pp.

HARTENSTEIN (H.). See BARDEY (E.).

HAUCK (G.). Lehrbuch der Stereometrie. Auf Grund von F. Kommerell's Lehrbuch neu bearbeitet und erweitert. 8te Auflage (7te der Neubearbeitung). Tübingen, Laupp, 1900. 8vo. 16 + 224 pp. Cloth. M. 2.90

LÜBSEN (H. B.). Ausführliches Lehrbuch der Elementargeometrie. Ebene und körperliche Geometrie. 29ste Auflage. Leipzig, 1900. 8vo. 4 + 179 pp. M. 3.00

——. Ausführliches Lehrbuch der ebenen und sphärischen Trigonometrie. 17te Auflage. Leipzig, 1900. 8vo. 4 + 115 pp. M. 2.40

MIELE (S.). Problema sulla costruzione d'un segmento circolare di data area. Firenze, Niccolai, 1900. 8vo. 24 pp.

NOZIONI elementari di geometria piana e solida. Bologna, Pongetti, 1900. Oblong. 24 pp.

PIETZKER (F.). See BARDEY (E.).

PORTA (F.). I complimenti di matematica per l'ammissione all'Accademia militare. 4ta edizione. Torino, 1900. 8vo. 218 pp. Fr. 4.00

PRESLER (O.). See BARDEY (E.).

ROSENBERG (K.). Methodisch geordnete Sammlung von Aufgaben aus der Arithmetik und Algebra für Lehrer- und Lehrerinnen-Bildungsanstalten sowie für andere gleichgestellte Lehranstalten. 2te Auflage. Wien, Hölder, 1900. 8vo. 4 + 255 pp. Cloth. M. 2.60

SAUERBECK (P.). Lehrbuch der Stereometrie, nebst zahlreichen Uebungen und einem Abschnitt über Krystallographie. Zum Gebrauch an höheren Lehranstalten, sowie für den Selbstunterricht bearbeitet. Stuttgart, Bergsträsser, 1900. 8vo. 7 + 291 pp. M. 5.40

SCHÜLKE (A.). Vierstellige Logarithmentafeln, nebst mathematischen, physikalischen und astronomischen Tabellen. Für den Schulgebrauch zusammengestellt. 3te Auflage. Leipzig, Teubner, 1900. 8vo. 2 + 18 pp. M. 0.60

SCHWERING (K.). Stereometrie für höhere Lehranstalten, nach den neuen Lehrplänen bearbeitet. 2te Auflage. Freiburg i. B., Herder, 1900. 8vo. 7 + 56 pp. M. 0.80

SENDLER (R.). Theorie und Praxis des geometrischen Unterrichts in der Volksschule. Im Anschluss an die "Raumlehre für Volksschulen" bearbeitet. Breslau, Handel, 1900. 8vo. 8 + 180 pp. M. 1.80

SIMON (M.). Geometrie für höhere Bürgerschulen (Realschulen) und Lehrerseminarien. Ein methodischer Leitfaden in heuristischer Darstellung. 5te Auflage. Breslau, Hirt, 1900. 8vo. 80 pp. Boards. M. 1.00

SPIEKER (T.). Kurze Anleitung zum Lösen der Uebungsaufgaben des Lehrbuchs der ebenen Geometrie für höhere Lehranstalten. 2te Auflage. Potsdam, Stein, 1900. 8vo. 4 + 68 pp. M. 1.20

WINTER (W.). Stereometrie. Lehrbuch und Aufgabensammlung für Schulen. 3te Auflage. München, Ackermann, 1900. 8vo. 4 + 116 pp. M. 1.60

WOOLWICH mathematical papers for admission into the Royal Military Academy for the years 1890–99. Edited by E. J. Brooksmith. London, Macmillan, 1900. 12mo. 6s.

III. APPLIED MATHEMATICS.

BLAKE (E. M.). The ellipsograph of Proclus. (*American Journal of Mathematics*, Vol. 22, pp. 146–153.) 4to.

DRUDE (P.). Lehrbuch der Optik. Leipzig, Hirzel, 1900. 8vo. 14 + 498 pp. M. 10.00.

EWERS (P.). Zur Mechanik der Kanal- und Kathodenstrahlen. (Diss.) München, 1899. 8vo. 52 pp.

FRITSCHE (C.). Ueber die Beeinflussung der Circumnutation durch verschiedene Faktoren. Leipzig, 1900. 8vo. 35 pp.

HATZIDAKIS (N. J.). Sur les équations cinématiques fondamentales des variétés dans l'espace à n dimensions. (*Comptes rendus*, 26 février, 1900.) 4to. 3 pp.

HERRMANN (G.). Die graphische Theorie der Turbinen und Kreiselpumpen. 2te Auflage. Berlin, Simion, 1900. 8vo. 7 + 213 pp., 7 plates. Cloth. M. 8.00

HOFF (J. H. VAN'T). Lectures on theoretical and physical chemistry. Translated by R. A. Lehfeldt. Part 2: Chemical Statics. London, Arnold, 1900. 8vo. 156 pp. 8s. 6d.

Keck (W.). Vorträge über Mechanik als Grundlage für das Bau- und Maschinenwesen. Teil I: Mechanik starrer Körper. 2te Auflage. Hannover, Helwing, 1900. 8vo. 7 + 330 pp. M. 10.00

Larmor (J.). Aether and matter; a development of the dynamical relations of the aether to material systems on the basis of the atomic constitution of matter, including a discussion of the influence of the earth's motion on optical phenomena; being an Adams prize essay in the University of Cambridge. Cambridge University Press, 1900. 8vo. 394 pp. 10s.

Lauenstein (R.). Die Mechanik. Elementares Lehrbuch für technische Mittelschulen und zum Selbstunterricht. 4te Auflage. Stuttgart, Bergsträsser, 1900. 8vo. 7 + 201 pp. M. 4.40

——. Die grapische Statik. Elementares Lehrbuch für technische Unterrichtsanstalten und zum Gebrauch in der Praxis. 6te Auflage. Stuttgart, Bergsträsser, 1900. 8vo. 6 + 251 pp. M. 5.40

Lehfeldt (R. A.). See Hoff (J. H. van't).

Obermayer (A. von). Leitfaden für den Unterricht in der Physik an der technischen Militär-Akademie mit besonderer Berücksichtigung ausgewählter Capitel, insbesondere der Mechanik. Wien, 1900. 8vo. 35 + 827 pp. M. 13.40

Pascal. Traité pratique des ponts métalliques. Calcul des poutres et des ponts à une ou à plusieurs travées par la méthode ordinaire et par la statique graphique; ponts en arc. Nouvelle édition, entièrement refondue et considérablement augmentée. Paris, Baudry, 1900. 8vo. 7 + 287 pp., 15 plates.

Reynolds (O.). Papers on mechanical and physical subjects. Reprinted from various transactions and journals. Vol. I. Cambridge University Press, 1900. 8vo. 15s.

Sonnet (H.). Dictionaire des mathématiques appliquées, comprenant les principales applications des mathématiques, à l'architecture, à l'arithmétique commerciale, à l'arpentage, à l'artillerie, aux assurances, à la balistique, à la banque, à la charpente, au chemins defer, etc , et l'explication d'un grand nombre de termes techniques usités dans les applications. 6^{e} édition. Paris, Hachette, 1900. 8vo. 4 + 1478 pp. Fr. 30.00

Sturm (R.). Elemente der darstellenden Geometrie. 2te Auflage. Leipzig, Teubner, 1900. 8vo. 5 + 157 pp., 7 plates. Cloth. M. 5.60

Suslov (G. K.). Analytical mechanics. Vol. I. Kiev, Ogloblin, 1900. 8vo. 14 + 543 pp. (Russian.) R. 3.75

Vecchi (S.). Saggio di un disegno polarimetrico; esercizio di geometria descrittiva. Parma, Rossi-Ubaldi, 1899. 4to. 8pp., 2 plates.

Zubov (I. M.). Moments of inertia, statical moments, and other elements of cross-sections of parts of metallic constructions. Moscow, 1900. 8vo. 127 pp. (Russian.) R. 1.75

NINTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

Aley, R. J. A Triangle Related to Nagel's Triangle. Read (Chicago) April 9, 1898. *Proceedings of the Indiana Academy of Science*, 1898, pp. 89–91.

—— A New Set of Three Collinear Points Connected with a Triangle. Read (Chicago) Dec. 28, 1899 *Bulletin of the American Mathematical Society*, vol. 6, No. 5, p. 193 ; Feb., 1900.

Benner, H. Bestimmung der Coefficienten welche bei der Berechnung der Integrale $\int \frac{x^n\,dx}{\sqrt{1+ax+bx^2}}$ und $\int \frac{x^n\,dx}{\sqrt{1+ax+bx^2+cx^3}}$ auftreten. Read (Chicago) Dec. 31, 1897. Doctor Dissertation, University of Erlangen, 1897, pp. 57. Boston, Ginn and Company, 1899.

Baker, A. L. Fundamental Algebraic Operations. Read Aug. 19, 1898. *Proceedings of the Rochester Academy of Science*, vol. 3, p. 162.

Blake, E. M. Linear Euthymorphic Functions of the First Order. Read (Chicago) Dec. 31, 1896. *Proceedings of the Indiana Academy of Science*. 1896, pp. 87–88.

—— The Ellipsograph of Proclus. Read (Chicago) Dec. 31, 1897, and April 9, 1898. *American Journal of Mathematics*, vol. 22, No. 2, pp. 146–153 ; April, 1900.

—— On the Ruled Surfaces Generated by the Plane Movements whose Centrodes are Congruent Conics Tangent at Homologous Points. Read Aug. 19, 1898. *American Journal of Mathematics*, vol. 21, No. 3, pp. 257–269 ; July, 1899.

Bôcher, M. On Regular Singular Points of Linear Differential Equations of the Second Order whose Coefficients are not Necessarily Analytic. Read Dec. 28, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 40–52 ; Jan., 1900.

—— Some Theorems Concerning Linear Differential Equations of the Second Order. Read Feb. 24, 1900. *Bulletin of the American Mathematical Society*, vol. 6, No. 7, pp. 279–280 ; April, 1900.

Bolza, O. The Elliptic Sigma-Functions Considered as a Special Case of the Hyperelliptic Sigma-Functions. Read (Chicago) Dec. 28, 1899. *Transactions of the American Mathematical Society*, Vol. 1, No. 1, pp. 53–65 ; Jan., 1900.

Brown, E. W. On the Particular Integrals of Linear Differential Equations, and Their Application to the Lunar Theory. Read Mar. 28, 1896. *Cambridge Philosophical Transactions*, vol. 18, pp. 94–106 ; 1899.

—— On Tide Currents in Estuaries and Rivers. Read Apr. 29, 1899. *Annals of Mathematics*, 2d series, vol. 1, No. 2, pp. 68–70 ; Jan., 1900.

CAJORI, F. Notes on the History of Logarithms. Read Aug. 25, 1899. *Zeitschrift für Mathematik und Physik*, vol. 44, supplement: *Abhandlungen zur Geschichte der Mathematik*, vol. 14, No. 9. pp. 31–39; 1899.

CAMPBELL, D. F. On Linear Differential Equations of the Third and Fourth Orders in whose Solutions Exist Certain Homogeneous Relations. Read Feb. 25, 1899. *Quarterly Journal of Mathematics*, vol. 31, No. 122, pp. 161–192; Sept., 1899.

CAMPBELL, J. E. On the Types of Linear Partial Differential Equations of the Second Order in Three Independent Variables Which are Unaltered by the Transformations of a Continuous Group. Read Dec. 28, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 2, pp. 243–258; April, 1900.

CHESSIN, A. S. On Relative Motion. Read Aug. 19, 1898, Oct. 29, 1898, and Aug. 25, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 2, pp. 116–169; April, 1900.

—— On the Development of the Perturbative Function in Terms of the Eccentric Anomalies. Read Dec. 28, 1898 *Astronomical Journal*, No. 446, vol. 20, No. 10, pp. 73–76; July 22, 1899.

COOLIDGE, J. L. A Purely Geometric Representation of all Points in the Projective Plane. Read Oct. 28, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 2, pp. 182–192; April, 1900.

COTTIER, J. On the Expression of the General Equations of Hydrodynamics in Terms of Curvilinear Coordinates. Read March. 27, 1897. *Mathematical Magazine*, vol. 1, No. 3.

DAVIS, E. W. Note on Special Regular Reticulations. Read (Chicago) Dec. 31, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 10, pp. 529–530; July, 1898.

—— The Group of the Trigonometric Functions. Read (Chicago) April 1, 1899. *Bulletin of the American Mathematical Society*, vol. 5, No. 8, pp. 380–381; May, 1899.

DICKSON, L. E. The Structure of Certain Linear Groups with Quadratic Invariants. Read Aug. 19, 1898. *Proceedings of the London Mathematical Society*, vol. 30, Nos. 662–664, pp. 70–98; Nov., 1898.

—— The Determination of the Structure of All Linear Homogeneous Groups in a Galois Field which are Defined by a Quadratic Invariant, with the Announcement of Two New Systems of Simple Groups. Read (Chicago) Dec. 30, 1898. *American Journal of Mathematics*, vol. 21, No. 3, pp. 193–256; July, 1899.

—— A New Definition of the General Abelian Linear Group. Read Feb. 25, 1899, and Aug. 26, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 91–96; Jan. 1900.

—— Concerning the Four Known Simple Linear Groups of Order 25920, with an Introduction to the Hyper-Abelian Linear Groups. Read (Chicago) April 1, 1899. *Proceedings of the London Mathematical Society*, vol. 31, Nos. 680–683, pp. 30–68; April, 1899.

—— Definition of the Abelian, the Two Hypoabelian, and Related Linear Groups as Quotient Groups of the Groups of Isomorphisms of Certain Elementary Groups. Read Aug. 26, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 30–38; Jan., 1900.

—— Canonical Form of a Linear Homogeneous Substitution in a General Galois Field. Read (Chicago) Dec. 28, 1899. *American Journal of Mathematics*, vol. 22, No. 2, pp. 121–137 ; April, 1900.

—— Proof of the Existence of the Galois Field of Order p^r for Every Integer r and Prime Number p. Read Dec. 28, 1899. *Bulletin of the American Mathematical Society*, vol. 6, No. 5, pp. 203–204 ; Feb., 1900.

—— Isomorphism between Certain Systems of Simple Linear Groups. Read Feb. 24, 1900. *Bulletin of the American Mathematical Society*, vol. 6, No. 8, pp. 323–328 ; May, 1900.

—— Proof of the Non-Isomorphism of the Simple Abelian Group on $2m$ Indices and the Simple Orthogonal Group on $2m+1$ Indices for $m > 2$. Read Feb. 24, 1900. *Quarterly Journal of Mathematics*, vol. 32, No. 1, pp. 42–63 ; June, 1900.

Emch, A. Circular Transformations. Read (Chicago) Jan. 1, 1897. *Annals of Mathematics*, 1st series, vol. 12, No. 5, pp. 141–160 ; Aug., 1898.

—— Illustration of the Elliptic Integral of the First Kind by a Certain Link-Work. Read Aug. 25, 1899. *Annals of Mathematics*, 2d series, vol. 1, No. 2, pp. 81–92 ; Jan., 1900.

Engberg, C. C. The Cartesian Oval and the Auxiliary Parabola. Read (Chicago) Dec. 30, 1898. *The Graduate Bulletin of the University of Nebraska*, vol. 1, No. 1, pp. 23–34 ; April, 1900.

Goodspeed, E. J. The Ayer Papyrus : A Mathematical Fragment. Read (Chicago) Dec. 31, 1897. *American Journal of Philology*, vol. 19, No. 1, pp. 25–39 ; April, 1898.

Gordan, Paul. Formentheoretische Entwickelung der in Herrn White's Abhandlung über Curven dritter Ordnung enthaltenen Sätze. Read Oct. 28, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 9–13; Jan., 1900.

Goursat, E. Sur la Définition Générale des Fonctions Analytiques, d'après Cauchy. Read April 29, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 14–16 ; Jan., 1900.

Hathaway, A. S. Alternate Processes. Read Aug. 17, 1897, and (Chicago) Dec. 30, 1897. *Proceedings of the Indiana Academy of Science*, 1897, pp. 117–127.

—— Orthogonal Surfaces. Read (Chicago) Dec. 30, 1897. *Proceedings of the Indiana Academy of Science*, 1896, pp. 85–86.

Hayes, E. Note on the Folium of Descartes. Read Aug. 17, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 1, p. 4 ; Oct , 1897.

Hedrick, E. R. On Three Dimensional Determinants. Read Feb. 25, 1899. *Annals of Mathematics*, 2d series, vol. 1, No. 2, pp. 49–67 ; Jan., 1900.

Hill, G. W. On the Extension of Delaunay's Method in the Lunar Theory to the General Problem of Planetary Motion. Read Feb. 24, 1900. *Transactions of the American Mathematical Society*, vol. 1, No. 2, pp. 205–242 ; April, 1900.

Hutchinson, J. I. The Hessian of the Cubic Surface. II. Read Feb. 24, 1900. *Bulletin of the American Mathematical Society*, vol. 6, No. 8, pp. 328-337 ; May, 1900.

Keyser, C. J. Some Theorems in n Dimensional Space. Read Dec. 29, 1897. *Bulletin of the American Mathematical Society*, vol. 4, No. 5, pp. 181-182 ; Feb., 1898.

Laves, K. The Ten Integrals of the Problem of n Bodies for Forces Involving the Coordinates and Their First and Second Differentials. Read (Chicago) April 9, 1898. *Astronomical Journal*, vol. 19, No. 13, pp. 97–104 ; Oct. 8, 1898.

Ling, G. H., and Miller, G. A. A Proof that there are no Simple Groups of Order 1440, 1512, 1680, or 1800. Read Feb. 26, 1899, and Aug. 26, 1899. *American Journal of Mathematics*, vol. 22, No. 1, pp. 13–26 ; Jan., 1900.

Lovett, E. O. Sur les Invariants Projectifs d'un Système de $m+1$ Points dans l'Espace à $n+1$ Dimensions Read Aug. 19, 1898. *Bulletin des Sciences Mathématiques*, ser. 2, vol. 22, No. 1, pp. 10–15 ; Jan , 1899.

—— Contact Transformations of Developable Surfaces. Read Oct. 29, 1898. *Rendiconti del Circolo Matematico di Palermo*, vol. 13, No. 4, pp. 210–224 ; July-Aug., 1899.

—— Contact Transformations and Optics. Read Dec. 28, 1898. *Cambridge Philosophical Transactions*, vol. 18, pp. 256–268 ; 1899.

—— Note on the Differential Invariants of Goursat and Painlevé. Read Feb. 25, 1899. *American Journal of Mathematics*, vol. 22, No. 1, pp. 41–45 ; Jan., 1900.

—— Families of Transformations of Straight Lines into Spheres. Read Oct. 28, 1899. *American Journal of Mathematics*, vol. 22, No. 2, pp. 138–145 ; April, 1900.

Maclay, J. On Certain Algebraic Double Minimal Surfaces. Doctor Dissertation, Columbia University, 1899.

Maschke, H. Beweis des Satzes dass diejenigen endlichen linearen Substitutionsgruppen, in welchen einige durchgehends verschwindende Coefficienten auftreten, intransitiv sind, Read (Chicago) Dec. 30, 1898. *Mathematische Annalen*, vol. 52, No. 3, pp. 363–368 ; July, 1899.

—— Note on the Unilateral Surface of Moebius. Read (Chicago) Dec. 29, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 1, p. 39 ; Jan., 1900.

—— A New Method of Determining the Differential Parameters and Invariants of Quadratic Differential Quantities. Read (Chicago) April 14, 1900. *Transactions of the American Mathematical Society*, vol. 1, No. 2, pp. 197–204 ; April, 1900.

Metzler, W. H. On the Excess of the Number of Combinations in a Set Which Have an Even Number of Inversions over Those Which Have an Odd Number. Read Feb. 26, 1898. *American Journal of Mathematics*, vol. 22, No. 1, pp. 55–59 ; Jan., 1900.

—— On the Roots of a Determinantal Equation. Read (Chicago) Apr. 9, 1898. *American Journal of Mathematics*, vol. 21, No. 4, pp. 367–368; Oct., 1899.

Miller, J. A. Concerning Certain Modular Functions of Square Rank. Read (Chicago) Dec. 28, 1899. Doctor Dissertation, The University of Chicago, 1900.

Miller, G. A. Memoir on Substitution Groups Whose Degree Does not Exceed Eight. Read Dec. 28, 1898. *American Journal of Mathematics*, vol. 21, No. 4, pp. 287–338; Oct., 1899.

—— On the Primitive Groups of Degree Seventeen. Read Feb. 25, 1899. *Quarterly Journal of Mathematics*, vol. 31, No. 1, pp. 49–57; 1899.

—— On the Simply Transitive Primitive Groups. Read Oct 28, 1899. *Bulletin of the American Mathematical Society*, vol. 6, No. 3, pp. 103–104; Dec., 1899.

—— On the Groups That are Direct Products of Subgroups. Read Aug. 26, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 66–71; Jan., 1900.

—— On the Product of Two Substitutions. Read Oct. 28, 1899. *American Journal of Mathematics*, vol. 22, No. 2, pp. 185-190; April, 1900.

—— Note on the Group of Isomorphisms. Read Feb. 24, 1900. *Bulletin of the American Mathematical Society*, vol. 6, No. 8, pp. 337–339; May, 1900.

—— See Ling, G. H.

Moore, E. H. On Certain Crinkly Curves. Read Aug. 25, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 72–90; Jan., 1900.

More, L. T. On Fresnel's Wave Surface. Read Aug. 26, 1899. *Philosophical Magazine*, 5th series, vol. 49, No. 3, pp. 262–274; March, 1900.

Morley, F. On the Limaçons Which Possess Poncelet Polygons. Read May 25, 1895. *Proceedings of the London Mathematical Society*, vol. 29, No. 617, pp. 83-97; Nov., 1897.

—— The Metric Geometry of the Plane *n* Line. Read Feb. 24, 1900. *Transactions of the American Mathematical Society*, vol. 1, No. 2, pp. 97–115; April, 1900.

Moulton, F. R. On a Class of Particular Solutions of the Problem of Four Bodies. Read Aug. 25, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 17–29; Jan., 1900.

Newson, H. B. On Singular Transformations in Real Projective Groups. Read April 28, 1900. *Bulletin of the American Mathematical Society*, vol. 6, No. 10, pp. 431–439; July, 1900.

Osgood, W. F. Beweis der Existenz einer Lösung der Differentialgleichung $\frac{dy}{dx} = (x,y)$ ohne Hinzunahme der Cauchy-Lipschitz'schen Bedingung. Read Feb. 26, 1898. *Monatshefte für Mathematik und Physik*, vol. 9, No. 4, pp. 331–345; Oct., 1898.

Pell, A. Evaluation of a Definite Integral. Read (Chicago) Dec. 28, 1899. *Annals of Mathematics*, 2d series, vol. 1, No. 3, pp. 144–146; April, 1900.

PHILBRICK, P. H. The True Transition Curve. Read Aug. 17, 1897. Field Manual for Railroad Engineers, John Wiley and Sons, New York, 1900.

PIERPONT, J. P. Galois' Theory of Algebraic Equations. Read Sept. 1, 1896. *Annals of Mathematics*, 2d series, vol. 1, No. 3, pp. 113–143; April, 1900.

PORTER, M. B. Note on the Enumeration of the Roots of the Hypergeometric Series between Zero and One. Read Feb. 24, 1900. *Bulletin of the American Mathematical Society*, vol. 6, No. 7, pp. 280–282; April, 1900.

PUPIN, M. I. Propagation of Long Electrical Waves. Read Feb. 25, 1899. *Transactions of the American Institute of Electrical Engineers*, vol. 16, No. 3, pp. 111–160; March, 1899.

REES, J. K. Results of Seven Years' Observations for Variations of Latitude and the Constant of Aberration, Made at the Columbia University Observatory. Read Feb. 24, 1900. *Bulletin of the American Mathematical Society*, vol. 6, No. 7, pp. 269-273; April, 1900. *Popular Astronomy*, vol. 8, No. 4, pp. 169–174; April, 1900.

ROE, E. D. On Symmetric Functions. Read Oct. 29, 1898. *American Mathematical Monthly*, vol. 6, Nos. 1, 2, 3, 4, 5, 6, pp. 1–6, 25–30, 53–58, 103–107, 129-135, 161–165; Jan.-June, 1899.

—— On the Transcendental Form of the Resultant. Read (Chicago) Dec. 28, 1899. *American Mathematical Monthly*, vol. 7, No. 3, pp. 59–66; March, 1900.

SCHOTTENFELS, I. M. On Groups of Order 8!/2. Read June 29, 1900. *Bulletin of the American Mathematical Society*, vol. 6, No. 10, pp. 440–443; July, 1900.

SCOTT, C. A. A Proof of Noether's Fundamental Theorem. Read April 29, 1899. *Mathematische Annalen*, vol. 52, No. 4, pp. 593–597; Dec., 1899.

—— The Status of Imaginaries in Pure Geometry. Read Oct. 28, 1899. *Bulletin of the American Mathematical Society*, vol. 6, No. 4, pp. 163–168; Jan., 1900.

SNYDER, V. Lines of Curvature on Annular Surfaces Having Two Spherical Directrices. Read Feb. 25, 1899. *American Journal of Mathematics*, vol. 22, No. 1, pp. 96-200; Jan., 1900.

—— Geometric Construction of the Elliptic Integral of the Second Kind, and of the Weierstrass Sigma-Function. Read Aug. 26, 1899. Schilling's Modell-Katalog, Halle, 1900.

—— On Cyclical Quartic Surfaces in Space of *n* Dimensions. Read Dec. 28, 1899. *Bulletin of the American Mathematical Society*, vol. 6, No. 5, pp. 194-198; Feb., 1900.

—— On the Geometry of the Circle. Read Dec. 28, 1899. *Bulletin of the American Mathematical Society*, vol. 6, No. 8, pp. 319–322; May, 1900.

STABLER, E. L. A Rule for Finding the Day of the Week Corresponding to a Given Date. Read Aug. 28, 1898. *Science*, new series, vol. 8, No. 200, p. 594; Oct. 28, 1898.

STARKWEATHER, G. P. Non-Quaternion Number Systems Containing No Skew Units. Read April 29, 1899. *American Journal of Mathematics*, vol. 21, No. 4, pp. 369–386 ; Oct., 1899.

STECKER, H. F. Non-Euclidean Properties of Plane Cubics. Read Aug. 20, 1898. *American Journal of Mathematics*, vol. 22, No. 1, pp. 31–40 ; Jan., 1900.

TABER, H. On the Singular Transformations of Groups Generated by Infinitesimal Transformations. Read Feb. 25, 1899, and Dec. 28, 1899. *Bulletin of the American Mathematical Society*, vol. 6, No. 5, pp. 199–203 ; Feb., 1900.

TIMERDING, H. E. Some Remarks on Tetrahedral Geometry. Read June 27, 1900. *Bulletin of the American Mathematical Society*, vol. 6, No. 10, pp. 417–430 ; July, 1900.

VAN VLECK, E. B. On the Determination of a Series of Sturm's Functions by the Computation of a Single Determinant. Read Feb. 25, 1899. *Annals of Mathematics*, ser. 2, vol. 1, No. 1, pp. 1–13 ; Oct., 1899.

WESTLUND, J. On a Class of Equations of Transformation. Read Aug. 19, 1898. *American Journal of Mathematics*, vol. 21, No. 4, pp. 339–353 ; Oct., 1899.

WHITE, H. S. Conics and Cubics Connected with a Plane Cubic by Certain Covariant Relations. Read Aug. 25, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 1, pp. 1–8 ; Jan., 1900.

—— Plane Cubics and Irrational Covariant Cubics. Read (Chicago) Dec. 28, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 2, pp. 170–181 ; April, 1900.

WILSON, E. B. Note on the Functions Satisfying the Functional Relation $\phi(x)\phi(y) = \varphi(x+y)$. Read Apr. 29, 1899. *Annals of Mathematics*, ser. 2, vol. 1, No. 1, pp. 47–48 ; Oct., 1899.

—— The Decomposition of the General Collineation of Space into Three Skew Reflections. Read Oct. 28, 1899. *Transactions of the American Mathematical Society*, vol. 1, No. 2, pp. 193–196 ; April, 1900.

WOODWARD, R. S. The Century's Progress in Applied Mathematics. [Presidential Address.] Read Dec. 28, 1899. *Bulletin of the American Mathematical Society*, vol. 6, No. 4, pp. 133–163. Jan., 1900. *Science*, new series, vol. 11, No. 263, pp. 41–51 ; Jan. 12, 1900 ; No. 264, pp. 81–92 ; Jan. 19, 1900.

—— An Elementary Method of Integrating Certain Linear Differential Equations. Read April 28, 1900. *Bulletin of the American Mathematical Society*, vol. 6, No. 9, pp. 369–371 ; June, 1900.

INDEX.

Abraham, M., 311.
Academy of Sciences, Berlin, 169, 358; Brussels, 216; Irish, 305; Madrid, 38, 358; Munich, 169; National, 117; Paris, 35, 216; Texas, 259; Turin, 119.
Adams Prize, 38.
Address, Presidential, 84, 168.
Address, Presidential: The Century's Progress in Applied Mathematics, R. S. Woodward, 133.
Algebraic Solution of Equations, Vogt's, J. Pierpont, 344.
American Association for the Advancement of Science, The Forty-eighth Annual Meeting of the, G. A. Miller, 57.
——— Association, 32, 408.
——— Mathematical Society, Sixth Summer Meeting, T. F. Holgate, 1; October Meeting, F. N Cole, 95; Sixth Annual Meeting, F. N. Cole, 177; February Meeting, F. N. Cole, 267; April Meeting, F. N. Cole, 365; December Meeting of Chicago Section, T. F. Holgate, 184; April Meeting of Chicago Section, T. F. Holgate, 373.
——— Mathematical Society, Amendments of Constitution and By-Laws, 267, 365; Annual Register, 117, 213; Chicago Section, 32; Election of Officers, 178; New Members Admitted, 2, 95, 177, 267, 365; Presidential Address, 84, 133, 168; Seventh Summer Meeting, 259, 408; Transactions, 213, 354.
——— Physical Society, 85, 95, 177, 267, 365.
Analysis, Kinematics, Mechanics and Astronomy. Villié's Problems in, E. W. Brown, 115.
Analysis, Méray's Infinitesimal, E. O. Lovett, 204.
Analytic Functions, Harkness and Morley's Introduction to the Theory of, O. Bolza, 63.
Analytic Theory of Partial Differential Equations of the First Order, Delassus's, E. O. Lovett, 31.
Andoyer, H., 35.
Andoyer's (H.) Theory of Forms and its Geometric Applications, H. S. White, 29.
Andrews, G., 359.
Annali di Matematica, 411.
Announcements, Publishers', 35, 87, 118, 169, 215-216, 260-261, 306-307, 310, 355.
Annuaire pour l' An 1900 du Bureau des Longitudes, E. W. Brown, 252.
Annual List of Published Papers, Ninth, 472.
——— Meeting of the American Association for the Advancement of Science, The Forty-Eighth, G. A. Miller, 57.
——— Meeting of the American Mathematical Society, The Sixth, F. N. Cole, 177.
——— Register, American Mathematical Society, 117, 213.
Applied Mathematics, The Century's Progress in, R. S. Woodward, 133.
——— Mathematics: Poincaré's Kinematics and Mechanisms, Potential and Mechanics of Fluids, E. W. Brown, 249.
April Meeting of the American Mathematical Society, The, F. N. Cole, 365.
——— Meeting of the Chicago Section, The, T. F. Holgate, 373.
Aronhold, S., 169.
Association, American, 32, 57, 259.
——— Australasian, 306.

——— British, 33, 305.
——— British Mathematical, 214, 260.
——— German, 33, 216, 260, 263, 282, 305.
——— German, for Mathematical Instruction, 408.
——— International, for Promoting the Study of Quaternions, 169.
Astronomical and Astrophysical Society, 85.
Astronomy, Villié's Problems in Analysis, Kinematics, Mechanics and, E. W. Brown, 115.
Austin, W. H., 39.
Australasian Association, 306.
Authors of Articles in the Bulletin :
  Bôcher, M., 279.
  Bolza, O., 63.
  Brown, E. W., 27, 28, 115, 116, 249, 252, 253, 402.
  Cajori, F., 255, 257, 404, 405.
  Cole, F. N., 95, 177, 267, 365.
  Dickson, L. E., 13. 203, 323, 348.
  Ford, W. B., 407.
  Hagen, J. G., 381.
  Hathaway, A. S., 74.
  Holgate, T. F., 1, 184, 373.
  Hutchinson, J. I., 328.
  Lovett, E. O., 30, 31, 78, 109, 204, 299
  Maclay, J., 400.
  Maddison, I., 113.
  Miller, G. A., 57, 103, 105, 337, 390.
  Morley, F., 254, 398.
  Newson, H. B., 431.
  Pierpont, J., 225, 282, 344, 452.
  Porter, M. B., 280.
  Schottenfels, I. M., 440.
  Scott, C. A., 163.
  Snyder, V., 194, 319, 464.
  Sommer, J., 287.
  Taber, H., 199.
  Timerding, H. E., 417.
  White, H. S., 29.
  Whittemore, J. K., 352.
  Woods, F. S., 339, 443.
  Woodward, R. S., 133.
Autonne, L., 216.
Baills, —, 217.
Baire, R , 35.
Ball, R. S., 35.
Barton's (S. M.) Theory of Equations, J. Maclay, 400.
Beltrami, E., 169, 312.
Benedict, H. Y., 40.
Bennett, G. T., 35.
Berlin Academy of Sciences, 169, 358.
——— University of, 308.
Bertrand, J. L. F., 312.
Bianchi, L., 411.
Bibliography, 41, 90, 120, 170, 219, 264, 312, 359, 413, 466.
Bibliotheca Mathematica, 118, 408.
Bigourdan, G., 216.
Birtwhistle, T., 39.
Bjerknes, V., 215.
Blake, E. M., 170, 412.
Bôcher (M) : Some Theorems Concerning Linear Differential Equations of the Second Order, 279.
Bolletino di Matematiche, 408.
Boltzmann, L., 40, 358, 411, 465.
Bolza (O.) : Harkness and Morley's Introduction to the Theory of Analytic Functions, 63.
Bonjour, —, 217.
Bonn, University of, 308.
Books Reviewed, *see* Reviews.
Bourget's (H.) On a Particular Class of Hyperabelian Groups, E. O. Lovett, 78.
Boyer, J., 35.
Boyer's (J.) History of Mathematics, F. Cajori, 405.

Braunmühl, A. von, 118.
Braunmühl's (A. von) History of Trigonometry, F. Cajori, 404.
Breslau, University of, 36, 308.
British Association, 33, 305.
——— Mathematical Association, 214, 260.
Brooks, —, 170.
Brooks, W. R., 217.
Brown (E. W.): Annuaire pour l'An. 1900 du Bureau des Longitudes, 252.
——— Applied Mathematics: Poincaré's Kinematics and Mechanisms, Potential and Mechanics of Fluids, 249.
——— Kepler's Traum vom Mond, 116.
——— Kirchhoff's Lectures on Mathematical Physics, Vol. I, Mechanics, 28.
——— Merriman's Precise Surveying and Geodesy, 253.
——— Peirce's Short Table of Integrals, 116.
——— Rice's Theory and Practice of Interpolation, 402.
——— Tisserand's Determination of Orbits, 27.
——— Villié's Problems in Analysis, Kinematics, Mechanics, and Astronomy, 115.
Brown, S. J., 170.
Bruce, C. W., 312.
Brückner, M., 118.
Brussels Academy of Sciences, 216.
Bullard, W. G., 465
Bunsen, R. W., 40.
Bureau des Longitudes, Annuaire pour l'An 1900, E. W. Brown, 252.
Burkhardt's (H.) Elliptic Functions, J. Pierpont, 452.
Burnside, W., 38, 85, 219, 390.
Burnside's (W.) Theory of Groups of a Finite Order, G. A. Miller, 390.
Burnside, W. S., 118, 390.
Cahen, E., 35.
Cairns, W. D., 170.
Cajori (F.): Boyer's History of Mathematics, 405.
——— Braunmühl's History of Trigonometry, 404.
——— Laisant's Philosophy and Teaching of Mathematics, 257.
——— Maupin's Opinions and Curiosities of Mathematics, 255.
Calculus, Young and Linebarger's Differential and Integral, L. E. Dickson, 348.
——— On the History of the Extensions of the, J. G. Hagen, 381.
——— of Generalization, Oltramare's, E. O. Lovett, 109.
——— of Variations, Pascal's, J. K. Whittemore, 352.
California, University of, 217, 464.
Cambridge Philosophical Society, 118, 259, 464.
——— University, 87, 119, 168, 218, 261, 311.
Cantor, G., 35.
Cantor, M., 85, 409.
Carslaw, H. S., 40.
Catalogue of Scientific Literature, International Conference on, 355.
Century's Progress in Applied Mathematics, The, R. S. Woodward, 133.
Charbonnier, —, 217.
Chicago Section, American Mathematical Society, 32.
——— Section, The December Meeting of the, T. F. Holgate, 184.
——— Section, The April Meeting of the, T. F. Holgate, 373.
——— University, 355, 409.
Chittenden, J. B., 359.
Christoffel, B., 170, 359.
Circle, On the Geometry of the, V. Snyder, 319.

Circolo Matematico di Palermo, 214.
Circular and Spherical Functions, Frischauf's Series in, W. B. Ford, 407.
Cole, F. N., 359.
Cole (F. N.): Reports of the Meetings of the American Mathematical Society: October Meeting, 95; Sixth Annual Meeting, 177; February Meeting, 267; April Meeting, 365.
Columbia University, 356, 359.
Commutators of a Given Group, On the, G. A. Miller, 105.
Congresses, International, at Paris, 34, 215, 259, 354.
Conté, A., 408.
Cornell University, 261, 356.
Correction, 464.
Cosserat, E., 217.
Craig, T., 410.
Crenshaw, B. H., 219.
Crew, H., 119.
Cubic Surface, The Hessian of the, II., J. I. Hutchinson, 328.
Curiosities of Mathematics, Maupin's Opinions and, F. Cajori, 255.
Curtze, M., 85.
Cyclical Quartic Surfaces in Space of N Dimensions, On, V. Snyder, 194.
Darboux, G., 169, 218, 465.
Davies, J. E., 219.
December Meeting of the Chicago Section, The, T. F. Holgate, 184.
Delassus's (É.) Analytical Theory of Partial Differential Equations of the First Order, E. O. Lovett, 31.
De Morgan, A., 310.
Desaint's (L.) Some Points in the Theory of Functions, E. O. Lovett, 78.
Determination of Orbits, Tisserand's, E. W. Brown, 27.
Deutsche Mathematiker-Vereinigung, Summer Meeting at Munich, September, 1899, J. Pierpont, 282.
Dickson, L. E., 39, 358.
Dickson (L. E.): Isomorphism between Certain Systems of Simple Linear Groups, 323.
——— Proof of the Existence of the Galois Field of Order p^r for Every Integer r and Prime Number p, 203.
——— Report on the Recent Progress in the Theory of Linear Groups, 13.
——— Young and Linebarger's Elements of the Calculus, 348.
Differential Equations and the Theory of Groups, Marotte's Linear, E. O. Lovett, 78.
——— Equations of the First Order, Delassus's Analytical Theory of Partial, E. O. Lovett, 31.
——— Equations of the Second Order, Some Theorems Concerning Linear, M. Bôcher, 279.
D'Ocagne's (M.) Nomography, F. Morley, 398.
Drach, J., 217.
Drude, P., 358.
Duncan, L., 90.
Duporcq's (E.) First Principles of Modern Geometry, F. Morley, 254.
Edinburgh Mathematical Society, 213, 259, 305.
——— Royal Society, 214, 465.
Election of Officers, American Mathematical Society, 178.
——— of New Members, American Mathematical Society, 2, 95, 177, 267, 365.
Elements of the Calculus, Young and Linebarger's, L. E. Dickson, 348.
——— of Precise Surveying and Geodesy, Merriman's, E. W. Brown, 253.
——— of the Theory of Forms and its Geometric Applications, Andoyer's, H. S. White, 29.
Elliptic Functions, Burkhardt's, J. Pierpont, 452.

Encyclopædia of the Mathematical Sciences, 215.
Eneström, G., 118, 408.
Engberg, C. C., 465.
Engel's (F.) Translation of Lobachevsky's Geometry, F. S. Woods, 339.
Enumeration of the Roots of the Hypergeometric Series between Zero and One, Note on the, M. B. Porter, 280.
Equations, Barton's Theory of, J. Maclay, 400.
——— of the First Order, Delassus's Analytical Theory of Partial Differential, E. O. Lovett, 31.
——— and the Theory of Groups, Marotte's Linear Differential, E. O. Lovett, 78.
——— of the Second Order, Some Theorems Concerning Linear Differential, M. Bôcher, 279.
——— Vogt's Algebraic Solution of, J. Pierpont, 344.
——— of Heat, Le Roy On the Integration of the, E. O. Lovett, 78.
Erlangen, University of, 36, 261.
Errata, 464.
Escott, E. B., 465.
Existence of the Galois Field of Order p_r for Every Integer r and Prime Number p, Proof of the, L. E. Dickson, 203.
Extensions of the Calculus, On the History of the, J. G. Hagen, 381.
Faculty of Sciences, Paris, 119.
Faught, J. B., 120.
February Meeting of the American Mathematical Society, The, F. N. Cole, 267.
Field of Order p^r for Every Integer r and Prime Number p, Proof of the Existence of the Galois, L. E. Dickson, 203.
Field, P., 39.
First Principles of Modern Geometry, Duporcq's, F. Morley, 254.
Fluids, Poincaré's Kinematics and Mechanisms, Potential and Mechanics of, E. W. Brown, 249.
Ford (W. B.): Frischauf's Series in Circular and Spherical Functions, 407.
Forms, Andoyer's Theory of Forms and its Geometric Applications, H. S. White, 29.
Förster, W., 411.
Forsyth, A. R., 35, 39, 306, 312, 465.
Forty Eighth Annual Meeting of the American Association for the Advancement of Science, The, G. A. Miller, 57.
Foundations of Geometry, Hilbert's, J. Sommer, 287.
Fox, A. D., 359.
France, Mathematical Instruction in, J. Pierpont, 225.
Frischauf's (J.) Series in Circular and Spherical Functions, W. B. Ford, 407.
Fuchs, L., 39, 120.
Functions, Burkhardt's Elliptic, J. Pierpont, 452.
——— Desaint's Some Points in the Theory of, E. O. Lovett, 78.
——— Harkness and Morley's Introduction to the Theory of Analytic, O. Bolza, 63.
Galois Field of Order p^r for Every Integer r and Prime Number p, Proof of the Existence of the, L. E. Dickson, 203.
Galy-Aché, —, 217.
Gauss, C. F., 34.
Generalization, Oltramare's Calculus of, E. O. Lovett, 109.
Gentry, R., 465.
Geodesy, Merriman's Elements of Precise Surveying and, E. W. Brown, 253.
Geometry, Duporcq's First Principles of Modern, F. Morley, 254.
——— Hilbert's Foundations of, J. Sommer, 287.

——— Lobachevsky's, F. S. Woods, 339, 443.
——— of the Circle, On the, V. Snyder, 319.
——— Some Remarks on Tetrahedral, H. E. Timerding, 417.
——— Steiner's Lectures on Synthetic, Part II., I. Maddison, 113.
——— The Status of Imaginaries in Pure, C. A. Scott, 163.
German Mathematical Association, 33, 216, 260, 263, 305.
——— Mathematical Association, Summer Meeting at Munich, September, 1899, J. Pierpont, 282.
Gibbs, J. W., 311, 465.
Giessen, University of, 88, 308.
Goettler, J., 39.
Gordan, P., 358.
Göttingen, University of, 36, 262, 263, 310.
Grassmann, H., 39.
Graves, C., 40.
Gray, A., 90.
Greifswald, University of, 36, 309.
Griess, M., 40.
Group of Isomorphisms, Note on the, G. A. Miller, 337.
——— On the Commutators of a Given, G. A. Miller, 105.
Groups, Bourget On a Particular Class of Hyperabelian, E. O. Lovett, 78
———Generated by Infinitesimal Transformations, On the Singular Transformations of, H. Taber, 199.
——— Isomorphism between Certain Systems of Simple Linear, L. E. Dickson, 323.
——— Marotte's Linear Differential Equations and the Theory of, E. O. Lovett, 78.
——— Note on the Simply Transitive Primitive, G. A. Miller, 103.
——— Of a Finite Order, Burnside's Theory of, G. A. Miller, 390.
——— Of Order 8!/2, On, I. M. Schottenfels, 440.
——— On Singular Transformations in Real Projective, H. B. Newson, 431.
——— Report on the Recent Progress in the Theory of Linear, L. E. Dickson, 13.
Günther's (L.) Edition of Kepler's Traum vom Mond, E. W. Brown, 116.
Günther, S., 85.
Guthrie, F., 219.
Gutzmer, A., 263, 305.
Hackley, C., 312.
Hagen, J. G., 409.
Hagen (J. G.): On the History of the Extensions of the Calculus, 381.
Halle, University of, 37, 309.
Halsted, G. B., 311.
Hancock, H., 311.
Harkness (J.) and Morley's (F.) Introduction to the Theory of Analytic Functions, O. Bolza, 63.
Harkness, W., 170.
Harnack, A., 169.
Harvard University, 357.
Hathaway, A. S., 169.
Hathaway (A. S.): McAulay's Octonions, 74.
Haviland, E., 90.
Heat, Le Roy On the Integration of the Equations of, E. O. Lovett, 78.
Heidelberg, University of, 37, 262.
Hessian of the Cubic Surface, The, II., J. I. Hutchinson, 328.
Hilbert, D., 263, 305.
Hilbert's (D.) Foundations of Geometry, J. Sommer, 287.

History of the Extensions of the Calculus, On the, J. G. Hagen, 381.
——— of Mathematics, Boyer's, F. Cajori, 405.
——— of Trigonometry, Braunmühl's, F. Cajori, 404.
Hölder, O., 307.
Holgate (T. F.): The Sixth Summer Meeting of the American Mathematical Society, 1.
——— The December Meeting of the Chicago Section, 184.
——— The April Meeting of the Chicago Section, 373.
Hopkins Prize, 311.
Horn, J., 358.
House, R. T., 170.
Hudson, J. F., 411.
Humbert, G., 216.
Hutchinson (J. I.): The Hessian of Cubic Surface. II., 328.
Hyperabelian Groups, Bourget's On a Particular Class of, E. O. Lovett, 78.
Hypergeometric Series, Note on the Enumeration of the Roots between Zero and One, M. B. Porter, 280.
Imaginaries in Pure Geometry, The Status of, C. A. Scott, 163.
Index, 479.
Infinitesimal Analysis, Méray's, E. O. Lovett, 204.
——— Transformations, On the Singular Transformations of Groups Generated by, H. Taber, 199.
Innsbruck, University of, 357.
Institute, Lombardy, Mathematical Prize, 410.
——— Venice, Mathematical Prize, 410.
Instruction in France, Mathematical, J. Pierpont, 225.
Integrals, Peirce's Short Table of, E. W. Brown, 116.
Integration of the Equations of Heat, LeRoy's, E. O. Lovett, 78.
International Conference on Catalogue of Scientific Literature, 355.
——— Congress of History of the Sciences at Paris, 215.
——— Congress of Philosophy at Paris, 34, 259.
——— Mathematical Congress at Paris, 354.
Interpolation, Rice's Theory and Practice of, E. W. Brown, 402.
Irish Royal Academy, 305.
Isomorphism between Certain Systems of Simple Linear Groups, L. E. Dickson, 323.
Isomorphisms, Note on the Group of, G. A. Miller, 337.
Jahresbericht of the German Mathematical Association, 216, 260.
Jena, University of, 37, 309, 410.
Johns Hopkins University, 464.
Jones, G. W., 355.
Kasner, E, 359.
Kazan, Physico-Mathematical Society of, 215.
Kelvin, Lord, 39, 84, 90, 311.
Kepler's (J.) Traum vom Mond, E. W. Brown, 116.
Keyser, C. J., 412.
Kiel, University of, 37, 262.
Kinematics and Mechanisms, Potential and Mechanics of Fluids, Poincaré's, E. W. Brown, 249.
——— Villié's Problems in Analysis, Kinematics, Mechanics, and Astronomy, E. W. Brown, 115.
——— Koenigs's Lectures on, E. O. Lovett, 299.
Kirchhoff's (G.) Lectures on Mathematical Physics, Vol. I., Mechanics, E. W. Brown, 28.
Klein, F., 40, 310.
Koenigsberg, University of, 37, 309.
Koenigs's (G.) Lectures on Kinematics, E. O. Lovett, 299.
Kowalewski, G., 39.

Laisant, C. A., 307.
Laisant's (C. A.) Philosophy and Teaching of Mathematics, F. Cajori, 257.
Lampe, E., 169.
Larmor, J., 38.
Laurent, H., 307.
Lawrence, R. S., 312.
Laws of Reciprocity, Stouff's, E. O. Lovett, 30.
Le Cordier, —, 217.
Legendre, A. M., 169.
Leipzig, University of, 38, 309.
Lehmer, D. N., 412.
Le Roy, E., 217.
Le Roy's (E.) On the Integration of the Equations of Heat, E. O. Lovett, 78.
Leuschner, A. O., 412.
Liebmann, H., 120, 264.
Linear Differential Equations and the Theory of Groups, Marotte's, E. O. Lovett, 78.
——— Differential Equations of the Second Order, Some Theorems Concerning, M. Bôcher, 267.
——— Groups, Isomorphism between Certain Systems of Simple, L. E. Dickson, 323.
——— Groups, Report on the Recent Progress in the Theory of, L. E. Dickson, 13.
Linebarger, C. E., 261.
Linebarger (C. E.) and Young's (J. W. A.) Elements of the Differential and Integral Calculus, L. E. Dickson, 348.
Ling, G. H., 412.
List of Members, American Mathematical Society, 117, 213.
——— of Published Papers, Ninth Annual, 472.
Liverpool Mathematical Society, 35.
Lobachevsky's (N. I.) Geometry, F. S. Woods, 339, 443.
Lombardy Institute, Mathematical Prize, 410.
London Mathematical Society, 35, 38, 84, 117, 214, 305, 311.
——— Royal Society, 120, 305.
Loria, G., 38, 408.
Love, A. E. H., 39.
Lovett, E. O., 412.
Lovett (E.O.): Delassus's Analytic Theory of Partial Differential Equations of the First Order, 31.
——— Koenigs's Lectures on Kinematics, 299.
——— Méray's Infinitesimal Analysis, 204.
——— Oltramare's Calculus of Generalization, 109.
——— Stouff's Laws of Reciprocity, 30.
——— Theses in Mathematics at the University of Paris : Desaint's Some Points in the Theory of Functions ; Bourget's On a Particular Class of Hyperabelian Groups ; Le Roy's On the Integration of the Equations of Heat ; Marotte's Linear Differential Equations and the Theory of Groups ; 78.
Lybyer, A. H., 412.
McAulay's (A.) Octonions, A. S. Hathaway, 74.
McCormick, L. J., 264.
McKean, J. F., 411.
Maclay, J., 400.
Maclay (J.): Barton's Theory of Equations, 400.
MacDonald, H. M., 39, 85.
Macfarlane, A., 218.

Mackenzie, A. S., 119.
Maddison (I.): Steiner's Lectures on Synthetic Geometry, Part II., 113.
Madrid Academy of Sciences, 38, 358.
Magie, W. F., 118.
Maltbie, W. H., 170.
Manchester Literary and Philosophical Society, 306.
Marburg, University of, 88, 310.
Markoff, A. A., 216.
Marotte, F., 35.
Marotte's (F.) Linear Differential Equations and the Theory of Groups, E. O. Lovett, 78.
Marshall, W., 465.
Mathematical Gazette, 260.
——— Instruction in France, J. Pierpont, 225.
——— Physics, Kirchhoff's Lectures on, vol. I., Mechanics, E. W. Brown, 29.
Mathematics, Applied: Poincaré's Kinematics and Mechanisms, Potential and Mechanics of Fluids, E. W. Brown, 249.
——— Boyer's History of, F. Cajori, 405.
——— Maupin's Opinions and Curiosities of, F. Cajori, 257.
——— Laisant's Philosophy and Teaching of, F. Cajori, 257.
——— The Century's Progress in Applied, R. S. Woodward, 133.
——— Theses in Mathematics at the University of Paris, E. O. Lovett, 78.
Maupin's (G.) Opinions and Curiosities of Mathematics, F. Cajori, 255.
Mechanics, Kinematics and Mechanisms, Potential and Mechanics of Fluids, Poincaré's, E. W. Brown, 249.
——— Kirchhoff's Lectures on Mathematical Physics, Vol. I., E. W. Brown, 28.
——— Villié's Problems in Analysis, Kinematics, Mechanics and Astronomy, E. W. Brown, 115.
Meetings, American Association, 57, 408.
——— American Mathematical Society, 1, 32, 84, 95, 177, 184, 259, 267, 365, 373, 408.
——— American Physical Society, 85, 95, 177, 267, 365.
——— Astronomical and Astrophysical Society, 85.
——— Australasian Association, 306.
——— British Association, 33, 305.
——— British Mathematical Association, 214.
——— Cambridge Philosophical Society, 118, 259, 464.
——— Circolo Matematico di Palermo, 214.
——— Edinburgh Mathematical Society, 213, 259, 305.
——— Edinburgh Royal Society, 214.
——— German Mathematical Association, 33, 282, 305.
——— German Association for Mathematical Instruction, 408.
——— Irish Royal Academy, 305.
——— Kazan Physico-Mathematical Society, 215.
——— London Mathematical Society, 85, 117, 214, 305.
——— London Royal Society, 305.
——— Manchester Literary and Philosophical Society, 306.
——— National Academy of Sciences, 117.
——— Paris International Congresses, 34, 215, 259, 354.
——— Texas Academy of Sciences, 259.
Members of the American Mathematical Society, List of, 117, 213.
——— Elected, 2, 95, 177, 267, 365.
Méray, C., 216.
Méray's (C.) Infinitesimal Analysis, E. O. Lovett, 204.
Merriman's (M.) Precise Surveying and Geodesy, E. W. Brown, 253.
Mertens, F., 358.

Miller (G. A.): Burnside's Theory of Groups, 390.
——— Note on the Group of Isomorphisms, 337.
——— Note on the Simply Transitive Primitive Groups, 103.
——— On the Commutators of a Given Group, 105.
——— The Forty-eighth Annual Meeting of the American Association for the Advancement of Science, 57.
Minot, C. S., 464.
Mitchell, H. B., 359.
Mittag-Leffler, G., 40, 263.
Modern Geometry, Duporcq's First Principles of, F. Morley, 254.
Moissan, H., 218.
Morley, F., 411.
Morley (F.): D'Ocagne's Nomography, 398.
——— Duporcq's First Principles of Modern Geometry, 254.
Morley's (F.) Introduction to the Theory of Analytic Functions, Harkness (J.) and, O. Bolza, 63.
Morse, R. L., 312.
Moutard, —, 217.
Müller, F., 409.
Muirhead, R. F., 120.
Munich, The Summer Meeting of the Deutsche Mathematiker-Vereinigung at, September, 1899, J. Pierpont, 282.
——— University of, 38, 262.
Myers, G. W., 311.
National Academy of Sciences, 117.
Nernst, W., 40.
Netto, E., 39.
Neuberg, J., 409.
Newcomb, S., 39, 85, 263, 465.
New Publications, 41, 90, 120, 170, 219, 264, 312, 359, 413, 466.
Newson (H. B.): On Singular Transformations in Real Projective Groups, 431.
Ninth Annual List of Published Papers, 472.
Niven, W. D., 311.
Nomography, D'Ocagne's, F. Morley, 398.
Note on the Enumeration of the Roots of the Hypergeometric Series between Zero and One, M. B. Porter, 280.
——— On the Group of Ismorphisms, G. A. Miller, 337.
——— On the Simply Transitive Primitive Groups, G. A. Miller, 103.
Notes, 32, 84, 117, 168, 213, 259, 305, 354, 408, 464.
Notices, Shorter, 27, 113, 252, 398.
Nouvelles Annales de Mathématique, 260.
Nyrén, M., 217.
October Meeting of the American Mathematical Society, The, F. N. Cole, 95.
Octonions, McAulay's, A. S. Hathaway, 74.
Officers of the American Mathematical Society, Elected at the Annual Meeting, 178.
Oltramare's (G.) Calculus of Generalization, E. O. Lovett, 109.
On Groups of Order 8!/2, I. M. Schottenfels, 440.
On Singular Transformations in Real Projective Groups, H. B. Newson, 431.
On the History of the Extensions of the Calculus, J. G. Hagen, 381.
Opdike, H., 40.
Opinions and Curiosities of Mathematics, Maupin's, F. Cajori, 255.
Orbits, Tisserand's Determination of, E. W. Brown, 27.
Oxford University, 87, 311.
Palermo, Circolo Matematico di, 214.

Panton, A. W., 118.
Papers and Communications Presented to the Society, Authors:
  Aley, R. J., 185.
  Blake, E. M., 179.
  Blichfeldt, H. F., 373.
  Bôcher, M., 96, 179, 268.
  Bolza, O., 185.
  Brown, E. W., 365.
  Cajori, F., 2, 365.
  Campbell, J. E., 179.
  Chessin, A. S., 2.
  Collins, J. V., 2.
  Coolidge, J. L., 96.
  Davies, J. E., 2.
  Davis, E. W., 373, 374.
  Dickson, L. E., 3, 179, 185, 268, 373.
  Emch, A., 2.
  Fite, W. B., 3, 179.
  Gale, A. S., 365.
  Gordan, P., 96.
  Hill, G. W., 268.
  Hinton, C. H., 366.
  Hutchinson, J. I., 179, 268.
  Kasner, E., 3.
  Krause, M., 373.
  Kuhn, H. W., 268.
  Laves, K., 373.
  Ling, G. H., 3.
  Loud, F. H., 365.
  Lovett, E. O., 96, 179, 366.
  Macfarlane, A., 268.
  Martin, E. N., 3.
  Maschke, H., 185, 373.
  Miller, G. A., 3, 96, 178, 268, 366
  Miller, J. A., 185.
  Moore, E. H., 3, 96, 373.
  More, L. T., 3.
  Moritz, R. E., 184, 373.
  Morley, F., 3, 96, 268.
  Moulton, F. R., 2, 185.
  Newson, H. B., 3, 366, 373.
  Pell, A., 3, 185.
  Porter, M. B., 268.
  Pupin, M. I., 179.
  Rees, J. K., 268.
  Reid, L. W., 268.
  Richmond, H. W., 2.
  Roe, E. D., 184.
  Scott, C. A., 96, 365.
  Shaw, J. B., 373.
  Smith, J. A., 374.
  Snyder, V., 3, 179, 365.
  Stecker, H. F., 3.
  Strong, W. M., 3.
  Taber, H., 179.
  Van Vleck, E. B., 268.
  Waldo, C. A., 185.
  Westfall, J. V., 185.
  White, H. S., 2, 185.
  Whittemore, J. K., 96.
  Wilczynski, E. J., 184, 365.
  Wilson, E. B., 96.
  Woodward, R. S., 179, 366.
Paranjpye, R. P., 39.
Paris Academy of Sciences, 35, 216.
——— Faculty of Sciences, 119.
——— International Congresses, 34, 215, 259, 354.
Paris, Theses in Mathematics at the University of, E. O. Lovett, 78.
Parrish, N. M., 170.
Partial Differential Equations of the First Order, Delassus's Analytical Theory of, E. O. Lovett, 31.
Partiot, —, 217.
Patillo, N. A., 170.
Pattingill, E. A., 359.
Pascal's (E.) Calculus of Variations, German of Schepp, J. K. Whittemore, 352.
Peirce's (B. O.) Short Table of Integrals, E. W. Brown, 116.
Pennsylvania, University of, 218.
Perchot (J.) Tisserand's Determination of Orbits, E. W. Brown, 27.
Perrin, E., 217.
Perry, N. L., 219.
Philosophy, Paris International Congress of, 34, 259.
——— and Teaching of Mathematics, Laisant's, F. Cajori, 257.
Physics, Kirchhoff's Lectures on Mathematical, Vol. I., Mechanics, E. W. Brown, 28.
Picard, É., 40, 261, 306.
Pierpont (J.): Burkhardt's Elliptic Functions, 452.

——— Mathematical Instruction in France, 225.
——— The Summer Meeting of the Deutsche Mathematiker-Vereinigung at Munich, September, 1899, 282.
——— Vogt's Algebraic Solution of Equations, 344.
Poincaré, H., 215, 218, 465.
Poincaré's (H.) Kinematics and Mechanisms, Potential and Mechanics of Fluids, E. W. Brown, 249.
Pollard, D. H., 412.
Porter (M. B.) : Note on the Enumeration of the Roots of the Hypergeometric Series between Zero and One, 280.
Portraits of Mathematicians, 216.
Potential, Kinematics and Mechanisms, Potential and Mechanics of Fluids, Poincaré's, E. W. Brown, 249.
Practice of Interpolation, Rice's Theory and, E. W. Brown, 402.
Prague, University of, 88, 357.
Presidential Address, 84, 168.
Presidential Address : The Century's Progress in Applied Mathematics, R. S. Woodward, 133.
Preston, T., 312.
Primitive Groups, Note on the Simply Transitive, G. A. Miller, 103.
Principles of Modern Geometry, Duporcq's First, F. Morley, 254.
Prizes, Adams, 38.
——— Brussels Academy, 216.
——— Cambridge University, 311.
——— Copley Medal, 120.
——— DeMorgan Medal, 38, 85.
——— Hopkins, 311.
——— Lombardy Institute, 410.
——— Madrid Academy of Sciences, 38, 358.
——— Paris Academy of Sciences, 216.
——— Royal Astronomical Society, 218.
——— Smith's, 39.
——— Turin Academy of Sciences, 119.
——— Venice Institute, 410.
Projective Groups, On Singular Transformations in Real, H. B. Newson, 431.
Proof of the Existence of the Galois Field of Order p^r for Every Integer r and Prime Number p, L. E. Dickson, 203.
Progress in the Theory of Linear Groups, Report on the Recent, L. E. Dickson, 13.
——— in Applied Mathematics, The Century's, R. S. Woodward, 133.
Publications, New, 41, 90, 120, 170, 219, 264, 312, 359, 413, 466.
Published Papers, Ninth Annual List of, 472.
Publishers' Announcements : Cambridge University Press, 35, 306 ; Carré and Naud, 35, 307 ; D. Appleton and Company, 261 ; Deighton Bell and Company, 261 ; W. Ernst and Son, 169 ; Gauthier-Villars, 35 ; Harper and Brothers, 118 ; Hermann, 35, 169, 216 ; Longmans, Green, and Co., 118 ; Macmillan and Company, 355 ; Teubner, 85, 118, 215, 216, 260, 306, 307, 355 ; Catalogues, 87, 169, 310, 355.
Pure Geometry, The Status of Imaginaries in, C. A. Scott, 163.
Purser, F., 215.
Putnam, I. N., 40.
Quaternions, Society for Promoting the Study of, 169.
Quartic Surfaces in Space of n Dimensions, On Cyclical, V. Snyder, 194.
Radau, J., 216.
Raffy, L., 35, 263.
Rateau, A., 217.
Rayleigh, Lord, 120.

Real Projective Groups, On Singular Transformations in, H. B. Newson, 431.
Recent Progress in the Theory of Linear Groups, Report on the, L. E. Dickson, 13.
Reciprocity, Stouff's Laws of, E. O. Lovett, 30.
Register, Annual, American Mathematical Society, 117, 213.
Reid, L. W., 40, 465.
Remarks on Tetrahedral Geometry, Some, H. E. Timerding, 417.
Report on the Recent Progress in the Theory of Linear Groups, L. E. Dickson, 13.
Reviews, List of Works Reviewed:
Andoyer (H.): Theory of Forms and its Geometric Applications, H. S. White, 29.
Annuaire du Bureau des Longitudes, 1900, E. W. Brown, 252.
Barton (S M.): Theory of Equations, J. Maclay, 400.
Boyer (J.): History of Mathematics, F. Cajori, 405.
Braunmühl (A. von): History of Trigonometry, F. Cajori, 404.
Burkhardt (H.): Elliptic Functions, J. Pierpont, 452.
Burnside (W.): Theory of Groups of a Finite Order, G. A. Miller, 390.
Delassus (É.): Analytical Theory of Partial Differential Equations of the First Order, E O. Lovett, 31.
D'Ocagne (M.): Treatise on Nomography, F. Morley, 398.
Duporcq (E.): First Principles of Modern Geometry, F. Morley, 254.
Frischauf (J.): Series in Circular and Spherical Functions, W. B. Ford, 407.
Harkness (J.) and Morley (F.): Introduction to the Theory of Analytic Functions, O. Bolza, 63.
Hilbert (D.): Foundations of Geometry, J. Sommer, 287.
Kepler (J.): Traum vom Mond, E. W. Brown, 116.
Kirchhoff (G.): Lectures on Mathematical Physics, Vol. I., Mechanics, E. W. Brown, 28.
Koenigs (G): Lectures on Kinematics, E. O. Lovett, 299.
Laisant (C. A.): Philosophy and Teaching of Mathematics, F. Cajori, 257.
Lobachevsky (N. I.): Geometry, F. S. Woods, 339, 443.
McAulay (A.): Octonions, A. S. Hathaway, 74.
Maupin (G.): Opinions and Curiosities of Mathematics, F. Cajori, 255.
Méray (C.): Infinitesimal Analysis and its Geometric Applications, E. O. Lovett, 204.
Merriman (M.): Precise Surveying and Geodesy, E. W. Brown, 253.
Oltramare (G.): Calculus of Generalization, E. O. Lovett, 109.
Pascal (E.): Calculus of Variations, J. K. Whittemore, 352.
Peirce (B. O.): Short Table of Integrals, E. W. Brown, 116
Poincaré (H.): Kinematics and Mechanisms, Potential and Mechanics of Fluids, E. W. Brown, 249.
Rice (H. L.): The Theory and Practice of Interpolation, E. W. Brown, 402.
Steiner (J.): Lectures on Synthetic Geometry, I. Maddison, 113.
Stouff (X.): Laws of Reciprocity, E. O. Lovett, 30.
Theses of Mathematics at the University of Paris: Desaint (L.): On Some Points in the Theory of Functions; Bourget (H.): On a Particular Class of Hyperabelian Groups; Le Roy (E.): On the Integration of the Equations of Heat; Marotte (F.): Linear Differential Equations and the Theory of Groups; E. O. Lovett, 78.
Tisserand (F.): Determination of Orbits, E. W. Brown, 27.
Villié (E.): Problems in Analysis, Kinematics, Mechanics, and Astronomy, E. W. Brown, 115.

Vogt (H.): Algebraic Solution of Equations, J. Pierpont, 344.
Young (J. W. A.) and Linebarger (C. E.): Differential and Integral Calculus, L. E. Dickson, 348.
Revue Philosophique, 408.
Rice's (H. L.) Theory and Practice of Interpolation, E. W. Brown, 402.
Ricci, G., 40.
Robin, G., 35.
Roe, E. D., Jr., 465.
Roots of the Hypergeometric Series between Zero and One, Note on the Enumeration of the, M. B. Porter, 280.
Ross, F. E., 219, 264.
Rowland, H. A., 85, 311.
Royal Astronomical Society, 218.
——— Irish Academy, 305, 311.
——— Society, Edinburgh, 214.
——— Society, London, 305.
Russell, B., 215.
Safford, F. H., 40.
Sawin, A. M., 264.
Schaeffer, H., 264.
Schapira, H., 169.
Schottenfels (I. M.): On Groups of Order 8!/2, 440.
Schottky, F., 358.
Schroeter's (H.) Edition of Steiner's Lectures on Synthetic Geometry, Part II., I. Maddison, 113.
Scott (C. A.): The Status of Imaginaries in Pure Geometry, 163.
Section of the American Mathematical Society, Chicago, 32.
——— The December Meeting of the Chicago, T. F. Holgate, 184.
——— The April Meeting of the Chicago, T. F. Holgate, 373.
Series, Note on the Enumeration of the Roots of the Hypergeometric Series Between Zero and One, M. B. Porter, 280.
——— in Circular and Spherical Functions, Frischauf's, W. B. Ford, 407.
Seventh Summer Meeting of the American Mathematical Society, 259.
Sherk, W. H., 170.
Shorter Notices, 27, 113, 252, 398.
Simple Linear Groups, Isomorphism between Certain Systems of, L. E. Dickson, 323.
Simart, G., 306.
Singular Transformations of Groups Generated by Infinitesimal Transformations, On the, H. Taber, 199.
——— Transformations in Real Projective Groups, On, H. B. Newson, 431.
Sixth Annual Meeting of the American Mathematical Society, The, F. N. Cole, 177.
——— Summer Meeting of the American Mathematical Society, The, T. F. Holgate, 1.
Slaught, H. E., 359.
Smith, C. P., 264.
Smith, P. F., 412.
Snyder (V.): On Cyclical Quartic Surfaces in Space of n Dimensions, 194.
——— On the Geometry of the Circle, 319; Correction, 464.
Solution of Equations, Vogt's Algebraic, J. Pierpont, 344.
Some Remarks on Tetrahedral Geometry, H. E. Timerding, 417.
Some Theorems Concerning Linear Differential Equations of the Second Order, M. Bôcher, 279.
Sommer (J.): Hilbert's Foundations of Geometry, 287.
Sommerfeld, A., 170.

Sonine, N., 216.
Space of n Dimensions, On Cyclical Quartic Surfaces in, V. Snyder, 194.
Spherical Functions, Frischauf's Series in Circular and, W. B. Ford, 408.
Stäckel, P., 169.
Stallo, J. B., 219.
Starkweather, G. P., 412.
Status of Imaginaries in Pure Geometry, The, C. A. Scott, 163.
Steiner's (J.) Lectures on Synthetic Geometry, Part II., I. Maddison, 113.
Stokes (G. G.), 217, 219, 311.
Stouff's (X.) Laws of Reciprocity, E. O. Lovett, 30.
Strassburg, University of, 89, 310.
Sturm (R.): Steiner's Lectures on Synthetic Geometry, Part II., I. Madison, 113.
Summer Meeting of the American Mathematical Society, The Sixth, T. F. Holgate, 1.
——— Meeting of the American Mathematical Society, Seventh, 259.
——— Meeting of the Deutsche Mathematiker-Vereinigung at Munich, September, 1899, The, J. Pierpont, 282.
Surface, The Hessian of the Cubic. II., J. I. Hutchinson, 328.
Surfaces in Space of n Dimensions, On Cyclical Quartic, V. Snyder, 194.
Surveying and Geodesy, Merriman's Precise, E. W. Brown, 253.
Sylvester, J. J., 310.
Synthetic Geometry, Steiner's Lectures on, Part II., I. Maddison, 113.
Systems of Simple Linear Groups, Isomorphism between Certain, L. E. Dickson, 323.
Taber (H.): On the Singular Transformations of Groups Generated by Infinitesimal Transformations, 199.
Table of Integrals, Peirce's Short, E. W. Brown, 116.
Tait, P. G., 311.
Tannenberg, W. de, 35.
Tannery, P., 215.
Tardy, P., 40.
Tchebychev, P. L., 216.
Teaching of Mathematics, Laisant's Philosophy and, F. Cajori, 257.
Teixeira, G., 38.
Tetrahedral Geometry, Some Remarks on, H. E. Timerding, 417.
Texas Academy of Sciences, 259.
Theorems Concerning Linear Differential Equations of the Second Order, Some, M. Bôcher, 279.
Theory of Analytic Functions, Harkness and Morley's Introduction to the, O. Bolza, 63.
——— of Equations, Barton's, J. Maclay, 400.
——— of Forms and its Geometric Applications, Andoyer's, H. S. White, 29.
——— of Functions, Desaint's On Some Points in the, E. O. Lovett, 78.
——— of Linear Groups, Report on the Recent Progress in the, L. E. Dickson, 13.
——— of Groups of a Finite Order, Burnside's, G. A. Miller, 390.
——— of Partial Differential Equations of the First Order, Delassus's Analytic, E. O. Lovett, 31.
——— of Groups, Marotte's Linear Differential Equations and the, E. O. Lovett, 78.
——— and Practice of Interpolation, Rice's, E. W. Brown, 402.
Theses in Mathematics at the University of Paris, E. O. Lovett, 78.
Thompson, J. D., 40.
Thomson, J. J., 311.
Thrasher, W. M., 359.
Timerding (H. E.): Some Remarks on Tetrahedral Geometry, 417.

Tisserand, F., 90.
Tisserand's (F.) Determination of Orbits, E. W. Brown, 27.
Transactions of the American Mathematical Society, 213, 354.
Transformations of Groups Generated by Infinitesimal Transformations, On the Singular, H. Taber, 199.
——— in Real Projection Groups, On Singular, H. B. Newson, 431.
Transitive Primitive Groups, Note on the Simply, G. A. Miller, 103.
Traum vom Mond, Kepler's, E. W. Brown, 116.
Trigonometry, Braunmühl's History of, F. Cajori, 404.
Tübingen, University of, 89, 263.
Turin Academy of Sciences, 119.
Turner, W., 305.
University of Paris, Theses in Mathematics at the, E. O. Lovett, 78.
Universities, Announcement of Courses in Mathematics:
  Berlin, 35, 308.
  Bonn, 308.
  Breslau, 36, 308.
  California, 217, 464.
  Cambridge, 87.
  Chicago, 355, 409.
  Columbia, 356.
  Cornell, 261, 356.
  Erlangen, 36, 261.
  Giessen, 88, 308.
  Göttingen, 36, 262, 263.
  Greifswald, 36, 309.
  Halle, 37, 309.
  Harvard, 357.
  Heidelberg, 37, 262.
  Innsbruck, 357.
  Jena, 37, 309.
  Johns Hopkins, 465.
  Kiel, 37, 262.
  Königsberg, 37, 309.
  Leipzig, 38, 309.
  Marburg, 88, 310.
  Munich, 38, 262.
  Oxford, 87.
  Paris, 119.
  Pennsylvania, 218.
  Prague, 88, 357.
  Strassburg, 89, 310.
  Tübingen, 89, 263.
  Vienna, 89, 357.
  Würzburg, 89, 310.
  Yale, 410.
  Zürich, 90.
Updegraff, M., 39.
Vargas y Aguirre, J., 38.
Variations, Pascal's Calculus of, J. K. Whittemore, 352.
Vasiliev, A., 215, 216.
Venice Institute, 410.
Veronese, G., 40.
Vienna, University of, 89, 357.
Villié's (E.) Problems in Analysis, Kinematics, Mechanics, and Astronomy, E. W. Brown, 115.
Vogt's (H.) Algebraic Solution of Equations, J. Pierpont, 344.
Von Ende, A., 219.
Walker, G. T., 38.
Walker, G. W., 39.
Walker, J. J., 312.
Wallon, E., 35.
Wangerin, A., 409.
Weber, H., 312.
Weber, W., 34.
Weingarten, J., 40.
Weiss, E., 411.
Westfall, J. V., 219.
White (H. S.): Andoyer's Theory of Forms and its Geometric Applications, 29.
Whittemore, J. K., Pascal's Calculus of Variations, German Edition by A. Schepp, 352.
Wien's (W.) Edition of Kirchhoff's Lectures on Mathematical Physics, Vol. I., Mechanics, E. W. Brown, 28.

Wilkinson, A. L., 465.
Wilson, A. H., 40, 412.
Wilson, E. B., 412.
Winston, M. F., 466.
Woods (F. S.): Lobachevsky's Geometry, 339, 443.
Woodward, R. S., 32, 84, 168.
Woodward (R. S.): The Century's Progress in Applied Mathematics (Presidential Address), 133.
Würzburg, University of, 89, 310.
Yale University, 410.
Young, J. W. A., 261.
Young (J. W. A.) and Linebarger's (C. E.) Differential and Integral Calculus, L. E. Dickson, 348.
Zeuthen, H. G., 263.
Zürich, University of, 90.